Belief Dispersion in the Stock Market*

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This Version: May 2017

Abstract

We develop a dynamic model of belief dispersion with a continuum of investors differing in beliefs. The model is tractable and qualitatively matches many of the empirical regularities in a stock price, its mean return, volatility, and trading volume. We find that the stock price is convex in cash-flow news and increases in belief dispersion, while its mean return decreases when the view on the stock is optimistic, and vice versa when pessimistic. Moreover, belief dispersion leads to a higher stock volatility and trading volume. We demonstrate that otherwise identical two-investor heterogeneous-beliefs economies do not necessarily generate our main results.

JEL Classifications: D53, G12.
Keywords: Asset pricing, belief dispersion, heterogeneous beliefs, stock price, mean return, volatility, trading volume, Bayesian learning.

*Email addresses: aatmaz@purdue.edu and sbasak@london.edu. We thank Andrea Buffa, Georgy Chabakauri, Francisco Gomes, Christian Heyerdahl-Larsen, Ralph Koijen, Hongjun Yan, and our discussants Daniel Andrei, Michael Brennan, Burton Hollifield, Philipp Illeditsch, Tim Johnson, Mina Lee, Bart Taub, as well as the seminar participants at the 2015 EFA meetings, 2015 International Moscow Finance Conference, 2015 SFS Finance Cavalcade, 2015 Wabash River Finance Conference, 2015 WFA meetings, 2017 Berlin-Princeton-Singapore Workshop on Quantitative Finance, London Mathematical Finance Seminar, Marti Subrahmanyam Festschrift, Workshop in Memory of Rick Green, and City University of Hong Kong, Graduate Institute Geneva, Hong Kong Polytechnic University, London Business School, Nanyang Technological University, University of Hong Kong, University of Hull, University of Lugano, University of New South Wales, University of Oxford, University of Southampton, University of Sydney, University of Technology Sydney for helpful comments. We have also benefited from the valuable suggestions of Ken Singleton (the editor) and two anonymous referees. All errors are our responsibility. The authors do not have any conflicts of interest as identified in the Disclosure Policy.

1 Introduction

The empirical evidence on the effects of investors' dispersion of beliefs on asset prices and their dynamics is vast and mixed. For example, several works find a negative relation between belief dispersion and a stock mean return (Diether, Malloy, and Scherbina (2002), Chen, Hong, and Stein (2002), Goetzmann and Massa (2005), Park (2005), Berkman, Dimitrov, Jain, Koch, and Tice (2009), Yu (2011)). Others argue that the negative relation is only valid for stocks with certain characteristics (e.g., small, illiquid, worst-rated or short sale constrained) and in fact, find either a positive or no significant relation (Qu, Starks, and Yan (2003), Doukas, Kim, and Pantzalis (2006), Avramov, Chordia, Jostova, and Philipov (2009)). Existing theoretical works (discussed below), on the other hand, do not provide satisfactory answers for these mixed results.

In this paper, we develop a tractable model of belief dispersion which is able to qualitatively match many of the empirical regularities in a stock price, its mean return, volatility, and trading volume. Towards that, we develop a dynamic general equilibrium model populated by a continuum of constant relative risk aversion (CRRA) investors who differ in their (dogmatic or Bayesian) beliefs and consume at a single consumption date. There are two key differences between our model and existing works, which typically employ two investors and a continuum of consumption dates. First, rather than considering the overall effects of belief heterogeneity, we isolate the effects of belief dispersion from the effects of other moments and conduct comparative statics analysis with respect to belief dispersion only, resulting in sharp results. Second, because of a continuum of investors in our model, no investor dominates the economy in relatively extreme states, leading to a non-vanishing belief dispersion, and hence to simple uniform behavior for economic quantities. Moreover, dynamic models with heterogeneous beliefs are generally hard to solve for long-lived assets beyond logarithmic preferences (e.g., Detemple and Murthy (1994), Zapatero (1998), Basak (2005)), nevertheless our model delivers fully closed-form expressions for quantities of interest.[1]

In our analysis, we summarize the wide range of investors' beliefs by two sufficient measures, the average bias and dispersion in beliefs, and demonstrate that equilibrium quantities are driven by these two key endogenous variables. We take the average bias to be the bias of the representative investor whereby how much an investor's belief contributes to the average bias depends on her wealth and risk attitude. Investors whose beliefs get supported by actual

[1]We explicitly obtain all quantities of interest in closed-form in all the economic settings considered in the paper, with the exception of trading volume in the multiple-stocks setting of Section 6.

cash-flow news become relatively wealthier through their investment in the stock, and therefore contribute more to the average bias. This leads to fluctuations in the average bias so that following good (bad) cash-flow news, the view on the stock becomes relatively more optimistic (pessimistic). On the other hand, consistently with empirical studies, we construct our belief dispersion measure as the cross-sectional standard deviation of investors' disagreement which also enables us to reveal its dual role. First, we uncover a novel role of belief dispersion in that it amplifies the average bias so that the same good (bad) news leads to more optimism (pessimism) when dispersion is higher. Second, we show that belief dispersion indicates how much the average bias fluctuates, and therefore measures the extra uncertainty investors face.

Turning to our model implications, we first find that in the presence of belief dispersion the stock price is convex in cash-flow news, indicating that the stock price is more sensitive to news in relatively good states. It also implies that the increase in the stock price following good news is more than the decrease following bad news, as supported by empirical evidence (Basu (1997), Xu (2007)). Convexity arises because, the better the cash-flow news, the higher the extra boost for the stock price coming from elevated optimism. Consequently, the stock price increases with belief dispersion when the view on the stock is relatively optimistic, and decreases otherwise, also consistent with empirical evidence (Yu (2011)). Our model also implies that the stock price may increase and its mean return may decrease in investors' risk aversion in relatively bad states. This is because in a more risk averse economy investors have less exposure to the stock which limits the wealth transfers to pessimistic investors in bad times, leading to a relatively optimistic view on the stock, hence to a higher stock price and a lower mean return.

We next examine the widely-studied relation between belief dispersion and a stock mean return. Since dispersion represents the extra uncertainty investors face, risk averse investors demand a higher return to hold the stock when dispersion is higher. However, dispersion also amplifies optimism and pushes up the stock price further following good news leading to a lower mean return in those states. When the view on the stock is relatively optimistic, the second effect dominates and we find a negative dispersion-mean return relation. As discussed earlier, empirical evidence on this relation is mixed, with some studies finding a negative while others finding a positive or no significant relation. Our model generates both possibilities and demonstrates that this relation is negative when the view on the stock is relatively optimistic, and positive otherwise. Diether, Malloy, and Scherbina (2002) provide supporting evidence to our finding by documenting an optimistic bias in their study overall, and by also showing that the negative effect of dispersion becomes stronger for more optimistic stocks. A similar evidence is also provided by Yu (2011).

We further find that the stock volatility increases monotonically in belief dispersion, consistent with empirical evidence (Ajinkya and Gift (1985), Anderson, Ghysels, and Juergens (2005), Banerjee (2011)). This is because the average bias in beliefs fluctuates more, and hence so does the stock price, when belief dispersion is higher. In addition to belief dispersion, the investors' Bayesian learning process also increases the fluctuations in the average bias, and hence leads to a higher stock volatility. This occurs because all investors become relatively more optimistic (pessimistic) following good (bad) news due to belief updating. Our closed-form stock volatility expression allows us to disentangle the respective effects of belief dispersion and Bayesian learning, and yields a novel testable implication that Bayesian learning induces less stock volatility when belief dispersion is higher. Moreover, we find that the stock trading volume is also increasing in belief dispersion, consistently with empirical evidence (Ajinkya, Atiase, and Gift (1991), Bessembinder, Chan, and Seguin (1996), Goetzmann and Massa (2005)). This finding is intuitive since when dispersion is higher, investors with relatively different beliefs, who also have relatively higher trading demands, are more dominant. We also find a positive relation between the stock volatility and trading volume due to the positive effect of dispersion on both quantities, also supported empirically (Gallant, Rossi, and Tauchen (1992), Banerjee (2011)).

We further demonstrate that most of our results discussed above do not necessarily obtain in an otherwise identical economy to ours but populated by two rather than a continuum of investors. In particular, we show that in this economy, the stock price is no longer convex in cash-flow news across all states of the world, and a higher belief dispersion can actually lead to a lower stock volatility and trading volume in some states of the world. This happens because in this economy, unlike in our model, belief heterogeneity effectively vanishes in relatively extreme states, since the pessimistic investor eventually controls almost all the wealth in the economy in very bad states, and the optimistic investor in very good states. The transition from the states in which belief heterogeneity is prevalent to the relatively extreme states in which belief heterogeneity vanishes generates irregular behavior for economic quantities across states of the world. Finally, we generalize our main model with a single stock to one with multiple stocks, on which investors have different beliefs. We demonstrate that all our main results and underlying economic mechanisms still go through in this more elaborate economy. We also provide an extension of our main model with a single stock payoff and a consumption date to feature multiple stock payoffs and consumption dates (Internet Appendix ID), where we demonstrate that our main insights remain valid, though the equilibrium characterization becomes more complex.

Our methodological contribution and the tractability of our model is in large part due to

a continuum of investor types having a Gaussian distribution. This assumption follows from the recent works by Cvitanić and Malamud (2011) and Atmaz (2014). Cvitanić and Malamud focus on the survival and portfolio impact of irrational investors and do not characterize the investor belief heterogeneity, and consequently express the equilibrium quantities, in terms of average bias and dispersion in beliefs as we do, while Atmaz does, but employs logarithmic preferences and focuses on short interest.

The literature on heterogeneous beliefs in financial markets is vast. One strand of this literature examines the relation between belief dispersion and stock mean return and typically finds this relation to be positive (Abel (1989), Anderson, Ghysels, and Juergens (2005), David (2008), Banerjee and Kremer (2010)). On the other hand, Chen, Hong, and Stein (2002) and Johnson (2004) establish a negative relation by imposing short selling constraints for certain type of investors and considering levered firms, respectively. Buraschi, Trojani, and Vedolin (2013) develop a credit risk model and show that an increasing heterogeneity of beliefs has a negative (positive) effect on the mean return for firms with low (high) leverage. However, this result does not hold for unlevered firms. Differently from these works, we show that the dispersion-mean return relation is negative when the view on the stock is relatively optimistic and positive otherwise. Moreover, in this literature, David (2008) shows that belief heterogeneity leads to a non-monotonic relation between the mean return and investors' risk aversion. We compliment David by providing the additional insight that the relation between the mean return and risk aversion depends on the level of the optimism/pessimism on the stock, and is non-monotonic only when the view is relatively pessimistic, but it is monotonic otherwise.

Another strand in the literature examines the impact of belief heterogeneity on stock volatility and typically finds a positive effect (Scheinkman and Xiong (2003), Buraschi and Jiltsov (2006), Li (2007), David (2008), Dumas, Kurshev, and Uppal (2009), Banerjee and Kremer (2010), Andrei, Carlin, and Hasler (2015)). Yet another strand in this literature employs belief dispersion models to explain empirical regularities in trading volume. Early works include Harris and Raviv (1993) and Kandel and Pearson (1995). This strand also includes works that find a positive relation between belief dispersion and trading volume, as in our work (Varian (1989), Shalen (1993), Cao and Ou-Yang (2008), Banerjee and Kremer (2010)). Even though our paper differs from each one of these works in several aspects, one common difference is that none of them generate the stock price convexity as in our model.[2]

[2]Other works studying the effects of heterogeneous beliefs in financial markets include Basak (2000, 2005), Kogan, Ross, Wang, and Westerfield (2006), Jouini and Napp (2007), Yan (2008), Xiong and Yan (2010), Zapatero (1998). Additionally, more general works study the effects of heterogeneous beliefs in the presence of other investor heterogeneities, including Cvitanić, Jouini, Malamud, and Napp (2012), Bhamra and Uppal

Finally, this paper is also related to the literature on parameter uncertainty and Bayesian learning. In this literature, Veronesi (1999) and Lewellen and Shanken (2002) show that learning leads to stock price overreaction, time-varying expected returns and higher volatility. In particular, Veronesi shows that the stock price overreaction leads to a convex stock price.[3] Timmermann (1993, 1996), Barsky and De Long (1993), Brennan and Xia (2001), Pástor and Veronesi (2003) show that learning increases volatility and generates predictability for stock returns. However, differently from our work, all these works employ homogeneous investors setups, and therefore are not suitable for studying the effects of belief dispersion. The works that study the effects of parameter uncertainty and Bayesian learning with heterogeneous beliefs include Nakov and Nuño (2015) and Collin-Dufresne, Johannes, and Lochstoer (2017). These works show that the learning bias among young and old investors leads to booms and busts in stock prices, long-run return predictability and variations in price-dividend ratios. These results are primarily driven by the standard mechanism of Bayesian updating in the presence of parameter uncertainty, which is also present in our analysis with Bayesian learning, where investors become optimistic (pessimistic) after a sequence of positive (negative) shocks. However, these works employ discrete-time settings and rely on numerical solutions for their results, in contrast to our continuous-time setting leading to analytical results.

Section 2 presents the main model, Section 3 analyzes the average bias and dispersion in beliefs, and Section 4 provides our results on the stock price, its dynamics and trading volume. Section 5 presents two-investor economy, Section 6 the multi-stock economy, and Section 7 our main model with Bayesian learning. Section 8 concludes. Appendix A contains the proofs of the main model, Appendix B discusses the parameter values employed in Figures. Internet Appendix IA contains the proofs of the two-investor economy, IB the proofs of the multi-stock economy, IC the proofs of the main model with Bayesian learning, ID provides an extension of our analysis that features multiple stock payoffs and consumption dates.

(2014) (risk aversion and time preference), Chabakauri (2015) (risk aversion and various portfolio constraints), Chabakauri and Han (2015) (risk aversion and non-pledgeable labor income), Detemple and Murthy (1997) (short-selling and borrowing constraints), Gallmeyer and Hollifield (2008) (short-selling constraints), Osambela (2015) (funding constraint).

[3] In Veronesi (1999) the stock price convexity arises due to parameter uncertainty and the learning process, whereas in our model the convexity follows from the stochastic average bias in beliefs and obtains even when there is no parameter uncertainty and learning. Relatedly, Xu (2007) develops a model in which the stock price is a convex function of the public signal. However, in his model no-short-sales constraints are needed to obtain this result.

2 Economy with Dispersion in Beliefs

We consider a simple and tractable pure-exchange security market economy with a finite horizon evolving in continuous time. The economy is assumed to be large as it is populated by a continuum of investors with heterogeneous beliefs and standard CRRA preferences. In the general specification of our model, investors optimally learn over time in a Bayesian fashion. However, to highlight that our results are not driven by parameter uncertainty and learning, we first consider the economy when all investors have dogmatic beliefs. The richer case when investors update their beliefs over time is relegated to Section 7 where we show that all our results hold in this more complex economy. Moreover, to demonstrate our main economic mechanism and results as clearly as possible, we first consider economies with a single risky stock. The generalization to the more elaborate economy with multiple stocks is undertaken in Section 6, where we again show that all our main predictions remain valid.

2.1 Securities Market

There is a single source of risk in the economy which is represented by a Brownian motion ω defined on the true probability measure $\mathbb{P}$. Available for trading are two securities, a risky stock and a riskless bond. The stock price S is posited to have dynamics

$$dS_t = S_t \left[\mu_{St} dt + \sigma_{St} d\omega_t\right], \tag{1}$$

where the stock mean return μ_S and volatility σ_S are to be endogenously determined in equilibrium. The stock is in positive net supply of one unit and is a claim to the payoff D_T, paid at some horizon T, and so $S_T = D_T$. This payoff D_T is the horizon value of the process D_t with dynamics

$$dD_t = D_t \left[\mu dt + \sigma d\omega_t\right], \tag{2}$$

where $D_0 = 1$, and μ and σ are constant, and represent the true mean growth rate of the expected payoff and the uncertainty about the payoff, respectively. The process D_t represents the arrival of news about D_T, and hence we refer to it as the cash-flow news. Moreover, this cash-flow news process can be mapped into the analyst forecasts about the long-term earnings growth rates, as we discuss in Appendix B. The bond is in zero net supply and pays a riskless interest rate r, which is set to 0 without loss of generality.[4]

[4]Since in this setting consumption can occur only at time T (i.e., no intermediate consumption), the interest rate can be taken exogenously. Our normalization of zero interest rate is for expositional simplicity and it is commonly employed in models with no intermediate consumption, see, for example, Pástor and Veronesi (2012)

2.2 Investors' Beliefs

There is a continuum of investors who commonly observe the same cash-flow news process D (2), but have different beliefs about its dynamics. The investors are indexed by their type θ, where a θ-type investor agrees with others on the stock payoff uncertainty σ but believes that the mean growth rate of the expected payoff is $\mu + \theta$ instead of μ. This allows us to interpret a θ-type investor as an investor with a bias of θ in her beliefs. Consequently, a positive (negative) bias for an investor implies that she is relatively optimistic (pessimistic) compared to an investor with true beliefs. Under the θ-type investor's beliefs, the cash-flow news process has dynamics

$$dD_t = D_t \left[(\mu + \theta)\, dt + \sigma d\omega_t (\theta) \right],$$

where $\omega(\theta)$ is her perceived Brownian motion with respect to her own probability measure $\mathbb{P}^\theta$, and is given by $\omega_t(\theta) = \omega_t - \theta t/\sigma$. Similarly, the risky stock price dynamics as perceived by the θ-type investor follows

$$dS_t = S_t \left[\mu_{St}(\theta)\, dt + \sigma_{St} d\omega_t (\theta) \right], \tag{3}$$

which together with the dynamics (1) yields the following consistency relation between the perceived and true stock mean returns for the θ-type investor

$$\mu_{St}(\theta) = \mu_{St} + \sigma_{St} \frac{\theta}{\sigma}. \tag{4}$$

The investor type space is denoted by Θ and it is taken to be the whole real line $\mathbb{R}$ to incorporate all possible beliefs including the extreme ones and to avoid having arbitrary bounds for investor biases. We assume a Gaussian distribution with mean $\tilde{m}$ and standard deviation $\tilde{v}$ for the relative frequency of investors over the type space Θ. A higher $\tilde{m}$ ($\tilde{v}$) implies that initially there are more investors with relatively optimistic (large) biases. This specification conveniently nests the benchmark homogeneous beliefs economy with no bias when $\tilde{m} = 0$ and $\tilde{v} \to 0$. Moreover, this assumption ensures that the investor population has a finite (unit) measure and admits much tractability, and can be justified on the grounds of the typical investor distribution observed in well-known surveys.[5] We further assume that all investors are initially

for a recent reference.

[5]See, for example, the Livingston survey and the survey of professional forecasters conducted by the Philadelphia Federal Reserve. Generally, the observed distributions are roughly symmetric, single-peaked and assign less and less people to the tails, resembling a Binomial distribution for a limited sample. For a large economy, these properties can conveniently be captured by our Gaussian distribution assumption, which also follows from the recent works by Cvitanić and Malamud (2011) and Atmaz (2014) in dynamic settings as discussed in Introduction. Söderlind (2009) also invokes this assumption but in a single-period static model, and obtains implications that are different from ours, since ours are much driven by the dynamic interactions between economic quantities as we demonstrate in the ensuing analysis.

endowed with an equal fraction of stock shares. Since a group of investors with the same beliefs and endowments are identical in every aspect, we represent them by a single investor with the same belief and whose initial endowment of stock shares is equal to the relative frequency of that group. This simplifies the analysis and provides the following initial wealth for each distinct θ-type investor

$$W_0(\theta) = S_0 \frac{1}{\sqrt{2\pi\tilde{v}^2}} e^{-\frac{1}{2}\frac{(\theta-\tilde{m})^2}{\tilde{v}^2}}, \tag{5}$$

where S_0 is the (endogenous) initial stock price.

2.3 Investors' Preferences and Optimization

Each distinct θ-type investor chooses an admissible portfolio strategy $\phi(\theta)$, the fraction of wealth invested in the stock, so as to maximize her CRRA preferences over the horizon value of her portfolio $W_T(\theta)$

$$\mathbb{E}^\theta \left[\frac{W_T(\theta)^{1-\gamma}}{1-\gamma} \right], \qquad \gamma > 0, \tag{6}$$

where $\mathbb{E}^\theta$ denotes the expectation under the θ-type investor's subjective beliefs $\mathbb{P}^\theta$, and the financial wealth of the θ-type investor $W_t(\theta)$ follows

$$dW_t(\theta) = \phi_t(\theta) W_t(\theta) \left[\mu_{St}(\theta)\, dt + \sigma_{St} d\omega_t(\theta) \right]. \tag{7}$$

In this setting investors' preferences are over the horizon value of their wealth/consumption rather than intermediate consumption, which would otherwise endogenize the interest rate in equilibrium. As the previous literature highlights, the presence of belief heterogeneity may have important effects on the interest rate in the economy (e.g., Detemple and Murthy (1994), David (2008)). However, in this paper, our focus is not on the interest rate, but on the marginal effects of belief dispersion on risky stocks, and as we demonstrate in Section 6.2, we can still calibrate our model and quantify these effects even though the interest rate is exogenous.

3 Equilibrium in the Presence of Belief Dispersion

To explore the implications of belief dispersion on the stock price and its dynamics, we first need a reasonable measure of it. In this Section, we define belief dispersion in a canonical way, to be the standard deviation of investors' biases in beliefs. Using the cross-sectional standard deviation of investors' disagreement as belief dispersion is also consistent with the commonly

employed belief dispersion measures in empirical studies.[6] However, for this, we first need to determine the average bias in beliefs from which the investors' biases deviate. The average bias is defined to be the bias of the representative investor in the economy. We then summarize the wide range of investors' beliefs in our economy by these two variables, the *average bias* and *dispersion in beliefs*, and determine their values in the ensuing equilibrium. As we also demonstrate in Section 4, the equilibrium quantities are driven by these two key (endogenous) variables, in addition to those in a homogeneous beliefs economy. Moreover, specifying the belief dispersion this way enables us to isolate its effects from the effects of other moments and conduct comparative statics analysis with respect to it only.

Equilibrium in our economy is defined in a standard way. The economy is said to be in equilibrium if equilibrium portfolios and asset prices are such that (i) all investors choose their optimal portfolio strategies, and (ii) stock and bond markets clear. We will often make comparisons with equilibrium in a *benchmark economy* where all investors have unbiased beliefs. We refer to this homogeneous beliefs economy as the economy with no belief dispersion.

Definition 1 (**Average bias and dispersion in beliefs**)**.** The time-t *average bias in beliefs*, m_t, is defined as the implied bias of the corresponding representative investor in the economy. Moreover, expressing the average bias in beliefs as the weighted average of the individual investors' biases

$$m_t = \int_{\Theta} \theta h_t(\theta)\, d\theta, \tag{8}$$

with the weights $h_t(\theta) > 0$ are such that $\int_{\Theta} h_t(\theta)\, d\theta = 1$, we define the *dispersion in beliefs*, v_t, as the standard deviation of investors' biases

$$v_t^2 \equiv \int_{\Theta} (\theta - m_t)^2 h_t(\theta)\, d\theta. \tag{9}$$

The extent to which an investor's belief is represented in the economy depends on her wealth and risk attitude. In our dynamic economy, the investors whose beliefs are supported by the actual cash-flow news become relatively wealthier. This increases the impact of their beliefs in the determination of equilibrium prices. Our definition of the average bias in beliefs captures

[6]See, for example, Diether, Malloy, and Scherbina (2002), Johnson (2004), Boehme, Danielsen, and Sorescu (2006), Sadka and Scherbina (2007), Avramov, Chordia, Jostova, and Philipov (2009) who employ the standard deviation of levels in analysts' earnings forecasts, normalized by the absolute value of the mean forecast. Anderson, Ghysels, and Juergens (2005), Moeller, Schlingemann, and Stulz (2007), Yu (2011) employ the standard deviation of (long-term) growth rates in analysts' earnings forecasts as the measure of belief dispersion. Since we define ours as the standard deviation of investors' biases, our belief dispersion measure is similar to those used in the latter works. As Moeller, Schlingemann, and Stulz (2007) argue, there are several advantages of using the standard deviation of growth rates rather than of levels as a measure of belief dispersion, since the timing of the forecasts affect levels but not growth rates, and since growth rates are easily comparable across firms whereas normalization introduces noise for the levels.

this mechanism by equating it to the bias of the representative investor who assigns more weight to an investor whose belief has more impact on the equilibrium prices. Finding the average bias this way is similar to representing heterogeneous beliefs in an economy by a consensus belief as in Rubinstein (1976), and more recently in Jouini and Napp (2007).[7]

The average bias in beliefs, by construction, implies that when it is positive the (average) view on the stock is optimistic, and when negative pessimistic. The weights, $h_t(\theta)$, are such that the weighted average of individual investors' biases is the bias of the representative investor. We also discuss alternative weights, average bias and dispersion measures in Remark 1. Importantly, it is these weights that allow us to define belief dispersion in an intuitive way. Proposition 1 presents the average bias and dispersion along with the corresponding unique weights in our economy in closed form.

Proposition 1. *The time-t average bias m_t and dispersion v_t in beliefs are given by*

$$m_t = m + \Big(\ln D_t - \Big(m + \mu - \frac{1}{2}\sigma^2\Big)t\Big)\frac{v_t^2}{\gamma\sigma^2}, \qquad v_t^2 = \frac{v^2\sigma^2}{\sigma^2 + \frac{1}{\gamma}v^2 t}, \tag{10}$$

where their initial values m and v are related to the initial mean $\tilde{m}$ and standard deviation $\tilde{v}$ of investor types as

$$m = \tilde{m} + \Big(1 - \frac{1}{\gamma}\Big)v^2 T, \qquad v^2 = \Big(\frac{\gamma}{2}\tilde{v}^2 - \frac{\gamma^2}{2T}\sigma^2\Big) + \sqrt{\Big(\frac{\gamma}{2}\tilde{v}^2 - \frac{\gamma^2}{2T}\sigma^2\Big)^2 + \frac{\gamma^2}{T}\tilde{v}^2\sigma^2}. \tag{11}$$

The weights $h_t(\theta)$ are uniquely identified to be given by

$$h_t(\theta) = \frac{1}{\sqrt{2\pi v_t^2}} e^{-\frac{1}{2}\frac{(\theta - m_t)^2}{v_t^2}}, \tag{12}$$

where m_t, v_t are as in (10)*.*

Consequently, a higher belief dispersion v_t leads to a higher average bias m_t for relatively good cash-flow news $D_t > \exp\left(m + \mu - \frac{1}{2}\sigma^2\right)t$, and to a lower average bias otherwise.

We see that the average bias in beliefs (10) is stochastic and depends on the cash-flow news D.[8] When there is good news, the relatively optimistic investors' beliefs get supported, and

[7]The main idea, as elaborately discussed in Jouini and Napp (2007), is to summarize the heterogeneous beliefs in the economy by a single consensus belief so that when the consensus investor has that consensus belief and is endowed with the aggregate consumption in the economy, the resulting equilibrium is as in the heterogeneous-investors economy. In a model with intertemporal consumption and finitely-many agents having CRRA preferences, Jouini and Napp show that when investors' preferences are not logarithmic, the consensus belief is not necessarily well-defined since the process which aggregates investors' beliefs is not a martingale, and hence not a proper belief process. Differently from their analysis, as we demonstrate in the proof of Proposition 1 in Appendix A, it turns out this issue does not arise in our setting and we obtain a well-defined consensus belief process for all risk aversion values due to the investors' preferences being over horizon wealth.

[8]For notational convenience, we denote the initial values of the average bias and dispersion in beliefs by m and v instead of m_0 and v_0, respectively. We note that the average bias can also be represented in terms of the

through their investment in the stock they get relatively wealthier. This in turn increases their weight in equilibrium and consequently makes the view on the stock more optimistic. The analogous mechanism makes the view on the stock more pessimistic following bad news.[9]

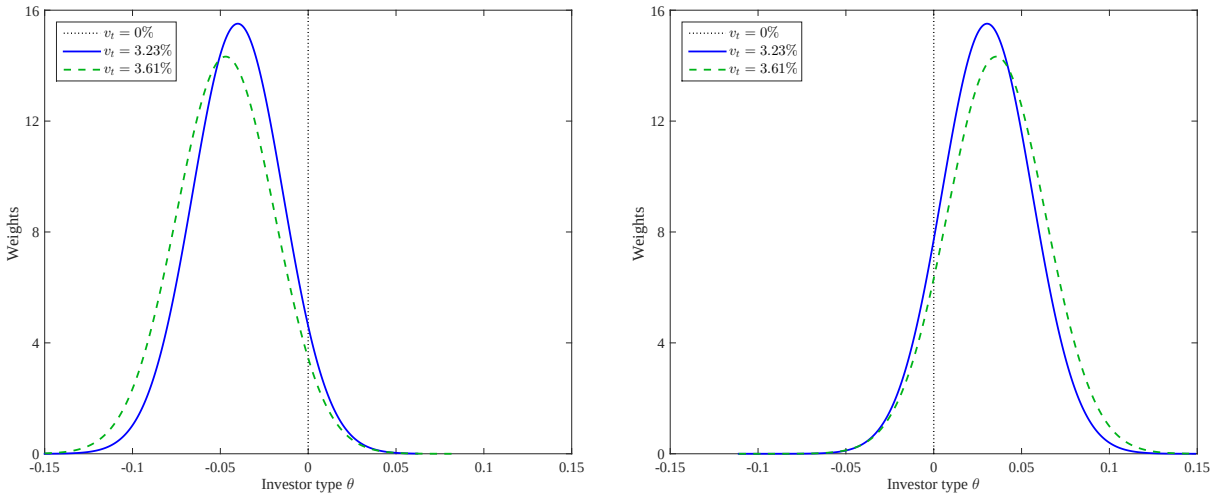
As highlighted in Proposition 1, a higher belief dispersion leads to a higher average bias for relatively good cash-flow news and to a lower average bias otherwise. This is notable since it reveals that the extent of optimism/pessimism depends crucially on the level of belief dispersion v_t. In particular, dispersion amplifies the effects of cash-flow news on the average bias, and hence the same level of good (bad) news leads to more optimism (pessimism) when dispersion is higher. We illustrate this feature in Figure 1, where we plot the weights $h_t(\theta)$ for different levels of dispersion in relatively bad (panel (a)) and good (panel (b)) cash-flow news states. The average bias is given by the point on the $x-$axis where the respective plot centers. We see that higher dispersion plots are flatter and center at a point further away from the origin, which shows that investors with relatively large biases are indeed assigned higher weights and optimism/pessimism is amplified under higher dispersion. Investors' attitude towards risk, γ, influences the average bias too. In a more risk averse economy, investors hold relatively less stock which limits the wealth transfers to the investors whose beliefs are supported. Consequently, this reduces the sensitivity of the average bias to cash-flow news, leading to less optimism (pessimism) for the same level of good (bad) news.

In the presence of heterogeneity in beliefs, the belief dispersion has a dual role. Besides amplifying the current average bias in beliefs m_t, the current belief dispersion v_t also drives the extent to which average bias fluctuates next instant, and hence represents the riskiness of average bias. Indeed, it can be shown from (10) that the dynamics of average bias is $dm_t = \mu_{mt}dt + \sigma_{mt}d\omega_t$, where the diffusion term is $\sigma_{mt} = v_t^2/\gamma\sigma$. As for the dynamics of belief dispersion itself, as (10) highlights, the dispersion is at its highest level initially and then decreases over time deterministically as investors with extreme beliefs tend to receive less and less weight over time due to their diminishing wealth and impact in equilibrium. We discuss the limiting behavior of dispersion in detail in Remark 3 of Section 5.

Equation (12) indicates that the time-t weights $h_t(\theta)$, which can be thought of as the time-t "effective" relative frequency of investors, have a convenient Gaussian form with mean m_t and

initial values by $m_t = \sigma\left(\sigma m + \frac{1}{\gamma}v^2\omega_t\right)/\left(\sigma^2 + \frac{1}{\gamma}v^2 t\right)$.

[9]The wealth transfers among investors is the main underlying mechanism in dynamic heterogeneous-beliefs models. We add to this literature by demonstrating in our subsequent analysis that these wealth transfers affect economic quantities also through the average bias and dispersion in beliefs, which are relatively easier to observe in the data.



(a) Relatively bad news

(b) Relatively good news

Figure 1: **Investors' weights.** These panels plot the weights $h_t(\theta)$ for each distinct θ-type investor for different levels of current belief dispersion v_t. The belief dispersion is $v_t = 3.23\%$ in solid blue and 3.61% in dashed green lines. The vertical dotted black lines correspond to the benchmark economy with no belief dispersion. The cash-flow news is relatively bad $D_t = 1.22$ in panel (a), and good $D_t = 2.50$ in panel (b). The baseline parameter values follow from Table 1 of Appendix B which are based on matching the model implied belief dispersion moments to the corresponding summary statistics in Yu (2011): $\tilde{m} = 0$, $\tilde{v} = 3.39\%$, $\mu = 14.23\%$, $\sigma = 8.25\%$, $\gamma = 2$, $t = 4.37$ and $T = 10$.

standard deviation v_t as also illustrated in Figure 1. This feature allows us to characterize the wide range of investor heterogeneity in our economy by the average bias and dispersion in beliefs since they are the first two (thus sufficient) central moments of Gaussian weights.

Remark 1 (Alternative average bias and dispersion in beliefs measures). In a dynamic economy such as ours, to characterize the equilibrium quantities in terms of the moments of belief heterogeneity the stochastic impact of investors' beliefs and wealth ought to be taken into account. To capture the larger impact of wealthier investors on equilibrium prices, one may alternatively define the average bias in beliefs as in (8) but using the wealth-share distribution $W(\theta)/S$ as the weights. This definition does not require the construction of the representative investor and yields alternative average bias and dispersion in beliefs measures denoted by $\tilde{m}_t$ and $\tilde{v}_t$, respectively, which can be shown to be given by

$$\tilde{m}_t \equiv \int_\Theta \theta \frac{W_t(\theta)}{S_t} d\theta = m_t - \left(1 - \frac{1}{\gamma}\right) v_t^2 (T-t), \quad \tilde{v}_t^2 \equiv \int_\Theta (\theta - \tilde{m}_t)^2 \frac{W_t(\theta)}{S_t} d\theta = \frac{1}{\gamma} v_t^2 + \left(1 - \frac{1}{\gamma}\right) v_T^2, \quad (13)$$

where m_t, v_t are as in (10).[10] As the expressions in (13) highlight, our average bias and dispersion in beliefs coincide with their respective wealth-share weighted counterparts when the preferences are logarithmic ($\gamma = 1$) and also at the horizon T. For non-logarithmic preferences,

[10] Above expressions are derived in the proof of Proposition 5 in Appendix A.

at any point in time, the wealth-share weighted average bias $\tilde{m}_t$ differs from the average bias m_t, but only by a constant. This constant arises since the distinct θ-type investor with the highest wealth is not the same investor whose bias has the highest impact on equilibrium quantities when $\gamma \neq 1$. However, since the difference between the two average bias measures is a constant, we obtain similar results and predictions if, instead of m_t and v_t, we use the wealth-share weighted average bias and dispersion measures as in (13).

4 Stock Price, Its Dynamics and Trading Volume

In this Section, we investigate how the stock price, its mean return, volatility and trading volume are affected by the average bias and dispersion in beliefs. In particular, we demonstrate that in the presence of belief dispersion, the stock price is convex in cash-flow news. A higher belief dispersion gives rise to a higher stock price and a lower mean return when the view on the stock is relatively optimistic, and vice versa when pessimistic. We further show that a higher belief dispersion leads to a higher stock volatility and trading volume. These findings are consistent with empirical evidence.

4.1 Equilibrium Stock Price

Proposition 2. *In the economy with belief dispersion, the equilibrium stock price is given by*

$$S_t = \overline{S}_t e^{m_t(T-t) - \frac{1}{2\gamma}(2\gamma-1)v_t^2(T-t)^2}, \tag{14}$$

where the average bias m_t and dispersion v_t in beliefs are as in Proposition 1, and the equilibrium stock price in the benchmark economy with no belief dispersion is given by $\overline{S}_t = D_t e^{\left(\mu-\gamma\sigma^2\right)(T-t)}$. Consequently, in the presence of belief dispersion,

i) The stock price is convex in cash-flow news D_t.

ii) The stock price is increasing in belief dispersion v_t when $m_t > \tilde{m} + (1/2\gamma)\,(2\gamma-1)\,v_t^2\,(T-t)$, and is decreasing otherwise.

iii) The stock price is decreasing in investors' risk aversion γ, as in the benchmark economy for relatively good cash-flow news. However, the stock price is increasing in investors' risk aversion for relatively bad cash-flow news and low levels of risk aversion.

The stock price in the benchmark economy is driven by cash-flow news D_t, whereby good news (higher D_t) leads to a higher stock price since investors increase their expectations of the

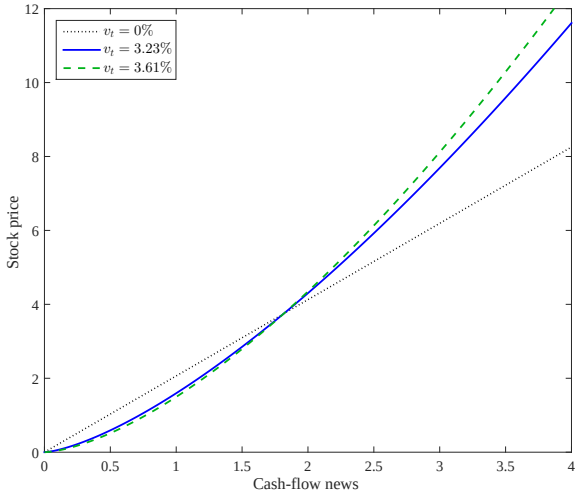
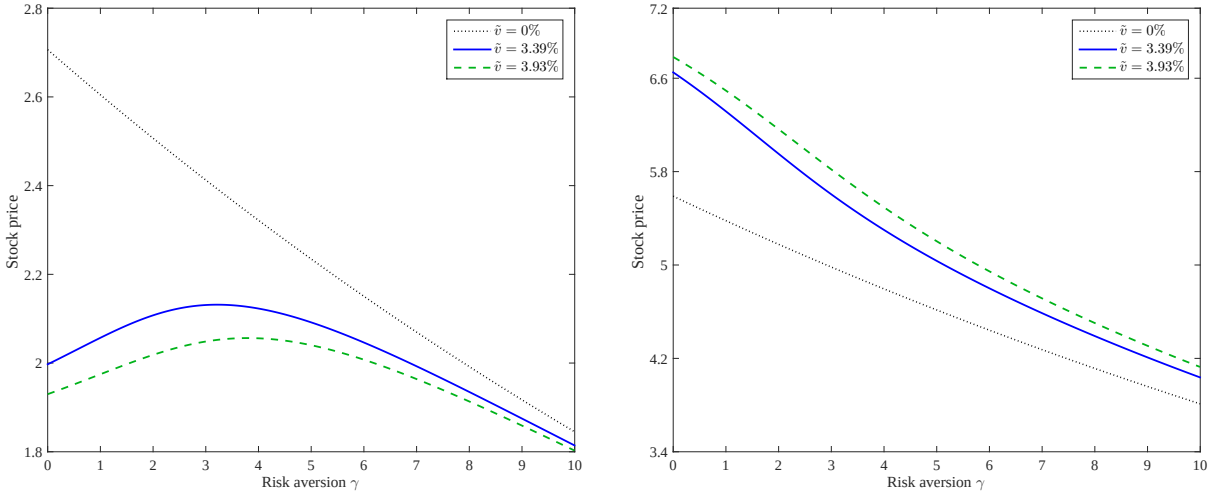


Figure 2: **Stock price convexity and effects of belief dispersion.** This figure plots the equilibrium stock price S_t against cash-flow news for different levels of current belief dispersion v_t. The dotted line corresponds to the equilibrium stock price in the benchmark economy with no belief dispersion. The baseline parameter values are as in Figure 1.

stock payoff D_T. The equilibrium stock price in the presence of belief dispersion has a simple structure, and is additionally driven by the average bias m_t and dispersion v_t in beliefs. The role of the average bias in beliefs is to increase the stock price further following good news, and conversely decrease following bad news. This is because, as discussed in Section 3, following good cash-flow news the view on the stock becomes relatively more optimistic which then leads to a further increase in the expectation of the stock payoff, and consequently in the stock price, and vice versa following bad news. Figure 2 plots the equilibrium stock price against cash-flow news for different levels of belief dispersion, illustrating above points.

Figure 2 also illustrates the extra boost in the stock price due to increased optimism following good news. The notable implication here is the convex stock price-news relation as opposed to the linear one in the benchmark economy (Property (i)). The convexity implies that the increase in the stock price following good news is more than the decrease following bad news (all else fixed), which is also supported empirically (Basu (1997), Xu (2007)). It also implies that the stock price is more sensitive to news (good or bad) in relatively good states. Conrad, Cornell, and Landsman (2002) document that bad news decreases the stock price more in good states which is also in line with our finding. As mentioned in the Introduction, a similar convexity property is obtained by Veronesi (1999), but due to parameter uncertainty in a model with homogeneous agents.

Turning to the role of belief dispersion v_t, we see that its influence on the stock price (14)



(a) Relatively bad news

(b) Relatively good news

Figure 3: **Effects of risk aversion on stock price.** These figures plot the equilibrium stock price S_t against relative risk aversion coefficient γ for different levels of standard deviation of investor types $\tilde{v}$. The dotted lines correspond to the equilibrium stock price in the benchmark economy with no belief dispersion. The cash-flow news is relatively bad $D_t = 1.22$ in panel (a) and good $D_t = 2.50$ in panel (b). The baseline parameter values are as in Figure 1.[11]

enters via two channels: directly (v_t^2 term) and indirectly (via average bias in beliefs m_t). The direct effect always decreases the stock price for plausible levels of risk aversion ($\gamma > 1/2$) since dispersion represents the riskiness of the average bias (as discussed in Section 3). The indirect effect, due to dispersion amplifying the average bias (Section 3), increases the stock price further following relatively good news and decreases it further following relatively bad news. Since both effects have a negative impact following bad news, the stock price always decreases in relatively bad states due to dispersion. On the other hand, for sufficiently good cash-flow news, the indirect effect of dispersion dominates and the stock price increases. These are also illustrated in Figure 2.

Consequently, a notable implication here is that the stock price increases in belief dispersion when the view on the stock is relatively optimistic, and decreases otherwise (Property (ii)). A higher belief dispersion leading to a higher stock price is often found to be somewhat surprising since, instead of requiring a premium for the extra uncertainty due to belief dispersion, investors appear to pay a premium for it. Our model reconciles with this seemingly counterintuitive finding by demonstrating that a higher dispersion may lead to a higher stock price when the

[11] We note that unlike earlier Figures, these plots are not for different levels of current belief dispersion v_t but for different levels of standard deviation of investor types $\tilde{v}$, since v_t depends on γ and therefore cannot be fixed across different levels of relative risk aversion.

stock price is driven by sufficiently optimistic beliefs. This is supported by the stock price evidence in Yu (2011). Yu provides evidence that a higher belief dispersion increases growth stock (low book-to-market) prices more than value stock prices, and associates growth stocks with optimism motivated by the findings of Lakonishok, Shleifer, and Vishny (1994), La Porta (1996). He also finds weak evidence that value stock prices in fact decrease under higher dispersion.

Figure 3 presents the effects of risk aversion on the equilibrium stock price and highlights that in the presence of belief dispersion the stock price may actually increase in investors' risk aversion γ (Property (iii)). In the benchmark economy, the stock price always decreases in investors' risk aversion. This is intuitive since in a more risk averse economy, investors demand a higher return to hold the risky stock and so push down its price. In the presence of belief dispersion, risk aversion has an additional stochastic impact on the stock price through the average bias in beliefs. As discussed in Section 3, a higher risk aversion makes the average bias less sensitive to news since it reduces the magnitude of wealth transfers among investors. Therefore, the same level of bad news generates less pessimism, which leads to a relatively higher stock price in a more risk averse economy. For a range of low risk aversion values this additional impact overrides the benchmark behavior resulting with the stock price actually being increasing in investors' risk aversion. On the other hand, for relatively good news, both the increased risk aversion and the accompanying reduced optimism induce investors to demand a higher return, which leads to the stock price being monotonically decreasing in investors' risk aversion as in the benchmark economy.

4.2 Equilibrium Mean Return

In our economy, the mean return perceived by each θ-type investor, $\mu_S(\theta)$, is different than the (observed) true mean return, μ_S, with the relation between them being given by (4). To make our results comparable to empirical studies, in this Section we present our results in terms of the true mean return (as observed in the data), henceforth, simply referred to as the mean return. Proposition 3 reports the equilibrium mean return and its properties.

Proposition 3. *In the economy with belief dispersion, the equilibrium mean return is given by*

$$\mu_{St} = \overline{\mu}_{St}\left(1 + \frac{1}{\gamma}\frac{v_t^2}{\sigma^2}(T-t)\right)^2 - m_t\left(1 + \frac{1}{\gamma}\frac{v_t^2}{\sigma^2}(T-t)\right), \tag{15}$$

where the average bias m_t and dispersion v_t in beliefs are as in Proposition 1, and the equilibrium mean return in the benchmark economy with no belief dispersion is given by $\overline{\mu}_{St} = \gamma\sigma^2$.

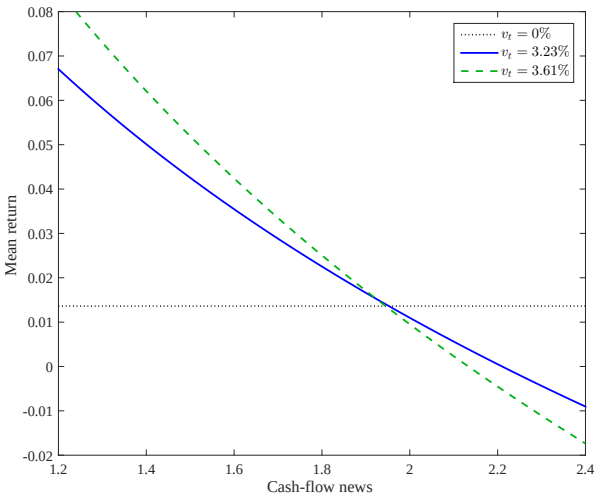


Figure 4: **Effects of belief dispersion on mean return.** This figure plots the equilibrium mean return μ_{St} against cash-flow news for different levels of current belief dispersion v_t. The dotted line corresponds to the equilibrium mean return in the benchmark economy with no belief dispersion. The baseline parameter values are as in Figure 1.

Consequently, in the presence of belief dispersion,

i) The mean return is decreasing in belief dispersion v_t when $m_t > v_t^2\left(\tilde{m} + 2v_t^2\left(T-t\right)\right) \times (2v_t^2 - v_T^2)^{-1}$, and is increasing otherwise.

ii) The mean return is increasing in investors' risk aversion γ, as in the benchmark economy for relatively good cash-flow news. However, the mean return is decreasing in investors' risk aversion for relatively bad cash-flow news and low levels of risk aversion.

The presence of belief dispersion makes the equilibrium mean return stochastic (a constant in benchmark economy) and strictly decreasing in the average bias in beliefs m_t.[12] This is because, the higher the average bias, the higher the stock price (Section 4.1), and therefore, the stock receives more negative subsequent news on average when the view on it is relatively optimistic, which in turn leads to a lower mean return.[13]

Figure 4 plots the equilibrium mean return against cash-flow news for different levels of belief dispersion and illustrates that a higher belief dispersion v_t leads to a lower mean return when the view on the stock is sufficiently optimistic, and to a higher mean return otherwise

[12]It may appear somewhat unusual to have the mean return expression (15) involve a term with the belief dispersion raised to the fourth power. This occurs because the equilibrium mean return is equal to the market price of risk times the stock volatility, as alternatively expressed in (A.36), and both these quantities involve the squared dispersion term v_t^2. A similar term also arises in our richer economy with Bayesian learning as (54) illustrates.

[13]The stock receiving more negative subsequent news on average when the view on it is relatively optimistic is due to the fact that the true data generating process, the cash-flow news, has constant parameters, which imply that the consecutive ratios (D_t/D_{t-h}) and (D_{t+h}/D_t) are i.i.d. lognormal.

(Property (i)). The intuition for this is similar to that for the stock price: dispersion represents additional risk for investors (Section 3), and therefore investors demand a higher return to hold the stock when dispersion is higher. However, we know that dispersion also amplifies the average bias in beliefs (Section 3), which in turn leads to a lower mean return when the view on the stock is optimistic and to a higher mean return when pessimistic. When there is sufficiently optimistic view on the stock, the latter effect dominates and produces the negative relation between belief dispersion and mean return.

As discussed in the Introduction, the empirical evidence on the relation between belief dispersion and mean return is vast and mixed, and existing theoretical works explain only one side of this relation. Our model generates both the negative and positive effects and implies that the documented negative relation must be due to the optimistic bias and it should be stronger, the higher the optimism. Diether, Malloy, and Scherbina (2002) provide supporting evidence for our implications by finding an optimistic bias in their study overall, and by also showing that the negative effect of dispersion is indeed stronger for more optimistic stocks. Similar evidence is also provided by Yu (2011) who documents that high dispersion stocks earn lower returns than low dispersion ones and this effect is more pronounced for growth (low book-to-market) stocks which tend to represent overly optimistic stocks (see, for example, Lakonishok, Shleifer, and Vishny (1994), La Porta (1996) and Skinner and Sloan (2002)).

Property (ii) highlights an interesting feature that the equilibrium mean return may decrease in investors' risk aversion for relatively bad news states over a range of risk aversion values. Analogous to the intuition given for the stock price (Section 4.1), this result is again due to bad news leading to less pessimism in more risk averse economies. We again note that for relatively good news, the mean return monotonically increases in investors' risk aversion as in the benchmark economy. This is because both the increased risk aversion and the accompanying reduced optimism induce investors to demand a higher return. A similar non-monotonic relation between the mean return and risk aversion is demonstrated by David (2008). Our result compliments David's by providing the additional insight that the relation between the mean return and risk aversion depends on the level of the optimism/pessimism on the stock, and is non-monotonic only when the view is relatively pessimistic, but it is monotonic otherwise.

4.3 Stock Volatility and Trading Volume

In our economy, investors manifest their differing beliefs by taking diverse stock positions, which in turn generate trade and wealth transfers among investors. As discussed in Section 3,

these wealth transfers make the average bias in beliefs stochastic, which then leads to extra uncertainty for investors. In this Section, we demonstrate how this extra uncertainty and investors' trading motives give rise to higher stock volatility and trading volume.

Proposition 4. *In the economy with belief dispersion, the equilibrium stock volatility is given by*

$$\sigma_{St} = \overline{\sigma}_{St} + \frac{v_t^2}{\gamma\sigma}(T-t), \tag{16}$$

where the dispersion in beliefs v_t is as in Proposition 1, and the equilibrium stock volatility in the benchmark economy with no belief dispersion is given by $\overline{\sigma}_{St} = \sigma$.

Consequently, in the presence of belief dispersion, the stock volatility is increasing in belief dispersion v_t.

The key implication of Proposition 4 is that the stock volatility increases monotonically in belief dispersion v_t. This is because, the higher the dispersion, the average bias in beliefs fluctuates more and hence so does the stock price (Section 4.1), and this additional fluctuation in the stock price across news states increases the stock volatility. Figure 5 illustrates this feature by plotting the equilibrium stock volatility against belief dispersion. This result is also consistent with the empirical evidence (Ajinkya and Gift (1985), Anderson, Ghysels, and Juergens (2005) and Banerjee (2011)). As we discuss in the Introduction, several other theoretical works find that a higher investor belief heterogeneity leads to a higher stock volatility (e.g., Scheinkman and Xiong (2003), Buraschi and Jiltsov (2006), Li (2007), David (2008), Dumas, Kurshev, and Uppal (2009), Banerjee and Kremer (2010), Andrei, Carlin, and Hasler (2015)). Our contribution here is to express the stock volatility and obtain this result in terms of belief dispersion itself (rather than overall belief heterogeneity), which is not straightforward to obtain in two-investor economies as we show in Section 5.

We now explore the aggregate trading activity in our economy. Towards this, we first express each θ-type investor's portfolio holdings in terms of the number of shares held in the stock, $\psi(\theta) = \phi(\theta) W(\theta)/S$, with dynamics $d\psi_t(\theta) = \mu_{\psi t}(\theta)\, dt + \sigma_{\psi t}(\theta)\, d\omega_t$, where $\mu_\psi(\theta)$ and $\sigma_\psi(\theta)$ are the drift and volatility of θ-type investor's portfolio process $\psi(\theta)$, respectively. Following recent works in continuous-time settings (e.g., Xiong and Yan (2010), Longstaff and Wang (2012)), we consider a trading volume measure V that sums over the absolute value of investors' portfolio volatilities,

$$V_t \equiv \frac{1}{2}\int_\Theta |\sigma_{\psi t}(\theta)|\, d\theta, \tag{17}$$

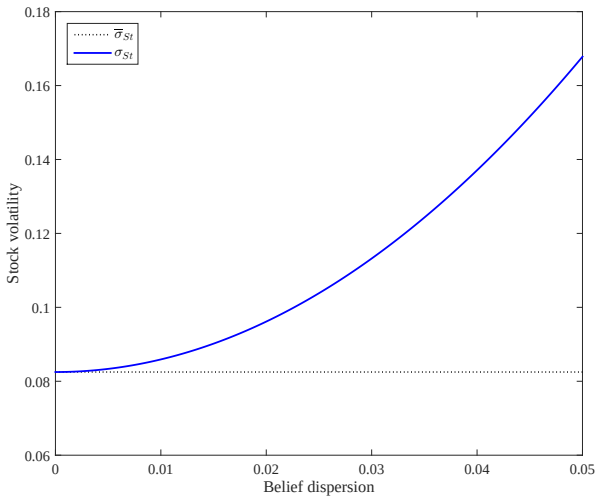


Figure 5: **Effects of belief dispersion on stock volatility.** This figure plots the equilibrium stock volatility σ_{St} against current belief dispersion v_t. The dotted line corresponds to the equilibrium stock volatility in the benchmark economy with no belief dispersion. The baseline parameter values are as in Figure 1.

where the adjustment $1/2$ is to prevent double summation of the shares traded across investors.[14] Proposition 5 reports the equilibrium trading volume measure in closed form and its properties.

Proposition 5. *In the economy with belief dispersion, the equilibrium trading volume measure is given by*

$$V_t = \frac{\sigma}{X_t^2}\frac{v_t^2}{v_T^2}\left[\left(\frac{X_t}{2}+\frac{\sqrt{X_t^2+4}}{2}\right)\boldsymbol{\phi}\left(\frac{X_t}{2}-\frac{\sqrt{X_t^2+4}}{2}\right)-\left(\frac{X_t}{2}-\frac{\sqrt{X_t^2+4}}{2}\right)\boldsymbol{\phi}\left(\frac{X_t}{2}+\frac{\sqrt{X_t^2+4}}{2}\right)\right], \tag{18}$$

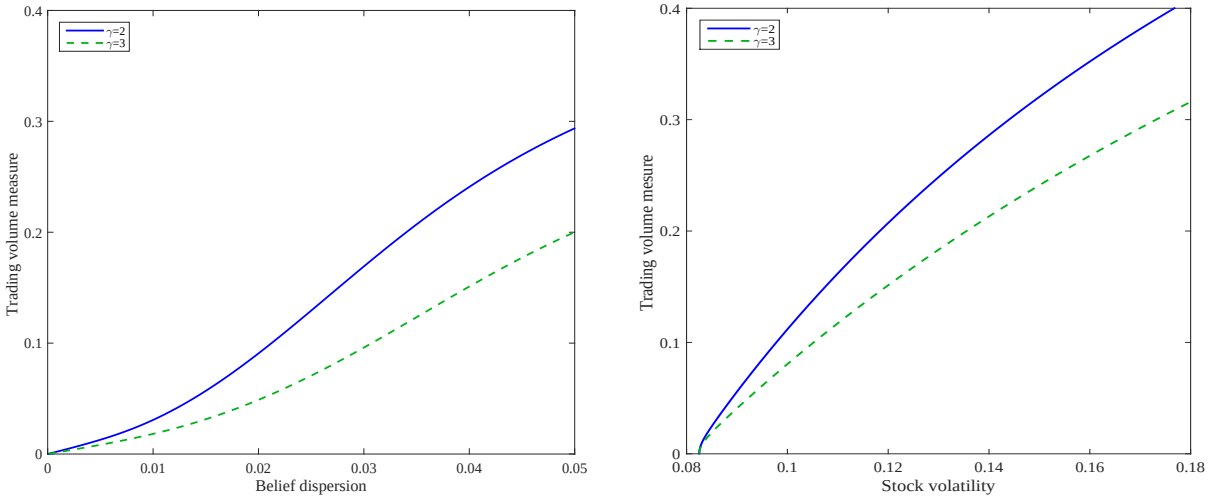
where the dispersion in beliefs v_t is as in Proposition 1, and $\boldsymbol{\phi}(.)$ is the probability density function of the standard normal random variable, and X is a (positive) deterministic process given by

$$X_t^2 = \gamma^2\frac{\sigma^4}{v_T^4}\left[\frac{1}{\gamma}v_t^2+\left(1-\frac{1}{\gamma}\right)v_T^2\right].$$

Consequently, in the presence of belief dispersion, the trading volume measure is increasing in belief dispersion v_t and is positively related to the stock volatility σ_{St}.

With belief dispersion, investors take diverse stock positions following cash-flow news, which in turn generate non-trivial trading activity. Naturally, the aggregate trading activity in the stock, which is captured by our trading volume measure V, increases as the belief dispersion increases. This is because, when dispersion is higher, investors with relatively different beliefs

[14] As is well recognized, employing the standard definition of trading volume, $\frac{1}{2}\int_\Theta |d\psi_t(\theta)|\,d\theta$ in a continuous-time setting is problematic since the local variation of the driving uncertainty, Brownian motion ω, and hence an investor's portfolio, is unbounded. The measure V defined in (17) does not suffer from this issue and indicates the unexpected trading volume by not taking into account of expected changes in investors' portfolio processes.



(a) Effects of belief dispersion

(b) Trading volume-stock volatility relation

Figure 6: **Effects of belief dispersion on trading volume measure.** These figures plot the equilibrium trading volume measure V_t against current belief dispersion v_t in panel (a) and against stock volatility σ_{St} in panel (b) for different relative risk aversion coefficients γ. The baseline parameter values are as in Figure 1.

have more weight and higher trading demand, which increase the stock trading volume. Figure 6a illustrates this feature by depicting the equilibrium trading volume measure against belief dispersion. This result is well-supported by empirical evidence (Ajinkya, Atiase, and Gift (1991), Bessembinder, Chan, and Seguin (1996) and Goetzmann and Massa (2005)). Figure 6b plots the equilibrium trading volume measure against stock volatility and illustrates the positive relation between these two economic quantities. This positive relation is intuitive since a higher dispersion leads to both a higher stock volatility and a higher trading volume measure. This result is also supported by empirical evidence; for example, Gallant, Rossi, and Tauchen (1992) document a positive correlation between the conditional stock volatility and trading volume, and more recently, Banerjee (2011) shows that stocks in high trading volume quintiles tend to have higher return variances.

Remark 2 (Multiple stock payoffs and consumption dates). To highlight our main insights as clearly as possible in a tractable setting, we model the stock as a claim to a single payoff. We provide an extension of our analysis that features additional stock payoffs and consumption dates in Internet Appendix ID. We first consider an economy with two stock payoffs and consumption dates in which investors have logarithmic preferences. We demonstrate that the equilibrium average bias and dispersion in beliefs are the same as in our main model for logarithmic preferences (Proposition 1). This is because in both economies investors have different beliefs about the cash-flow news process, and hence the introduction of additional stock

payoffs does not alter investors' beliefs. Consequently, the underlying economic mechanism revealed in Section 3 is also present in this setting, where the belief dispersion amplifies the average bias so that the same level of good (bad) news leads to more optimism (pessimism) when dispersion is higher. The stock price in this extended economy simply becomes the sum of the values of each payoff, but with a deterministic adjustment term for the longer-term payoff which accounts for the drop in the stock price due to the interim payoff. Since each value term is as in our main model, the stock price inherits our earlier key properties of the stock price being convex in cash-flow news and being increasing in belief dispersion when the view on the stock is relatively optimistic, and decreasing otherwise (Proposition 2). Although we determine the stock price dynamics of mean return and volatility in closed form, we are unable to analytically obtain their comparative statics for all periods (elaborated on in Appendix ID). Investigating these numerically, however, reveals that the stock mean return is decreasing in belief dispersion when the view on the stock is relatively optimistic, and increasing otherwise, and the stock volatility is increasing in belief dispersion, as in our main model (Propositions 3–4).

We then extend the setting further and add a third stock payoff and consumption date. We show that the average bias and dispersion in beliefs are identical to those in our main model, and the stock price is again the sum of the values of each payoff, but with more complex deterministic adjustment terms for the longer-term payoffs accounting additionally for the second drop in the stock price. Therefore, all the properties discussed for the economy with two stock payoffs and consumption dates remain valid in this setting. The extended setting with multiple stock payoffs and consumption dates, however, is less tractable as we are only able to solve it with logarithmic preferences. For CRRA preferences, we are not able to determine the average bias and dispersion in beliefs explicitly due to the inability to obtain the investors' weights explicitly, as we elaborate on in Appendix ID. Nevertheless, our main economic mechanisms at play and results remain valid in the multiple stock payoff settings that we analyze.

5 Comparisons with Two-Investor Economy

So far, we have investigated an economy with a continuum of investors having heterogeneous beliefs. In this Section, we consider an otherwise identical two-investor economy with heterogeneous beliefs and a single consumption date. The related work of Kogan, Ross, Wang, and Westerfield (2006) employs such a setup. We solve for the equilibrium stock price, its dynamics

and trading volume and demonstrate that most of our earlier results do not necessarily obtain in this setting. In this regard, we first show that in the two-investor economy it does not appear to be possible to neither write the average bias in terms of belief dispersion nor express the equilibrium quantities in terms of these two quantities as we do in Sections 3 and 4, respectively. We also show that, in contrast to our main model's implications, the stock price is no longer convex in cash-flow news across all states of the world, and a higher belief dispersion has ambiguous effects on the stock volatility and trading volume.

Now we consider a variant of our economy in Section 2 in which there are two investors instead of a continuum of them. The other features remain the same. In particular, the securities market is as in Section 2.1 and the investors' beliefs are as in Section 2.2. That is, under the θ_n-type investor's beliefs, $n = 1, 2$, the cash-flow news process has dynamics $dD_t = (\mu + \theta_n) D_t dt + \sigma D_t d\omega_{nt}$, where ω_n is her perceived Brownian motion with respect to her own probability measure $\mathbb{P}^{\theta_n}$, and is given by $\omega_{nt} = \omega_t - \theta_n t/\sigma$. We again index each θ_n-type investor by her bias θ_n, with the type space now becoming $\Theta = \{\theta_1, \theta_2\}$ rather than $\Theta = \mathbb{R}$ as in our main model. Without loss of generality we assume $\theta_1 < \theta_2$, hence we interpret the first investor as the relatively pessimistic investor with a bias θ_1, and the second investor as the relatively optimistic investor with a bias θ_2. We assume that investors are initially endowed with equal shares of the stock, $\psi_{10} = \psi_{20} = 0.5$. Investors' preferences are as in Section 2.3, however for tractability we take the investors' relative risk aversion coefficient γ to be a positive integer, as is usually assumed in this literature (e.g., Yan (2008), Dumas, Kurshev, and Uppal (2009), Dumas, Lewis, and Osambela (2017)).

We again proceed by first constructing the average bias and dispersion in beliefs following Definition 1 in our main model.[15] That is, the time-t average bias in beliefs, m_t, is the implied bias of the corresponding representative investor, expressed as the weighted average of the individual investors' biases

$$m_t = \sum_{n=1}^{2} \theta_n h_{nt}, \tag{19}$$

with the weights $h_{nt} > 0$ satisfying $\sum_{n=1}^{2} h_{nt} = 1$, and the dispersion in beliefs, v_t, is the

[15]Other, alternative dispersion measures employed in the literature for two-investor economies include the simple difference in (possibly stochastic) biases $(\theta_2 - \theta_1)$ (e.g., Basak (2005), Dumas, Kurshev, and Uppal (2009), Xiong and Yan (2010)) and the relative likelihood ratio process (η_{2t}/η_{1t}) (e.g., David (2008), Bhamra and Uppal (2014)). However, both of these measures capture the overall effects of belief heterogeneity rather than decomposing its effects due to average bias and dispersion in beliefs, as we do. Moreover, these dispersion measures are hard to generalize when there are more than two investors in the economy, since one still needs to find a suitable, economically justified, functional form for the dispersion measure which takes all the differences in biases, or likelihood ratios, as inputs and delivers a single dispersion measure.

standard deviation of investors' biases

$$v_t^2 \equiv \sum_{n=1}^{2} (\theta_n - m_t)^2 h_{nt}. \tag{20}$$

We then determine the ensuing equilibrium weights, average bias and belief dispersion, and solve for the equilibrium stock price, its dynamics and the trading volume measure in this economy. Proposition 6 reports these quantities.[16] Kogan, Ross, Wang, and Westerfield (2006) study the stock price and dynamics in a similar setting with a focus on the long-run survival and price impact of investors with biases. Our analysis here complements theirs by providing closed-form expressions for the stock price, its dynamics and the trading volume measure.

Proposition 6. *In the two-investor economy with heterogeneous beliefs, the time-t average bias and investors' corresponding equilibrium weights are given by*

$$m_t = \sum_{k=0}^{\gamma} \frac{G_{t,k} e^{\gamma\sigma\left(\frac{k}{\gamma}\frac{\theta_1}{\sigma}+\frac{\gamma-k}{\gamma}\frac{\theta_2}{\sigma}\right)(T-t)}}{\sum_{j=0}^{\gamma} G_{t,j} e^{\gamma\sigma\left(\frac{j}{\gamma}\frac{\theta_1}{\sigma}+\frac{\gamma-j}{\gamma}\frac{\theta_2}{\sigma}\right)(T-t)}} \left(\frac{k}{\gamma}\theta_1 + \frac{\gamma-k}{\gamma}\theta_2\right), \tag{21}$$

$$h_{1t} = \sum_{k=0}^{\gamma} \frac{G_{t,k} e^{\gamma\sigma\left(\frac{k}{\gamma}\frac{\theta_1}{\sigma}+\frac{\gamma-k}{\gamma}\frac{\theta_2}{\sigma}\right)(T-t)}}{\sum_{j=0}^{\gamma} G_{t,j} e^{\gamma\sigma\left(\frac{j}{\gamma}\frac{\theta_1}{\sigma}+\frac{\gamma-j}{\gamma}\frac{\theta_2}{\sigma}\right)(T-t)}} \frac{k}{\gamma}, \qquad h_{2t} = 1 - h_{1t}, \tag{22}$$

and the dispersion in beliefs by (20) *with* (21)–(22) *substituted in, where* $G_{t,k}$ *is as in* (27)*. The equilibrium stock price, mean return and volatility are given by*

$$S_t = e^{\left(\mu-\gamma\sigma^2\right)(T-t)} D_t \sum_{k=0}^{\gamma} g_{t,k} e^{\sigma\left(\frac{k}{\gamma}\frac{\theta_1}{\sigma}+\frac{\gamma-k}{\gamma}\frac{\theta_2}{\sigma}\right)(T-t)}, \tag{23}$$

$$\mu_{St} = \left[\gamma\sigma - \sum_{k=0}^{\gamma} g_{t,k}\left(\frac{k}{\gamma}\frac{\theta_1}{\sigma} + \frac{\gamma-k}{\gamma}\frac{\theta_2}{\sigma}\right)\right]\left[\sigma + \sum_{k=0}^{\gamma} p_{t,k}\left(\frac{k}{\gamma}\frac{\theta_1}{\sigma} + \frac{\gamma-k}{\gamma}\frac{\theta_2}{\sigma}\right) - \sum_{k=0}^{\gamma} g_{t,k}\left(\frac{k}{\gamma}\frac{\theta_1}{\sigma} + \frac{\gamma-k}{\gamma}\frac{\theta_2}{\sigma}\right)\right], \tag{24}$$

$$\sigma_{St} = \sigma + \sum_{k=0}^{\gamma} p_{t,k}\left(\frac{k}{\gamma}\frac{\theta_1}{\sigma} + \frac{\gamma-k}{\gamma}\frac{\theta_2}{\sigma}\right) - \sum_{k=0}^{\gamma} g_{t,k}\left(\frac{k}{\gamma}\frac{\theta_1}{\sigma} + \frac{\gamma-k}{\gamma}\frac{\theta_2}{\sigma}\right), \tag{25}$$

where

$$g_{t,k} = \frac{G_{t,k}}{\sum_{j=0}^{\gamma} G_{t,j}}, \qquad p_{t,k} = \frac{G_{t,k} e^{\sigma\left(\frac{k}{\gamma}\frac{\theta_1}{\sigma}+\frac{\gamma-k}{\gamma}\frac{\theta_2}{\sigma}\right)(T-t)}}{\sum_{j=0}^{\gamma} G_{t,j} e^{\sigma\left(\frac{j}{\gamma}\frac{\theta_1}{\sigma}+\frac{\gamma-j}{\gamma}\frac{\theta_2}{\sigma}\right)(T-t)}}, \tag{26}$$

$$G_{t,k} = \frac{\gamma!}{k!\,(\gamma-k)!} \eta_{1t}^{\frac{k}{\gamma}} \eta_{2t}^{\frac{\gamma-k}{\gamma}} e^{(\gamma-1)\sigma\left(\frac{k}{\gamma}\frac{\theta_1}{\sigma}+\frac{\gamma-k}{\gamma}\frac{\theta_2}{\sigma}\right)T} e^{-\frac{1}{2}\left[\frac{k}{\gamma}\frac{\theta_1^2}{\sigma^2}+\frac{\gamma-k}{\gamma}\frac{\theta_2^2}{\sigma^2}+2\gamma\sigma\left(\frac{k}{\gamma}\frac{\theta_1}{\sigma}+\frac{\gamma-k}{\gamma}\frac{\theta_2}{\sigma}\right)-\left(\frac{k}{\gamma}\frac{\theta_1}{\sigma}+\frac{\gamma-k}{\gamma}\frac{\theta_2}{\sigma}\right)^2\right](T-t)}, \tag{27}$$

$$\eta_{1t} = e^{\frac{\theta_1}{\sigma}\omega_t - \frac{1}{2}\frac{\theta_1^2}{\sigma^2}t}, \qquad \eta_{2t} = e^{\frac{\theta_2}{\sigma}\omega_t - \frac{1}{2}\frac{\theta_2^2}{\sigma^2}t}. \tag{28}$$

[16] For clarity, throughout this section, we use the same notation for equilibrium quantities as in our main model.

The equilibrium trading volume measure is given by

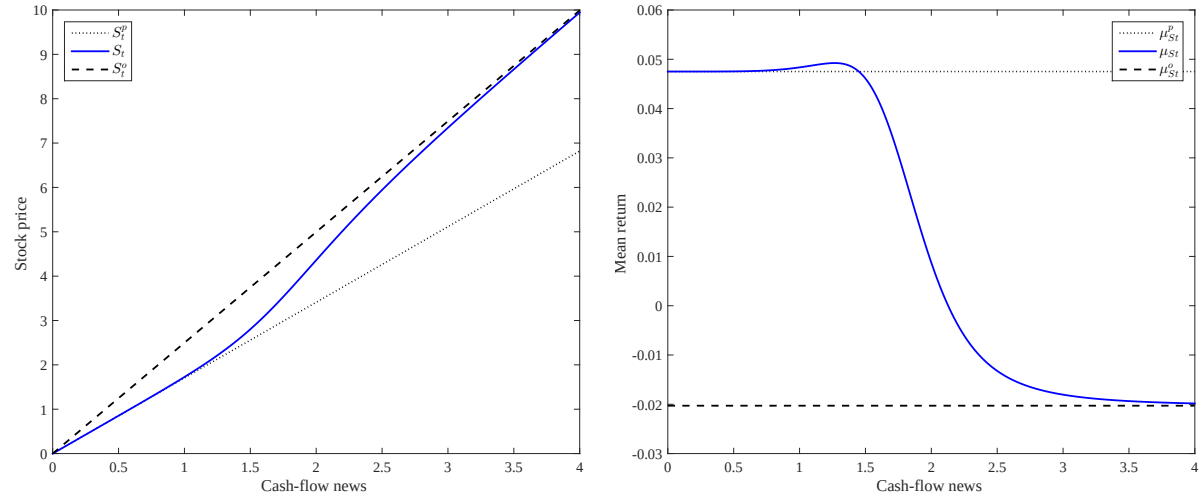
$$V_t = \frac{1}{2}\sum_{n=1}^{2} \frac{W_{nt}}{S_t} \left|\phi_{nt}\sigma_{W_n/St} + \sigma_{\phi_n t}\right|, \tag{29}$$

where the investors' wealth-share W_n/S, the portfolio strategies ϕ_n, as well as their corresponding diffusion terms, $\sigma_{W_n/S}$, and σ_{ϕ_n}, are provided in the Internet Appendix IA, for $n = 1, 2$.

Proposition 6 reveals that in the two-investor economy, the average bias in beliefs and belief dispersion have more complex structures as compared to their counterparts (10) in our main model.[17] In particular, it does not appear to be possible to write the average bias in terms of dispersion and obtain its amplification effect as we do in Proposition 1. Moreover, in this setting the distribution of investors' equilibrium weights, h_{nt}, do not have the convenient Gaussian form as in our main model. Therefore, we cannot characterize the investor belief heterogeneity, and consequently express the equilibrium quantities, in terms of the first two central moments, the average bias and dispersion in beliefs. Instead, the expressions for the equilibrium stock price and its dynamics are now more involved and are in terms of various weighted-average quantities, $g_{t,k}$ and $p_{t,k}$, which are not straightforward to interpret economically. This is in contrast to our main model presented in Section 4, where the corresponding equilibrium quantities are in terms of easily interpretable moments, the average bias and dispersion in beliefs, which also enable us to isolate the effects of dispersion and conduct comparative statics with respect to dispersion only.

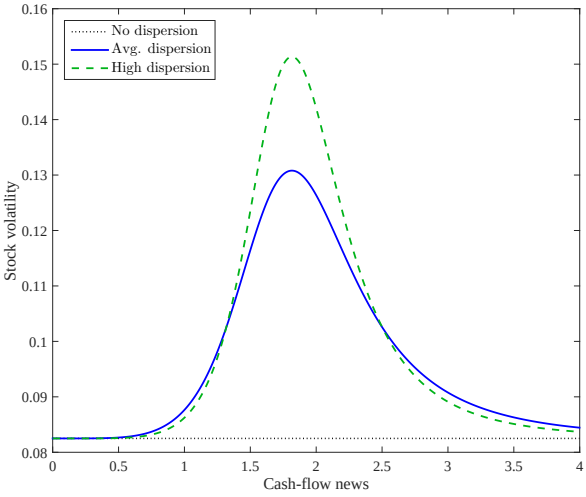
We now look at the effects of belief dispersion in the two-investor economy. To illustrate these, Figure 7 plots the equilibrium quantities against cash-flow news in this economy with one investor optimistic and the other pessimistic, as well as presenting the corresponding plots for single-investor economies. Figure 7a reveals that the stock price is no longer convex in cash-flow news across all states as opposed to that in our model (discussed in Section 4.1).

[17] One apparent difference in the expressions of Proposition 6 from the corresponding ones in our main model of Propositions 1–5 is that the two-investor economy quantities are driven by the stochastic likelihood ratios η_{1t} and η_{2t} (through (27)), capturing belief heterogeneity. In our main model, investors' likelihood ratios do not appear in equilibrium quantities because summing (integrating) across all the investors in the equilibrium market clearing condition yields a compact exponential function that embeds investors' likelihood ratios. As we demonstrate in the proof of Proposition 1 in Appendix A, this compact function is related to the average bias and dispersion and allows us to express the equilibrium quantities in terms of these moments. Furthermore, in this two-investor economy as well as in our main model, the likelihood ratio processes η turns out to be path-independent primarily because of the cash-flow news process (2) having constant coefficients in its dynamics which in turn leads to path-independent stock price and dynamics. This is in contrast to a setting with more complex cash-flow news process dynamics which may lead to path-dependent likelihood ratio processes (e.g., Basak (2000)), and hence path-dependent stock price and dynamics.

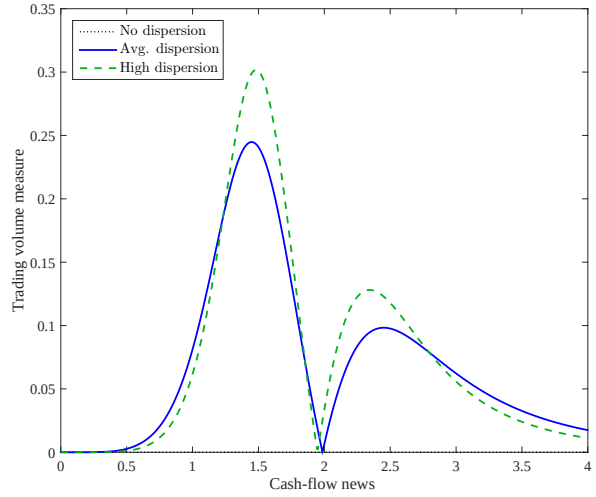


(a) Stock price

(b) Mean return



(c) Stock volatility



(d) Trading volume measure

Figure 7: **Effects of belief dispersion in the two-investor economy.** These figures plot the equilibrium stock price S_t in panel (a), the mean return μ_{St} in panel (b), the stock volatility in panel (c), and the trading volume measure in panel (d) against cash-flow news in an otherwise identical two-investor economy with an optimistic and a pessimistic investor. The dotted black lines in panels (a) and (b) correspond to the stock price S_t^p and mean return μ_{St}^p in an economy with a single pessimistic investor with a bias in beliefs -3.39%. The dashed black lines in panels (a) and (b) correspond to the stock price S_t^o and mean return μ_{St}^o in an economy with a single optimistic investor with a bias in beliefs 3.39%. The dotted black lines in panels (c) and (d) correspond to the stock volatility and trading volume measure in an economy with a single investor. The dashed green lines in panels (c) and (d) correspond to the stock volatility and trading volume measure in an economy with a one standard deviation higher belief dispersion than the average. The other applicable parameter values are as in Figure 1.

Likewise, Figure 7b illustrates that the mean return does not always decrease in cash-flow news, but in fact may even increase in moderate states. These different implications in the two-investor economy occur because belief heterogeneity effectively vanishes in relatively extreme states (as also elaborated on in Remark 3 below). The pessimistic (optimistic) investor controls almost all the wealth in the economy in very bad (good) news states, leading to equilibrium behavior similar to that in single pessimistic (optimistic) investor economies in those states. The transition from the moderate states in which belief heterogeneity is still prevalent to relatively extreme states leads to the irregular behavior in the plots. In contrast in our main model with a continuum of investors having all possible beliefs, investor heterogeneity does not vary across states of the world (since belief dispersion v_t is deterministic) and does not vanish in relatively extreme states, leading to simple uniform economic behavior. For example, following very good news, a relatively pessimistic investor would lose much wealth both in our model and in the two-investor economy. However, since there are numerous optimistic investors in our main model, the wealth transfer does not accumulate to one type of optimistic investor and make her dominate the economy, but rather shared among relatively optimistic investors.

The belief heterogeneity effectively vanishing in relatively extreme states also leads to nonuniform behavior for the stock volatility and trading volume in the two-investor economy. In particular, Figure 7c reveals that a higher belief dispersion increases the stock volatility only for moderate news states in which neither investor dominates the economy. However, for relatively extreme states the stock volatility actually decreases with higher belief dispersion, as the effective investor heterogeneity vanishes and the single-investor benchmark economy prevails. Similarly, Figure 7d reveals that the trading volume measure may actually decrease with higher belief dispersion in the two-investor economy, in contrast to our uniformly increasing trading volume belief dispersion result.[18]

In sum, by keeping investor heterogeneity the same across states, our main model is able to generate intuitive, simple and uniform results, which are not immediately possible in the two-investor economy, as Figure 7 illustrates. We note that the discussion above is not specific to a two-investor economy, i.e., $\Theta = \{\theta_1, \theta_2\}$. Our conclusions would be equally valid in a more general model with finitely-many investors, i.e., $\Theta = \{\theta_1, \ldots, \theta_N\}$ where N can be a large number. This is because in this more general model also there is necessarily a lower bound and an upper bound for investors' biases, say $\underline{\theta} \equiv \min\{\theta_1, \ldots, \theta_N\}$ and $\overline{\theta} \equiv \max\{\theta_1, \ldots, \theta_N\}$.

[18]While we have identified these implications, perhaps it is helpful to highlight that there could well be quantities other than the ones we focus on (e.g., stock price dynamics and trading volume) that one of two settings (a continuum of investors vs two investors) generates implications that are better supported by the empirical evidence. We leave that for future research.

Hence belief heterogeneity would again vanish in relatively extreme states because now the most pessimistic $\underline{\theta}$-type investor would eventually control almost all the wealth in the economy in very bad states, and the most optimistic $\overline{\theta}$-type investor in very good states. This would again lead to equilibrium behavior similar to that in single-investor economies in those extreme states, implying the irregular behavior depicted in the plots.

Remark 3 (Survival across states and over time). As the above discussion highlights, in the two-investor economy only one type of investor survives in extreme states of the world where survival is defined as an investor's wealth ratio (e.g., W_{1t}/W_{2t}) not vanishing in the limit, or equivalently in our analysis an investor's equilibrium weight (e.g., h_{1t}) not vanishing. Formally, as shown in the proof of Proposition 6 in Internet Appendix IA, we obtain the following limiting behavior of equilibrium weights h_{nt} in (22) and belief dispersion v_t:

$$\text{as } D_t \to \ 0 \qquad h_{1t} \to 1, \qquad v_t \to 0, \tag{30}$$

$$\text{as } D_t \to \infty \qquad h_{1t} \to 0, \qquad v_t \to 0. \tag{31}$$

The two-investor economy collapses to a single-investor economy in the limit of extreme states ($D_t \to 0$ or $D_t \to \infty$) as the belief dispersion vanishes. This is in sharp contrast to our main model with a continuum of investors for which belief dispersion never vanishes

$$\text{as } D_t \to 0 \ \text{ or } D_t \to \infty \qquad v_t = v\sigma(\sigma^2 + v^2 t/\gamma)^{-1/2} > 0, \tag{32}$$

because belief dispersion in (10) does not depend on the cash-flow news D_t – this is also illustrated in Figure 1 depicting that changes in D_t only shifts the equilibrium weight schedule $h_t(\theta)$ without scaling it. The reason for this behavior is that our model has a continuum of investors who have all possible beliefs, that is, the type space is unbounded, i.e., $\Theta = \mathbb{R}$. Hence even in the extreme states, the wealth transfer does not accumulate to one type of investor and make her dominate the economy, but rather is shared among investors, leading to a non-vanishing belief dispersion across all states.[19]

Even though our main model and the two-investor economy yield different implications for survival across states, they imply the same behavior for survival over time. In particular, both models imply that in the long run as $T \to \infty$, only the investor with a bias closest to $(\gamma - 1)\sigma^2$ survives, a finding consistent with earlier works in similar settings (e.g., Kogan, Ross, Wang, and Westerfield (2006), Cvitanić and Malamud (2011)). In particular, in our two-investor

[19] That being said, our model can be reached in the limit of a finitely-many-investor setting by letting $\underline{\theta} \to -\infty$, $\overline{\theta} \to \infty$, $N \to \infty$, along with a suitable relative frequency of investors which tends to a Gaussian distribution in the limit. Of course, in the limiting case there are an uncountable number of investors (a continuum) as in our model rather than finitely-many investors, which again leads to a non-vanishing belief dispersion.

economy we obtain (Internet Appendix IA)

$$\text{as } T \to \infty \qquad \begin{cases} h_{1T} \to 1, & v_T \to 0 \qquad \text{if} \quad \frac{\theta_1+\theta_2}{2} > (\gamma-1)\,\sigma^2, \\ h_{1T} \to 0, & v_T \to 0 \qquad \text{if} \quad \frac{\theta_1+\theta_2}{2} < (\gamma-1)\,\sigma^2. \end{cases} \tag{33}$$

Only in the knife-edge case when both investors' biases are equally distanced from $(\gamma-1)\,\sigma^2$ both investors survive, that is when $(\theta_1+\theta_2)/2 = (\gamma-1)\,\sigma^2$, as also demonstrated by Kogan, Ross, Wang, and Westerfield (2006). In our main model with a continuum of investors with all possible beliefs this knife-edge case does not arise since an investor with a bias of $(\gamma-1)\,\sigma^2$ always exists and becomes the only surviving investor in the long-run, implying

$$\text{as } T \to \infty \qquad v_T \to 0. \tag{34}$$

It is also important to note that even though investor heterogeneity disappears in the long-run, it may take a very long time to do so. For example, in our main model using the belief dispersion equation in (10), it can be shown that the half-life of dispersion at time t is $3\gamma\sigma^2/v_t^2$. Using the parameter values as in Figure 1, this half-life expression would indicate that it takes 837.5 years for a belief dispersion of 3.20% to decrease to 0.40% (halving 3 times) and it takes another 2552.3 years from 0.40% to halve and become 0.20%, consistent with Yan (2008).

6 Multiple Stocks Economy

Our results so far have been presented in the context of a single-stock economy to highlight our insights as clearly as possible. However, given that the documented empirical evidence is primarily based on cross-sectional studies, in this Section, we generalize our main model to feature multiple stocks, on which investors have dispersed beliefs. We demonstrate that all our main results and underlying economic mechanisms still go through in this more elaborate economy.

The multi-stock economy we consider here is the simplest and the most straightforward extension of our single-stock setting of Section 2, also admitting much tractability.[20] In this setting, there are instead N risky stocks and N sources of risk, generated by a standard N-dimensional Brownian motion $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_{N-1}, \omega)^\intercal$ defined on the true probability measure

[20]Solving multi-stock pure-exchange economies is typically a daunting task, but there has been some recent successes in the literature (e.g., Cochrane, Longstaff, and Santa-Clara (2007), Martin (2013), Chabakauri (2013)). Introducing belief dispersion on individual stock payoffs in these settings would add even more complexity. One recent work accomplishing tractability in a related two-country international finance setting is Dumas, Lewis, and Osambela (2017).

$\mathbb{P}$. Each stock price S_n, $n = 1, \ldots, N$, is posited to have dynamics $dS_{nt} = S_{nt}\left[\mu_{S_n t}dt + \sigma_{S_n t}d\boldsymbol{\omega}_t\right]$, where the stock mean return μ_{S_n} and the N-dimensional stock volatility vector σ_{S_n} are to be determined in equilibrium. The stocks are in positive net supply of one unit and are claims to the payoffs D_{nT}, paid at some horizon T. For $n = 1, \ldots, N-1$, these payoffs D_{nT} are the horizon value of the cash-flow news processes with dynamics

$$dD_{nt} = D_{nt}\left[\mu_n dt + \sigma_n d\omega_{nt} + \sigma d\omega_t\right], \tag{35}$$

where μ_n, σ_n, σ are constants, and represent the true mean growth rate of the expected payoff and the uncertainty about the payoff due to ω_n, ω, respectively. To maintain tractability, the cash-flow news for the last (residual) stock N is chosen so that the aggregate cash-flow news $D = \sum_{n=1}^{N} D_n$ has the dynamics (2) as in our single-stock setting. In this setting, we focus on the price and dynamics of the first $N-1$ stocks, which includes all I dispersed stocks, and not the N^{th} stock whose payoff has been left unspecified and theoretically can be negative.[21]

There is a continuum of investors who commonly observe all cash-flow news processes D_n (35), but have different beliefs about the dynamics of the first $I < N$ of them. We refer to the first I stocks as dispersed stocks, and the ones that investors agree on as the non-dispersed stocks. The investors are again indexed by their type θ, where a θ-type investor is now associated with an I-dimensional bias vector $\theta = (\theta_1, \ldots, \theta_I)^\intercal$ with its i^{th} element representing the investor's bias on the mean growth rate of stock i expected payoff. Hence, under the θ-type investor's beliefs, the cash-flow news processes have dynamics

$$dD_{it} = D_{it}\left[\left(\mu_i + \theta_i\right)dt + \sigma_i d\omega_{it}\left(\theta\right) + \sigma d\omega_t\right], \qquad \text{for } i = 1, \ldots, I, \tag{36}$$

$$dD_{nt} = D_{nt}\left[\mu_n dt + \sigma_n d\omega_{nt} + \sigma d\omega_t\right], \qquad \text{for } n = I+1, \ldots, N-1, \tag{37}$$

where $\omega_i\left(\theta\right)$, $i = 1, \ldots, I$, are her perceived Brownian motions with respect to her own probability measure $\mathbb{P}^\theta$, and is given by $\omega_{it}\left(\theta\right) = \omega_{it} - \theta_i t/\sigma_i$.[22]

The investor type space, denoted by Θ, is taken to be the whole I-dimensional Euclidean space $\mathbb{R}^I$ to incorporate all possible beliefs on the dispersed stocks. Accordingly, we now assume a multivariate Gaussian distribution with an I-dimensional mean vector $\tilde{m} = (\tilde{m}_1, \ldots, \tilde{m}_I)^\intercal$ and a diagonal variance matrix $\tilde{v}^2 = \mathrm{diag}\left(\tilde{v}_1^2, \ldots, \tilde{v}_I^2\right)$ whose main diagonal entries starting in the upper left corner are $\tilde{v}_1^2, \ldots, \tilde{v}_I^2$ and the entries outside the main diagonal are all zero, for the

[21] Modeling individual and aggregate cash-flow news as geometric Brownian motions is somewhat in the spirit of Brennan and Xia (2001). To prevent the N^{th} stock payoff potentially taking on negative values, one could consider a setup with cash-flow news processes having stochastic volatilities as in Menzly, Santos, and Veronesi (2004), Longstaff and Piazzesi (2004), but this would much complicate our analysis and is beyond the scope of our goal in this section.

[22] The dynamics of the N^{th} stock could be derived from (36)–(37) and (2) using the identity $D_t = \sum_{n=1}^{N} D_{nt}$.

relative frequency of investors over the type space Θ.[23] As before, we assume that all investors are initially equally endowed in all the stocks, implying that the initial total wealth of the group of investors having the bias vector θ, denoted as the distinct θ-type investor, as

$$W_0(\theta) = W_0(\theta_1, \ldots, \theta_I) = S_0 \prod_{i=1}^{I} \frac{1}{\sqrt{2\pi\tilde{v}_i^2}} e^{-\frac{1}{2}\frac{(\theta_i - \tilde{m}_i)^2}{\tilde{v}_i^2}},$$

where S_0 is the (endogenous) initial aggregate stock price $S_0 = \sum_{n=1}^{N} S_{n0}$. In this setting, each distinct θ-type investor chooses an admissible N-dimensional portfolio strategy (investment in each stock) so as to maximize her CRRA preferences over the horizon value of her portfolio $W_T(\theta)$ as in (6) subject to the corresponding budget constraint.

We proceed by first constructing the average bias and dispersion in beliefs for dispersed stocks following Definition 1 in our main model. The time-t average bias in beliefs, m_{it}, on stock $i = 1, \ldots, I$ is the implied bias of the corresponding representative investor, expressed as the weighted average of the individual investors' biases

$$m_{it} = \int_{\Theta} \theta_i h_t(\theta)\, d\theta, \tag{38}$$

where the weights $h_t(\theta) > 0$ are such that $\int_{\Theta} h_t(\theta)\, d\theta = 1$, while the dispersion in beliefs, v_{it}, is the standard deviation of investors' biases

$$v_{it}^2 \equiv \int_{\Theta} (\theta_i - m_{it})^2 h_t(\theta)\, d\theta. \tag{39}$$

Proposition 7 reports the average bias and dispersion in beliefs, along with the corresponding equilibrium stock prices, mean returns and volatilities in this economy in closed form.

Proposition 7. *The time-t average bias m_{it} and dispersion v_{it} in beliefs of dispersed stock $i = 1, \ldots, I$, are given by*

$$m_{it} = m_i + \Big(\ln N + \ln\frac{D_{it}}{D_t} - \Big(m_i + \mu_i - \mu - \frac{1}{2}\sigma_i^2\Big)t\Big)\frac{1}{\gamma}\frac{v_{it}^2}{\sigma_i^2}, \qquad v_{it}^2 = \frac{v_i^2\sigma_i^2}{\sigma_i^2 + \frac{1}{\gamma}v_i^2 t}, \tag{40}$$

where their initial values m_i and v_i are related to the mean $\tilde{m}_i$ and standard deviation $\tilde{v}_i$ of investor types as

$$m_i = \tilde{m}_i + \Big(1 - \frac{1}{\gamma}\Big)v_i^2 T, \qquad v_i^2 = \Big(\frac{\gamma}{2}\tilde{v}_i^2 - \frac{\gamma^2}{2T}\sigma_i^2\Big) + \sqrt{\Big(\frac{\gamma}{2}\tilde{v}_i^2 - \frac{\gamma^2}{2T}\sigma_i^2\Big)^2 + \frac{\gamma^2}{T}\tilde{v}_i^2\sigma_i^2}. \tag{41}$$

Moreover, the equilibrium stock price, mean return and volatility of dispersed stock $i = 1, \ldots, I$,

[23] This simplifying assumption is the most straightforward natural extension of our main model and ensures that the investor population again has a finite (unit) measure. While admitting much tractability, it however rules out potential correlations across biases on individual stocks.

are given by

$$S_{it} \;=\; \overline{S}_{it} e^{m_{it}(T-t) - \frac{1}{2\gamma}(2\gamma-1)v_{it}^2(T-t)^2 + v_{it}^2(T-t)^2}, \tag{42}$$

$$\mu_{S_it} \;=\; \overline{\mu}_{S_it} - m_{it}\left(1 + \frac{v_{it}^2}{\gamma\sigma_i^2}\,(T-t)\right), \qquad \sigma_{S_it} = \sqrt{\overline{\sigma}_{S_it}^2 + \sigma_i^2\left(\left(1 + \frac{v_{it}^2}{\gamma\sigma_i^2}\,(T-t)\right)^2 - 1\right)}, \tag{43}$$

while the corresponding quantities for non-dispersed stocks $n = I+1, \ldots, N-1$, *are given by*

$$S_{nt} = D_{nt} e^{\left(\mu_n - \gamma\sigma^2\right)(T-t)}, \qquad \mu_{S_nt} = \gamma\sigma^2, \qquad \sigma_{S_nt} = \sqrt{\sigma^2 + \sigma_n^2}, \tag{44}$$

where the equilibrium stock price, mean return and volatility in the benchmark economy with no belief dispersion are given by

$$\overline{S}_{it} = D_{it} e^{\left(\mu_i - \gamma\sigma^2\right)(T-t)}, \qquad \overline{\mu}_{S_it} = \gamma\sigma^2, \qquad \overline{\sigma}_{S_it} = \sqrt{\sigma^2 + \sigma_i^2}. \tag{45}$$

Consequently, in the presence of belief dispersion, for a dispersed stock $i = 1, \ldots, I$,

i) A higher belief dispersion v_{it} *leads to a higher average bias* m_{it} *for relatively good cash-flow news* $D_{it} > (D_t/N)\exp(m_i + \mu_i - \mu - \frac{1}{2}\sigma_i^2)t$, *and to a lower average bias otherwise.*

ii) The stock price is convex in its cash-flow news D_{it}.

iii) The stock price is increasing in belief dispersion v_{it} *when* $m_{it} > \tilde{m}_i - (1/2\gamma)\, v_{it}^2\,(T-t)$, *and is decreasing otherwise.*

iv) The mean return is decreasing in belief dispersion v_{it} *when* $m_{it} > v_{it}^2 \tilde{m}_i \left(2v_{it}^2 - v_{iT}^2\right)^{-1}$, *and is increasing otherwise.*

v) The stock volatility is increasing in belief dispersion v_{it}.

The average bias and dispersion in beliefs (40) for dispersed stocks are multivariate versions of the single-stock case, but now adjusted to incorporate individual stock-specific quantities, μ_i, σ_i, $\tilde{m}_i$, $\tilde{v}_i$. The fluctuations in the average bias are due to the (relative) cash-flow news D_{it}/D_t, which are driven only by the Brownian motion ω_i that investors' beliefs differ on. Consequently, the underlying economic mechanisms revealed in Section 3 for our main model are also present in this setting with multiple stocks. In particular, for an individual dispersed stock, the belief dispersion amplifies its average bias and the effective investor belief heterogeneity on it does not vanish in relatively extreme states. Similarly, the equilibrium stock price, mean return and volatility (42)–(43) for dispersed stocks are multi-stock versions of the corresponding single-stock economy quantities with similar structures, but now incorporating individual stock specific average bias m_i and belief dispersion v_i.[24]

[24]These is a slight difference in the exact appearance of the belief dispersion in these equilibrium expressions

Since the underlying economic mechanisms and the structures of the economic quantities in this setting are as in our main model, all the implications for a dispersed stock's average bias, its price, mean return and volatility, are also as before, as highlighted in the properties (i)–(v) of Proposition 7. One difference from the single-stock case is that we are unable to obtain the equilibrium trading volume measure in this multiple stock setting. This is because there is a convoluted interaction of investors' views on dispersed stocks which prevents us identifying investors' portfolio reaction to changes in cash-flow news, which is required to compute the trading volume. Nevertheless, to see whether or not our model supports the documented empirical evidence on trading volume, in the subsequent analysis we use our single-stock economy trading volume measure (18) of Section 4.

7 Economy with Bayesian Learning

So far, we have studied an economy where investors have dogmatic beliefs, which not only simplified the analysis, but also demonstrated that our results are not driven by investors' learning. In this Section, we consider a setting with parameter uncertainty and more rational behavior for investors who optimally update their beliefs in a Bayesian fashion over time as more data becomes available. This setting is also tractable. We again obtain fully-closed form solutions for all quantities of interest, and show that all our results remain valid in this richer, incomplete information economy. This specification also enables us disentangle the effects of belief dispersion and parameter uncertainty on stock volatility, and establish the result that the investors' Bayesian learning induces less stock volatility when belief dispersion is higher.

To incorporate Bayesian learning in our main model, we make the following modification to investors' beliefs in Section 2.2. The investors are again indexed by their type θ, but instead of believing the mean growth rate of the expected payoff is $\mu + \theta$ at all times $0 \leq t \leq T$, now the θ-type investor at time 0 believes that the mean growth rate of the expected payoff is normally distributed with mean $\mu + \theta$ and variance s^2, $\mathcal{N}(\mu + \theta, s^2)$. This allows us to

as compared to the corresponding single-stock economy quantities. This is because in this setting investors only disagree on the dynamics of some of the stocks and the average bias on each dispersed stock i is driven only by the stock-specific single Brownian motion ω_i rather than the Brownian motion ω that determines the aggregate payoff. Hence, there is no additional risk arising from the covariance between the average bias and the aggregate payoff since the Brownian motions ω_i and ω are independent. This, in particular, leads to a simpler mean return expression (43), which in terms of factors $\ln D_t$ and $\ln(D_{it}/D_t)$ for $i = 1, ..., I$, can be expressed as $\mu_{S_i t}dt = \gamma \text{Cov}_t [d \ln D_t, d(S_{it}/S_{it})] - \sum_{j=1}^{I} \left(m_{jt}/\sigma_j^2\right) \text{Cov}_t [d \ln(D_{jt}/D_t), d(S_{it}/S_{it})]$. However, these small differences turn out to be economically immaterial for our implications of belief dispersion as the properties (i)–(v) of Proposition 7 highlight. This is because the main driving force behind our results is the fact that the belief dispersion enters into and amplifies the average bias, which is still present in this multiple-stocks setting.

interpret a θ-type investor as an investor with an initial bias of θ in her beliefs. The identical prior variance s^2 for all investors ensures that our results are not driven by heterogeneity in confidence of their estimates. The normal prior and the Bayesian updating rule imply that the time-t posterior distribution of μ, conditional on the information set $\mathcal{F}_t = \{D_s : 0 \leq s \leq t\}$, is also normally distributed, $\mathcal{N}(\mu + \widehat{\theta}_t, s_t^2)$, where the time-$t$ bias of θ-type investor $\widehat{\theta}_t$ (difference between her mean estimate and the true μ) and the type-independent mean squared error s_t^2, which represents the level of *parameter uncertainty*, are reported in Proposition 8. Therefore, under the θ-type investor's beliefs, the posterior cash-flow news process has dynamics

$$dD_t = D_t[(\mu + \widehat{\theta}_t)dt + \sigma d\omega_t(\theta)],$$

where $\omega(\theta)$ is her perceived Brownian motion with respect to her own probability measure $\mathbb{P}^\theta$, and is given by $d\omega_t(\theta) = d\omega_t - (\widehat{\theta}_t/\sigma)dt$. We note that this specification conveniently nests the earlier dogmatic beliefs economy when $s^2 = 0$.

We proceed by first constructing the average bias and dispersion in beliefs following Definition 1 in Section 3. The time-t *average bias in beliefs*, m_t, is the implied bias of the corresponding representative investor, expressed as the weighted average of the individual investors' biases

$$m_t = \int_\Theta \widehat{\theta}_t h_t(\theta)\, d\theta, \tag{46}$$

with the weights $h_t(\theta) > 0$ are such that $\int_\Theta h_t(\theta)\, d\theta = 1$, while the *dispersion in beliefs*, v_t, is the standard deviation of investors' biases

$$v_t^2 \equiv \int_\Theta (\widehat{\theta}_t - m_t)^2 h_t(\theta)\, d\theta. \tag{47}$$

Proposition 8 reports the average bias and dispersion in beliefs along with the corresponding unique weights in this economy with belief dispersion and parameter uncertainty in closed form.

Proposition 8. *The time-t average bias m_t and dispersion v_t in beliefs are given by*

$$m_t = m + \Big(\ln D_t - \Big(m + \mu - \frac{1}{2}\sigma^2\Big)t\Big)\Big(\frac{1}{\gamma}v^2 + s^2\Big)\frac{1}{\sigma^2}\frac{v_t^2}{v^2}\frac{s^2}{s_t^2}, \qquad v_t^2 = \frac{v^2\sigma^2}{\sigma^2 + \Big(\frac{1}{\gamma}v^2 + s^2\Big)\,t}\frac{s_t^2}{s^2}, \tag{48}$$

where the investors' time-t parameter uncertainty s_t is given by

$$s_t^2 = \frac{s^2\sigma^2}{\sigma^2 + s^2 t}, \tag{49}$$

and the initial values m and v are related to the initial mean $\tilde{m}$ and standard deviation $\tilde{v}$ of

investor types as

$$m=\tilde{m}+\left(1-\frac{1}{\gamma}\right)v^2T,\quad v^2=\left(\frac{\gamma}{2}\tilde{v}^2-\frac{\gamma^2}{2T}\left(\sigma^2+s^2T\right)\right)+\sqrt{\left(\frac{\gamma}{2}\tilde{v}^2-\frac{\gamma^2}{2T}\left(\sigma^2+s^2T\right)\right)^2+\frac{\gamma^2}{T}\tilde{v}^2\left(\sigma^2+s^2T\right)}. \tag{50}$$

The weights $h_t(\theta)$ are uniquely identified to be given by

$$h_t(\theta)=\frac{1}{\sqrt{2\pi v_t^2}}e^{-\frac{1}{2}\frac{\left(\widehat{\theta}_t-m_t\right)^2}{v_t^2}}\frac{s_t^2}{s^2}, \tag{51}$$

where m_t, v_t and s_t are as in (48)–(49) *and the time-t bias of θ-type investor $\widehat{\theta}_t$ is given by*

$$\widehat{\theta}_t=\frac{s_t^2}{s^2}\theta+\frac{s_t^2}{\sigma}\omega_t. \tag{52}$$

Consequently, in the presence of belief dispersion and parameter uncertainty, for economies with the same initial average bias m and dispersion v,

i) The average bias in beliefs is increasing in parameter uncertainty s_t when $D_t>\exp\left(\left(m+\mu-\frac{1}{2}\sigma^2\right)t\right)$, and is decreasing otherwise.

ii) The dispersion in beliefs is decreasing in parameter uncertainty s_t.

The average bias and dispersion in beliefs (48) are generalizations of the earlier dogmatic beliefs case and are now additionally driven by parameter uncertainty s_t.[25] We see that, similarly to the effect of dispersion, a higher parameter uncertainty leads to a relatively more optimistic (pessimistic) view on the stock following good (bad) news (Property (i)), and this then amplifies the volatility of the average bias relative to the dogmatic beliefs case. However, the underlying mechanisms of belief dispersion and parameter uncertainty are notably different. In the case of dispersion, the view on the stock becomes more optimistic following good news, because the optimistic investors, whose beliefs are supported, become wealthier and this increases their impact on the average bias in beliefs. In the case of parameter uncertainty, the view on the stock becomes more optimistic following good news, because all Bayesian investors increase their estimates of the mean growth rate of the expected payoff μ. Proposition 8 also reveals that a higher parameter uncertainty leads to a lower dispersion in beliefs (Property (ii)). This is intuitive because investors' estimates of μ is a weighted average of their prior and the data (cash-flow news). The higher the parameter uncertainty, the more weight investors place on the data, which in turn reduces the differences in their estimates and the belief dispersion.

We remark that in this Section, we consider the effects of parameter uncertainty s_t only for economies with the same initial average bias m and dispersion v, as highlighted in Proposition

[25] Note that when $s^2=0$, the ratio $s_t^2/s^2=1$ for all t, and the expressions in Proposition 8 collapse down to the dogmatic beliefs economy expressions in Proposition 1.

8. This way, economies only differ in their initial level of parameter uncertainty s and our results are not driven by the indirect effects through the initial average bias and dispersion.[26] Proposition 9 reports the equilibrium stock price, its dynamics and the trading volume measure in this economy with belief dispersion and parameter uncertainty in closed form.

Proposition 9. *In the economy with belief dispersion and parameter uncertainty, the equilibrium stock price, mean return and volatility are given by*

$$S_t = \overline{S}_t e^{m_t(T-t)-\frac{1}{2}(2\gamma-1)\left(\frac{1}{\gamma}v^2+s^2\right)\frac{v_t^2}{v^2}\frac{s^2}{s_t^2}(T-t)^2}, \tag{53}$$

$$\mu_{St} = \overline{\mu}_{St}\left(1+\frac{1}{\sigma^2}\left(\frac{1}{\gamma}v^2+s^2\right)\frac{v_t^2}{v^2}\frac{s^2}{s_t^2}(T-t)\right)^2 - m_t\left(1+\frac{1}{\sigma^2}\left(\frac{1}{\gamma}v^2+s^2\right)\frac{v_t^2}{v^2}\frac{s^2}{s_t^2}(T-t)\right), \tag{54}$$

$$\sigma_{St} = \overline{\sigma}_{St}+\frac{1}{\sigma}\left(\frac{1}{\gamma}v^2+s^2\right)\frac{v_t^2}{v^2}\frac{s^2}{s_t^2}(T-t), \tag{55}$$

where the average bias m_t, dispersion v_t in beliefs and parameter uncertainty s_t are as in Proposition 8 and the equilibrium stock price $\overline{S}_t$, mean return $\overline{\mu}_{St}$, and stock volatility $\overline{\sigma}_{St}$ in the benchmark economy with no belief dispersion are as in Propositions 2–4, respectively.
The equilibrium trading volume measure is given by

$$V_t = \frac{\sigma}{X_t^2}\frac{v_t^2}{v_T^2}\frac{s_T^2}{s_t^2}\left[\left(\frac{X_t}{2}+\frac{\sqrt{X_t^2+4}}{2}\right)\boldsymbol{\phi}\left(\frac{X_t}{2}-\frac{\sqrt{X_t^2+4}}{2}\right)-\left(\frac{X_t}{2}-\frac{\sqrt{X_t^2+4}}{2}\right)\boldsymbol{\phi}\left(\frac{X_t}{2}+\frac{\sqrt{X_t^2+4}}{2}\right)\right], \tag{56}$$

where $\boldsymbol{\phi}(.)$ is the probability density function of the standard normal random variable, and X is a (positive) deterministic process given by

$$X_t^2 = \frac{\gamma^2\sigma^4}{v_T^4}\left[\frac{1}{\gamma}v_t^2\frac{s_T^4}{s_t^4}+\left(1-\frac{1}{\gamma}\right)v_T^2\right]. \tag{57}$$

Consequently, in the presence of belief dispersion and parameter uncertainty, in addition to the properties in Propositions 2–5, for economies with the same initial average bias m and dispersion v, the stock volatility is increasing in parameter uncertainty s_t but this effect is decreasing in belief dispersion v_t.

Proposition 9 confirms that our earlier implications for the stock price, its dynamics and the trading volume remain valid with Bayesian learning. However, these equilibrium quantities are now also driven by the parameter uncertainty s_t. More notably, the additional effect due to the parameter uncertainty now makes the stock price even more volatile as compared with the dogmatic beliefs case. This is because a higher parameter uncertainty makes the average bias more volatile, which leads to a higher stock price following relatively good news,

[26] This is established by letting the initial indirect effect of parameter uncertainty fall on the mean $\tilde{m}$ and standard deviation $\tilde{v}$ of investor types using the monotonic relations between m and $\tilde{m}$, and v and $\tilde{v}$ in (50).

and to a lower stock price otherwise, compared to the dogmatic beliefs case. Therefore, in this economy, the stock volatility is not only increasing in belief dispersion as in our earlier analysis, but also in parameter uncertainty. Importantly, our stock volatility expression in (55) allows us to disentangle the effects of belief dispersion from those of parameter uncertainty and yields a novel testable implication that the parameter uncertainty (and the subsequent Bayesian learning) induces less stock volatility when belief dispersion is higher.[27] This is because fluctuations in the average bias, and hence in the stock price, due to parameter uncertainty is lower when dispersion is higher. In the literature, both channels are shown to generate higher stock volatility. By disentangling these effects, our result may help future works to measure the relative contributions of parameter uncertainty and belief dispersion in stock volatility better.

8 Conclusion

In this paper, we have developed a dynamic model of belief dispersion which qualitatively matches many of the empirical regularities in a stock price, its mean return, volatility, and trading volume. In our analysis, we have determined two sufficient measures, the average bias and dispersion in beliefs, to summarize the wide range of investors' beliefs and have demonstrated that the equilibrium quantities are driven by these two key variables. Our model is tractable and delivers exact closed-form expressions for quantities of interest.

We have found that the stock price increases in cash-flow news in a convex manner. We have also shown that the stock price increases and its mean return decreases in belief dispersion when the view on the stock is optimistic, and vice versa when pessimistic. We have found that the presence of belief dispersion leads to a higher stock volatility, trading volume, and a positive relation between these two quantities. We have disentangled the effects of belief dispersion and learning on stock volatility, and found that the effects of learning is reduced when dispersion is higher. Furthermore, we have demonstrated how otherwise identical two-investor economies with heterogeneous beliefs and a single consumption date do not necessarily generate most of our main results, particularly convexity.

In models such as ours where investors have preferences only over horizon wealth, the discount factor is determined by the anticipation of future consumption. In contrast, in a model with continuous consumption the discount factor is determined by market clearing in

[27] The parameter uncertainty channel is shut down by setting $s^2 = 0$ in (55), which implies $s_t^2/s^2 = 1$, and yields $\sigma_{St} = \overline{\sigma}_{St} + (v_t^2/\sigma\gamma)(T-t)$. Similarly, the belief dispersion channel is shut down by setting $v^2 = 0$ in (55), which implies $v_t^2/v^2 = s_t^4/s^4$, and yields $\sigma_{St} = \overline{\sigma}_{St} + (s_t^2/\sigma)(T-t)$.

the current consumption good. In such a model it is not immediately clear that all our results would obtain. For example, when preferences in that setting are logarithmic there are no asset pricing effects for the stock (e.g., Detemple and Murthy (1997), Atmaz (2014)), therefore the model would not explain the empirical regularities observed in the stock market. On the other hand, considering more general power preferences with intertemporal consumption leads to the issue of the representative investor's belief not being well-defined since the process that aggregates investors' beliefs is not a martingale, and hence not a proper belief process (see, Jouini and Napp (2007)). Therefore, in that setting we may not obtain the average bias, then back out the investors' equilibrium weights, and define the belief dispersion as we do in our model. Our setting turns out to not suffer from this issue and yields a well-defined belief process for the representative investor for all risk aversion values due to her preferences being over horizon wealth. We leave the analysis with continuous consumption for future research. In the Internet Appendix ID we make some progress in that direction by extending our main model with a single consumption date to one with multiple consumption dates.

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A Appendix: Proofs of Main Model

Proof of Proposition 1. We proceed by first solving each θ-type investor's problem, and determining the equilibrium horizon prices and investors' Lagrange multipliers, which are used in the representative investor construction to infer her implied bias and to hence define the average bias in beliefs, m_t. We next identify the unique weights $h_t(\theta)$ which along with the average bias allow us to determine the belief dispersion, v_t.

Dynamic market completeness implies a unique state price density process ξ under the true measure $\mathbb{P}$, such that the time-t value of a payoff X_T at time T is given by $\mathbb{E}_t[\xi_T X_T]/\xi_t$, where ξ_T/ξ_t represents the stochastic discount factor. Accordingly, the dynamic budget constraint (7) of each θ-type investor under $\mathbb{P}$ can be restated as

$$\mathbb{E}_t[\xi_T W_T(\theta)] = \xi_t W_t(\theta). \tag{A.1}$$

We also rewrite each θ-type investor's expected utility function (6) under $\mathbb{P}$ as

$$\mathbb{E}\left[\eta_T(\theta)\frac{W_T(\theta)^{1-\gamma}}{1-\gamma}\right], \tag{A.2}$$

where $\eta_T(\theta)$ is the Radon-Nikodým derivative of the subjective measure $\mathbb{P}^\theta$ with respect to $\mathbb{P}$,

$$\eta_T(\theta) = \frac{d\mathbb{P}^\theta}{d\mathbb{P}} = e^{\frac{\theta}{\sigma}\omega_T - \frac{1}{2}\frac{\theta^2}{\sigma^2}T}. \tag{A.3}$$

Maximizing each (distinct) θ-type investor's expected objective function (A.2) subject to (A.1) evaluated at time $t=0$ leads to the optimal horizon wealth of each θ-type as

$$W_T(\theta) = \left(\frac{\eta_T(\theta)}{y(\theta)\xi_T}\right)^{\frac{1}{\gamma}}, \tag{A.4}$$

where the Lagrange multiplier $y(\theta)$ solves (A.1) evaluated at time $t=0$

$$y(\theta)^{-\frac{1}{\gamma}} = \mathbb{E}\left[\eta_T(\theta)^{\frac{1}{\gamma}}\xi_T^{1-\frac{1}{\gamma}}\right]^{-1}\frac{\xi_0 S_0}{\sqrt{2\pi\tilde{v}^2}}e^{-\frac{1}{2}\frac{(\theta-\tilde{m})^2}{\tilde{v}^2}}. \tag{A.5}$$

We next determine the time-T equilibrium state price density ξ_T. Substituting (A.4) into the market clearing condition $\int_\Theta W_T(\theta)\,d\theta = D_T$ yields $\xi_T^{-\frac{1}{\gamma}}\int_\Theta y(\theta)^{-\frac{1}{\gamma}}\eta_T(\theta)^{\frac{1}{\gamma}}\,d\theta = D_T$, which after rearranging we obtain the time-T equilibrium state price density

$$\xi_T = D_T^{-\gamma}M_T^\gamma, \tag{A.6}$$

where the auxiliary process M and the likelihood ratio process $\eta(\theta)$ are given by

$$M_t \equiv \int_\Theta\left(\frac{\eta_t(\theta)}{y(\theta)}\right)^{\frac{1}{\gamma}}d\theta, \qquad \eta_t(\theta) = \mathbb{E}_t[\eta_T(\theta)] = e^{\frac{\theta}{\sigma}\omega_t - \frac{1}{2}\frac{\theta^2}{\sigma^2}t}. \tag{A.7}$$

As we show below, $y(\theta)^{-\frac{1}{\gamma}}$ is (scaled) Gaussian over the type space Θ for some mean α_o, variance β_o^2 and a constant K:

$$y(\theta)^{-\frac{1}{\gamma}} = K\frac{1}{\sqrt{2\pi\beta_o^2}}e^{-\frac{1}{2}\frac{(\theta-\alpha_o)^2}{\beta_o^2}}. \tag{A.8}$$

Substituting $\eta_t(\theta)$ in (A.7) along with (A.8) into the definition of M in (A.7) yields

$$M_t = K\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi\beta_o^2}}e^{-\frac{1}{2}\frac{(\theta-\alpha_o)^2}{\beta_o^2}}e^{\frac{1}{\gamma}\frac{\theta}{\sigma}\omega_t-\frac{1}{2}\frac{1}{\gamma}\frac{\theta^2}{\sigma^2}t}d\theta = K\frac{\beta_t}{\beta_o}e^{-\frac{1}{2}\frac{\alpha_o^2}{\beta_o^2}+\frac{1}{2}\frac{\alpha_t^2}{\beta_t^2}}, \tag{A.9}$$

where the last equality follows by completing the square and integrating, and the processes α and β, with their initial values $\alpha_0 = \alpha_o$ and $\beta_0 = \beta_o$, respectively, defined as

$$\alpha_t \equiv \sigma\frac{\sigma\alpha_o + \frac{1}{\gamma}\beta_o^2\omega_t}{\sigma^2 + \frac{1}{\gamma}\beta_o^2 t}, \qquad \beta_t^2 \equiv \frac{\beta_o^2\sigma^2}{\sigma^2 + \frac{1}{\gamma}\beta_o^2 t}. \tag{A.10}$$

We now verify $y(\theta)^{-\frac{1}{\gamma}}$ is as in (A.8). Substituting (A.6) into (A.5) gives

$$y(\theta)^{-\frac{1}{\gamma}} = \left(\mathbb{E}\left[\eta_T(\theta)^{\frac{1}{\gamma}}D_T^{1-\gamma}M_T^{\gamma-1}\right]\right)^{-1}\frac{\xi_0 S_0}{\sqrt{2\pi\tilde{v}^2}}e^{-\frac{1}{2}\frac{(\theta-\tilde{m})^2}{\tilde{v}^2}}, \tag{A.11}$$

where M_T is equal to (A.9) evaluated at time T. From Lemma A2 at the end of this appendix, evaluated at $t = 0$, the expectation in (A.11) is equal to

$$\mathbb{E}\left[\eta_T(\theta)^{\frac{1}{\gamma}}D_T^{1-\gamma}M_T^{\gamma-1}\right] = \xi_0 S_0 K^{-1}\left(\frac{1}{\sqrt{2\pi\beta_o^2}}e^{-\frac{1}{2}\frac{(\theta-\alpha_o)^2}{\beta_o^2}}\right)^{-1}\frac{1}{\sqrt{2\pi\beta_o^2 A^2}}e^{-\frac{1}{2}\frac{\left(\theta-\left[\alpha_o-\left(1-\frac{1}{\gamma}\right)\beta_o^2 T\right]\right)^2}{\beta_o^2 A^2}}, \tag{A.12}$$

where the constant A^2 is given by $A^2 = \left(\sigma^2 + \frac{1}{\gamma^2}\beta_o^2 T\right)/\left(\sigma^2 + \frac{1}{\gamma}\beta_o^2 T\right)$. Substituting (A.12) into (A.11) and manipulating terms yields (A.8) with

$$\alpha_o = \tilde{m} + \left(1 - \frac{1}{\gamma}\right)\beta_o^2 T, \qquad \beta_o^2 = \left(\frac{\gamma}{2}\tilde{v}^2 - \frac{\gamma^2}{2T}\sigma^2\right) + \sqrt{\left(\frac{\gamma}{2}\tilde{v}^2 - \frac{\gamma^2}{2T}\sigma^2\right)^2 + \frac{\gamma^2}{T}\tilde{v}^2\sigma^2}. \tag{A.13}$$

We note that when $\gamma = 1$ the constants α_o and β_0^2 coincide with $\tilde{m}$ and $\tilde{v}^2$, respectively.

We now construct the representative investor in our dynamically complete market economy to infer her implied bias in beliefs. The representative investor solves

$$U(D_T;\boldsymbol{\lambda}) = \max\int_\Theta \lambda(\theta)\,\eta_T(\theta)\frac{W_T(\theta)^{1-\gamma}}{1-\gamma}d\theta, \qquad \text{s.t.} \qquad \int_\Theta W_T(\theta)\,d\theta = D_T, \tag{A.14}$$

for some strictly positive weights $\lambda(\theta)$ for each θ-type investor, where the collection of weights is denoted by $\boldsymbol{\lambda} = \{\lambda(\theta)\}_{\theta\in\Theta}$. The first order conditions of (A.14) yield

$$\frac{W_T(\theta)}{W_T(0)} = \left(\frac{\lambda(0)\,\eta_T(0)}{\lambda(\theta)\,\eta_T(\theta)}\right)^{-\frac{1}{\gamma}},$$

where $\lambda(0)$ and $\eta_T(0)$ denote the 0-type investor's weight and the Radon-Nikodým derivative,

respectively. Imposing $\int_\Theta W_T(\theta)\,d\theta = D_T$ yields the equilibrium horizon wealth allocations as

$$W_T(\theta) = \frac{[\lambda(\theta)\,\eta_T(\theta)]^{\frac{1}{\gamma}}}{\int_\Theta [\lambda(\theta)\,\eta_T(\theta)]^{\frac{1}{\gamma}}\,d\theta} D_T. \tag{A.15}$$

Substituting (A.15) into (A.14) gives the representative investor's utility function as

$$U(D_T;\boldsymbol{\lambda}) = \frac{\int_\Theta [\lambda(\theta)\,\eta_T(\theta)]\,[\lambda(\theta)\,\eta_T(\theta)]^{\frac{1-\gamma}{\gamma}}\,d\theta}{\left(\int_\Theta [\lambda(\theta)\,\eta_T(\theta)]^{\frac{1}{\gamma}}\,d\theta\right)^{1-\gamma}} \frac{D_T^{1-\gamma}}{1-\gamma}, \tag{A.16}$$

which after rearranging becomes

$$U(D_T;\boldsymbol{\lambda}) = \left(\int_\Theta [\lambda(\theta)\,\eta_T(\theta)]^{\frac{1}{\gamma}}\,d\theta\right)^{\gamma} \frac{D_T^{1-\gamma}}{1-\gamma}. \tag{A.17}$$

We next identify, from the second welfare theorem, the weights as $\lambda(\theta) = 1/y(\theta)$ where $y(\theta)$ is the θ-type investor's Lagrange multiplier given by (A.8). By substituting these weights into (A.17) we observe that the parenthesis term as M_T in (A.7). We define the martingale Z as the conditional expectation of M_T^γ under the true measure

$$Z_t \equiv \mathbb{E}_t[M_T^\gamma] = M_t^\gamma \left(\frac{\beta_T}{\beta_t}\right)^{\gamma-1}, \tag{A.18}$$

where the last equality above follows from Lemma A1 at the end of this appendix by taking $a = 0$ and $b = \gamma$. Since equilibrium is unique up to a constant, without loss of generality we set the constant $K = (\beta_T/\beta_o)^{\frac{1}{\gamma}-1}$ in (A.8) so that when substituted into (A.18) we obtain $Z_0 = 1$. Applying Itô's Lemma to (A.18) gives the dynamics of Z as

$$\frac{dZ_t}{Z_t} = \frac{\alpha_t}{\sigma} d\omega_t. \tag{A.19}$$

Therefore, we obtain the representative investor's utility function as

$$U(D_T;\boldsymbol{\lambda}) = Z_T \frac{D_T^{1-\gamma}}{1-\gamma}, \tag{A.20}$$

and identify Z_T as being the Radon-Nikodým derivative of the representative investor's subjective belief $\mathbb{P}^R$ with respect to the true belief $\mathbb{P}$, that is $d\mathbb{P}^R/d\mathbb{P} = Z_T$.[28] Moreover, (A.19) implies that α_t is the time-t (stochastic) bias of the representative investor, and so is the time-t

[28] Alternatively, we can derive the representative investor's utility function in (A.20) by applying Itô's Lemma to M_t^γ using (A.9) to obtain the dynamics $dM_t^\gamma/M_t^\gamma = -\frac{1}{2}(1-1/\gamma)\left(\beta_t^2/\sigma^2\right)dt + (\alpha_t/\sigma)\,d\omega_t$. Since the drift term is deterministic, we may write M_t^γ as $M_t^\gamma = K^\gamma Y_t Z_t$ where Y is a deterministic process and Z is a martingale process with dynamics as in (A.19) with initial values $Y_0 = Z_0 = 1$. The solution to Y is given by $Y_t = (\beta_t/\beta_o)^{\gamma-1}$. Setting the constant as above $K = (\beta_T/\beta_o)^{(1/\gamma)-1}$ yields the representative investor's utility function as in (A.20).

average bias in beliefs, as denoted by m_t,

$$m_t = \alpha_t = \sigma \frac{\sigma\alpha_o + \frac{1}{\gamma}\beta_o^2\omega_t}{\sigma^2 + \frac{1}{\gamma}\beta_o^2 t} = \int_\Theta \theta \left(\frac{y(\theta)^{-\frac{1}{\gamma}} \eta_t(\theta)^{\frac{1}{\gamma}}}{\int_\Theta y(\theta)^{-\frac{1}{\gamma}} \eta_t(\theta)^{\frac{1}{\gamma}} d\theta} \right) d\theta. \tag{A.21}$$

Substituting $\sigma\omega_t$ by $\ln D_t - \left(\mu - \frac{1}{2}\sigma^2\right) t$ yields the expression stated in (10).

From the last equality in (A.21) we identify the unique weights $h_t(\theta)$ such that the weighted-average of investors' biases equals to the average bias in beliefs as

$$h_t(\theta) = \frac{y(\theta)^{-\frac{1}{\gamma}} \eta_t(\theta)^{\frac{1}{\gamma}}}{\int_\Theta y(\theta)^{-\frac{1}{\gamma}} \eta_t(\theta)^{\frac{1}{\gamma}} d\theta} = \frac{1}{\sqrt{2\pi\beta_t^2}} e^{-\frac{1}{2}\frac{(\theta-\alpha_t)^2}{\beta_t^2}}, \tag{A.22}$$

where the last equality follows from substituting $\eta_t(\theta)$ into (A.7) and (A.8) into (A.22) and rearranging where α_t and β_t are as in (A.10).

Finally, to determine the belief dispersion, we use the definition in (9) with the average bias in beliefs (A.21) and weights (A.22) substituted in to obtain

$$v_t^2 = \int_\Theta (\theta - m_t)^2 h_t(\theta) d\theta = \int_\Theta (\theta - m_t)^2 \frac{1}{\sqrt{2\pi\beta_t^2}} e^{-\frac{1}{2}\frac{(\theta-m_t)^2}{\beta_t^2}} d\theta = \beta_t^2. \tag{A.23}$$

By equating the initial values α_o and β_o^2 to m and v^2 in (A.13) we obtain the (squared) dispersion and the weights as in (10) and (12).

The condition for the property that a higher belief dispersion leads to a higher average bias follows from the positivity of the partial derivative of (10) with respect to v_t

$$\frac{\partial}{\partial v_t} m_t = \left(\ln D_t - \left(m + \mu - \frac{1}{2}\sigma^2 \right) t \right) 2\frac{v_t}{\gamma\sigma^2} > 0. \tag{A.24}$$

Rearranging the term in the bracket gives the desired condition $D_t > \exp\left(m + \mu - \frac{1}{2}\sigma^2\right) t$. □

Proof of Proposition 2. By no arbitrage, the stock price in our economy is given by

$$S_t = \frac{1}{\xi_t} \mathbb{E}_t \left[\xi_T D_T \right]. \tag{A.25}$$

To determine the stock price, we first compute the equilibrium state price density at time t by using the fact that it is a martingale, $\xi_t = \mathbb{E}_t[\xi_T]$. The equilibrium state price density at time T is as in the proof of Proposition 1, given by (A.6). Hence,

$$\xi_t = \mathbb{E}_t \left[D_T^{-\gamma} M_T^\gamma \right] = D_t^{-\gamma} M_t^\gamma \left(\frac{v_T}{v_t} \right)^{\gamma-1} e^{-\gamma\left(\mu - \frac{1}{2}\sigma^2\right)(T-t)} e^{-\gamma m_t (T-t)} e^{\frac{1}{2}\gamma^2\sigma^2 \frac{v_t^2}{v_T^2}(T-t)}, \tag{A.26}$$

where the last equality follows from Lemma A1 by taking $a = -\gamma$ and $b = \gamma$ and using the equalities $m_t = \alpha_t$ and $v_t = \beta_t$ to express the equation in terms of model parameters.

Next, we substitute (A.6) into (A.25) and obtain the expectation $\mathbb{E}_t[\xi_T D_T] = \mathbb{E}_t\left[D_T^{1-\gamma} M_T^\gamma\right]$.

Again employing Lemma A1 with $a = 1-\gamma$ and $b = \gamma$, we obtain

$$\mathbb{E}_t\left[D_T^{1-\gamma}M_T^{\gamma}\right] = D_t^{1-\gamma}M_t^{\gamma}\left(\frac{v_T}{v_t}\right)^{\gamma-1} e^{(1-\gamma)\left(\mu-\frac{1}{2}\sigma^2\right)(T-t)}e^{(1-\gamma)m_t(T-t)}e^{\frac{1}{2}(1-\gamma)^2\sigma^2\frac{v_t^2}{v_T^2}(T-t)}. \quad \text{(A.27)}$$

Substituting (A.26) and (A.27) into (A.25) and manipulating yields the stock price expression (14) in Proposition 2. To determine the benchmark economy stock price, we set $\tilde{m} = \tilde{v} = 0$, which yields $m_t = v_t = 0$. Substituting into (14) gives the benchmark stock price.

Property (i) that the stock price is convex in cash-flow news follows once we substitute (10) into the stock price equation (14) and differentiate with respect to D_t.

The condition for property (ii) that the stock price is increasing in belief dispersion follows from the partial derivative of (14) with respect to v_t. This property holds when

$$\frac{\partial}{\partial v_t}m_t > \frac{1}{2\gamma}\left(2\gamma-1\right)\left(T-t\right)\frac{\partial}{\partial v_t}v_t^2. \quad \text{(A.28)}$$

Taking the partial derivative of m_t and v_t^2 using the expression (10) while taking account of the dependency on v and m, yields

$$\frac{\partial}{\partial v_t}m_t = \frac{2}{v_t}\left(m_t-\tilde{m}\right), \qquad \frac{\partial}{\partial v_t}v_t^2 = 2v_t, \quad \text{(A.29)}$$

which after substituting into (A.28) and rearranging gives the desired condition.

Finally, property (iii) that the stock price is increasing in investors' risk aversion for relatively bad cash-flow states and low values of γ follows from the partial derivative of (14) with respect to γ. This property holds when

$$\frac{\partial}{\partial\gamma}m_t > \sigma^2 + \left[\frac{1}{2\gamma^2}v_t^2 + \left(1-\frac{1}{2\gamma}\right)\left(\frac{\partial}{\partial\gamma}v_t^2\right)\right]\left(T-t\right). \quad \text{(A.30)}$$

In this regard, using (11), we first compute $\partial v^2/\partial\gamma$ and $\partial m/\partial\gamma$, and to simplify notation denote them by C and D, respectively

$$C \equiv \frac{\partial}{\partial\gamma}v^2 = \left(\frac{\tilde{v}^2}{2}-\frac{\gamma}{T}\sigma^2\right) + \frac{\left(\frac{\gamma}{2}\tilde{v}^2-\frac{\gamma^2}{2T}\sigma^2\right)\left(\frac{\tilde{v}^2}{2}-\frac{\gamma}{T}\sigma^2\right)+\frac{\gamma}{T}\tilde{v}^2\sigma^2}{\sqrt{\left(\frac{\gamma}{2}\tilde{v}^2-\frac{\gamma^2}{2T}\sigma^2\right)^2+\frac{\gamma^2}{T}\tilde{v}^2\sigma^2}}, \quad \text{(A.31)}$$

$$D \equiv \frac{\partial}{\partial\gamma}m = \frac{1}{\gamma^2}v^2T + \left(1-\frac{1}{\gamma}\right)CT. \quad \text{(A.32)}$$

Using the expressions in (10), we then obtain the required partial derivatives as

$$\frac{\partial}{\partial\gamma}m_t = \frac{v_t^2}{v^2}\left[D-\left(\frac{1}{\gamma}-\frac{C}{v^2}\right)\left(m_t-m\right)\right], \qquad \frac{\partial}{\partial\gamma}v_t^2 = \frac{v_t^4}{v^2}\left(\frac{C}{v^2}+\frac{v^2t}{\sigma^2\gamma^2}\right). \quad \text{(A.33)}$$

Substituting (A.33) into (A.30) and rearranging yields the condition as

$$m_t < m + \left(\frac{1}{\gamma} - \frac{C}{v^2}\right)^{-1} \left\{ D - \frac{v^2}{v_t^2}\sigma^2 - \left[\frac{v^2}{2\gamma^2} + \left(1 - \frac{1}{2\gamma}\right)\left(\frac{C}{v^2} + \frac{v^2 t}{\sigma^2\gamma^2}\right) v_t^2\right](T-t)\right\}. \quad \text{(A.34)}$$

For any time t, the right hand side of (A.34) is constant while its left hand side is the average bias in beliefs, which is a normally distributed random variable, hence for sufficiently low levels of m_t (A.34) always holds. □

Proof of Proposition 3. Applying Itô's Lemma to the stock price (14) yields

$$\frac{dS_t}{S_t} = \left[\gamma\sigma + \frac{v_t^2}{\sigma}(T-t) - \frac{m_t}{\sigma}\right]\left[\sigma + \frac{1}{\gamma}\frac{v_t^2}{\sigma}(T-t)\right]dt + \left[\sigma + \frac{1}{\gamma}\frac{v_t^2}{\sigma}(T-t)\right]d\omega_t, \quad \text{(A.35)}$$

where rearranging its drift term gives the equilibrium mean return as

$$\mu_{St} = \gamma\sigma^2\left(1 + \frac{1}{\gamma}\frac{v_t^2}{\sigma^2}(T-t)\right)^2 - m_t\left(1 + \frac{1}{\gamma}\frac{v_t^2}{\sigma^2}(T-t)\right). \quad \text{(A.36)}$$

The benchmark economy mean return is obtained by setting $m_t = v_t = 0$ in (A.36), which along with (A.36) gives (15).

The condition for property that the mean return is decreasing in belief dispersion follows from the partial derivative of (15) with respect to v_t. This property holds when

$$\frac{\partial}{\partial v_t}\mu_{St} = 2\gamma\sigma^2\left(\frac{v_t^2}{v_T^2}\right)\frac{\partial}{\partial v_t}\left(\frac{v_t^2}{v_T^2}\right) - m_t\frac{\partial}{\partial v_t}\left(\frac{v_t^2}{v_T^2}\right) - \frac{v_t^2}{v_T^2}\frac{\partial}{\partial v_t}m_t < 0. \quad \text{(A.37)}$$

Substituting the partial derivatives (A.29) into (A.37) and using the equality

$$\sigma + \frac{1}{\gamma}\frac{v_t^2}{\sigma}(T-t) = \sigma\frac{v_t^2}{v_T^2}, \quad \text{(A.38)}$$

and rearranging gives the desired condition.

Finally, property (ii) that the mean return is decreasing in investors' risk aversion for relatively bad cash-flow news and low levels of risk aversion follows from the partial derivative of (15) with respect to γ. This property holds when

$$\frac{\partial}{\partial\gamma}\mu_{St} = \sigma^2\left(\frac{v_t^2}{v_T^2}\right)^2 + \left[2\gamma\sigma^2\left(\frac{v_t^2}{v_T^2}\right) - m_t\right]\left(\frac{\partial}{\partial\gamma}\frac{v_t^2}{v_T^2}\right) - \frac{v_t^2}{v_T^2}\left(\frac{\partial}{\partial\gamma}m_t\right) < 0. \quad \text{(A.39)}$$

Substituting the partial derivatives (A.33) into (A.39) and using (A.38) yields

$$\frac{\partial}{\partial\gamma}\mu_{St} = \sigma^2\left(\frac{v_t^2}{v_T^2}\right)^2 + \left[2\gamma\sigma^2\left(\frac{v_t^2}{v_T^2}\right) - m_t\right]E - \frac{v_t^2}{v_T^2}\frac{v_t^2}{v^2}\left[D - \left(\frac{1}{\gamma} - \frac{C}{v^2}\right)(m_t - m)\right], \quad \text{(A.40)}$$

where we have defined E as

$$E \equiv \frac{\partial}{\partial\gamma}\frac{v_t^2}{v_T^2} = -\frac{1}{\gamma^2}\frac{v_t^2}{\sigma^2}(T-t) + \frac{1}{\gamma}\frac{1}{\sigma^2}\frac{v_t^4}{v^2}\left(\frac{C}{v^2} + \frac{v^2 t}{\sigma^2\gamma^2}\right)(T-t).$$

Rearranging (A.40) yields the condition as

$$m_t < \left[\frac{v_t^4}{v_T^2}\left(\frac{1}{\gamma v^2}-\frac{C}{v^4}\right)-E\right]^{-1}\left\{\frac{v_t^4}{v_T^2}\left[\frac{D}{v^2}+\left(\frac{1}{\gamma}-\frac{C}{v^2}\right)\frac{m}{v^2}\right]-\sigma^2\frac{v_t^4}{v_T^4}-2\gamma\sigma^2\frac{v_t^2}{v_T^2}E\right\}. \quad \text{(A.41)}$$

We note that, for any time t, the right hand side of (A.41) is a constant while its left hand side is the average bias in beliefs, which is a normally distributed random variable, hence for sufficiently low levels of m_t (A.41) always holds. □

Proof of Proposition 4. The volatility of the stock is given by the diffusion term of the dynamics (A.35). The benchmark stock volatility readily is obtained by setting $v_t = 0$ in the diffusion term of (A.35). The property that the stock volatility is increasing in belief dispersion is immediate from (16). □

Proof of Proposition 5. To compute the trading volume measure V, we proceed by first determining the dynamics of each θ-type investor's equilibrium wealth-share, $W(\theta)/S$ and portfolio, $\phi(\theta)$, the fraction of wealth invested in the stock. Then, applying the product rule to $\psi(\theta) = \phi(\theta)(W(\theta)/S)$, we obtain the dynamics of $\psi(\theta)$. Finally, using the definition in (17) we obtain the trading volume measure V for the stock in closed form.

To compute each investor's wealth share, we first consider her time-t wealth satisfying (A.1)

$$\xi_t W_t(\theta) = \mathbb{E}_t\left[\xi_T W_T(\theta)\right] = \frac{K}{\sqrt{2\pi v^2}}e^{-\frac{1}{2}\frac{(\theta-m)^2}{v^2}}\mathbb{E}_t\left[\eta_T(\theta)^{\frac{1}{\gamma}}D_T^{1-\gamma}M_T^{\gamma-1}\right], \quad \text{(A.42)}$$

where the second equality follows by substituting (A.4), (A.6) and (A.8). We also employed the equalities $\alpha_o = m$, $\beta_o^2 = v^2$ (see, proof of Proposition 1) to express (A.42) in terms of model parameters. Using Lemma A2 with $\alpha_t = m_t$, $\beta_t^2 = v_t^2$ substituted in, we have

$$\mathbb{E}_t\left[\eta_T(\theta)^{\frac{1}{\gamma}}D_T^{1-\gamma}M_T^{\gamma-1}\right] = \xi_t S_t K^{-1}\left(\frac{1}{\sqrt{2\pi v^2}}e^{-\frac{1}{2}\frac{(\theta-m)^2}{v^2}}\right)^{-1}\frac{1}{\sqrt{2\pi v_t^2 A_t^2}}e^{-\frac{1}{2}\frac{\left(\theta-\left[m_t-\left(1-\frac{1}{\gamma}\right)v_t^2(T-t)\right]\right)^2}{v_t^2A_t^2}}, \quad \text{(A.43)}$$

where A_t^2 is as defined in (A.60) with $\beta_t^2 = v_t^2$, and m_t and v_t are as in (10). Substituting (A.43) into (A.42) and rearranging gives the wealth-share of each θ-type investor as

$$\frac{W_t(\theta)}{S_t} = \frac{1}{\sqrt{2\pi v_t^2 A_t^2}}e^{-\frac{1}{2}\frac{\left(\theta-\left[m_t-\left(1-\frac{1}{\gamma}\right)v_t^2(T-t)\right]\right)^2}{v_t^2A_t^2}}. \quad \text{(A.44)}$$

We note that the time-t wealth-share distribution is Gaussian with mean and variance as reported in (13) of Remark 1. We then apply Itô's lemma to (A.44) to obtain the dynamics

$$d\frac{W_t(\theta)}{S_t} = \ldots dt + \frac{W_t(\theta)}{S_t}\frac{v_t^2}{\gamma\sigma\tilde{v}_t^2}(\theta-\tilde{m}_t)\,d\omega_t. \quad \text{(A.45)}$$

To obtain each θ-type investor's optimal portfolio $\phi(\theta)$ as a fraction of wealth invested in

the stock, we match the volatility term in (A.45) with the corresponding one in

$$d\frac{W_t(\theta)}{S_t} = \ldots dt + \frac{W_t(\theta)}{S_t}(\phi_t(\theta) - 1)\sigma_{St}d\omega_t,$$

which is obtained by using (1) and (7). This yields investors' equilibrium portfolios as

$$\phi_t(\theta) = 1 + \frac{v_T^2}{\gamma\sigma^2\tilde{v}_t^2}(\theta - \tilde{m}_t). \tag{A.46}$$

We then apply the product rule to $\psi(\theta) = \phi(\theta)(W(\theta)/S)$ using (A.44) and (A.46) to obtain the portfolio dynamics in terms of number of shares invested in the stock $\psi(\theta)$

$$d\psi_t(\theta) = \ldots dt + \frac{W_t(\theta)}{S_t}\frac{v_t^2}{\gamma\sigma\tilde{v}_t^2}\left[(\theta - \tilde{m}_t)\phi_t(\theta) - \frac{v_T^2}{\gamma\sigma^2}\right]d\omega_t,$$

which after substituting (A.46) yields the portfolio volatility of each θ-type investor $\sigma_\psi(\theta)$ as

$$\sigma_{\psi t}(\theta) = \frac{W_t(\theta)}{S_t}\frac{v_t^2}{\gamma\sigma\tilde{v}_t^2}\left[\frac{v_T^2}{\gamma\sigma^2}\left(\left(\frac{\theta - \tilde{m}_t}{\tilde{v}_t}\right)^2 - 1\right) + \tilde{v}_t\left(\frac{\theta - \tilde{m}_t}{\tilde{v}_t}\right)\right]. \tag{A.47}$$

We now compute our trading volume measure (17), obtained by summing the absolute value of investors' portfolio volatilities. To find this absolute value, we need to identify the types for whom the portfolio volatility is negative at time t. From (A.47), this occurs when the square bracket term is negative which is a quadratic in types θ. Therefore at time-t, the types for whom the portfolio volatility $\sigma_{\psi t}(\theta)$ is negative lies between two critical types θ_{c1} and θ_{c2} for which $\sigma_{\psi t}(\theta_{c1}) = \sigma_{\psi t}(\theta_{c2}) = 0$. Solving the quadratic equation yields the critical types as

$$\theta_{c1} = \tilde{m}_t + \tilde{v}_t\left(-\frac{1}{2}X_t - \frac{1}{2}\sqrt{X_t^2 + 4}\right), \qquad \theta_{c2} = \tilde{m}_t + \tilde{v}_t\left(-\frac{1}{2}X_t + \frac{1}{2}\sqrt{X_t^2 + 4}\right), \tag{A.48}$$

where $X_t = \gamma\sigma^2\tilde{v}_t/v_T^2$. From definition (17), the trading volume measure is

$$V_t \equiv \frac{1}{2}\int_\Theta |\sigma_{\psi t}(\theta)|d\theta = \frac{1}{2}\left[\int_{-\infty}^{\theta_{c1}}\sigma_{\psi t}(\theta)d\theta - \int_{\theta_{c1}}^{\theta_{c2}}\sigma_{\psi t}(\theta)d\theta + \int_{\theta_{c2}}^{\infty}\sigma_{\psi t}(\theta)d\theta\right] = -\int_{\theta_{c1}}^{\theta_{c2}}\sigma_{\psi t}(\theta)d\theta, \tag{A.49}$$

where the last equality follows from the fact $\int_\Theta \sigma_{\psi t}(\theta)d\theta = 0$. Substituting (A.44) and (A.47) into (A.49) yields

$$V_t = -\frac{v_t^2}{\gamma\sigma\tilde{v}_t^2}\int_{\theta_{c1}}^{\theta_{c2}}\left[\frac{v_T^2}{\gamma\sigma^2}\left(\left(\frac{\theta - \tilde{m}_t}{\tilde{v}_t}\right)^2 - 1\right) + \tilde{v}_t\left(\frac{\theta - \tilde{m}_t}{\tilde{v}_t}\right)\right]\frac{1}{\sqrt{2\pi\tilde{v}_t^2}}e^{-\frac{1}{2}\frac{(\theta-\tilde{m}_t)^2}{\tilde{v}_t^2}}d\theta.$$

Changing the variable of integration to $z = \frac{\theta - \tilde{m}_t}{\tilde{v}_t}$ and using the facts that

$$\int (z^2 - 1)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz = -z\boldsymbol{\phi}(z) + C, \qquad \text{and} \qquad \int z\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz = -\boldsymbol{\phi}(z) + C,$$

where $\boldsymbol{\phi}(.)$ is the standard normal density function and C is a constant, we obtain the trading

volume measure as

$$V_t = \frac{v_t^2}{\gamma\sigma\tilde{v}_t^2}\left[\frac{v_T^2}{\gamma\sigma^2}\left(\frac{\theta_{c2}-\tilde{m}_t}{\tilde{v}_t}\right)+\tilde{v}_t\right]\boldsymbol{\phi}\left(\frac{\theta_{c2}-\tilde{m}_t}{\tilde{v}_t}\right) - \frac{v_t^2}{\gamma\sigma\tilde{v}_t^2}\left[\frac{v_T^2}{\gamma\sigma^2}\left(\frac{\theta_{c1}-\tilde{m}_t}{\tilde{v}_t}\right)+\tilde{v}_t\right]\boldsymbol{\phi}\left(\frac{\theta_{c1}-\tilde{m}_t}{\tilde{v}_t}\right). \quad \text{(A.50)}$$

Finally, substituting (A.48) into (A.50) and rearranging gives (18).

The condition for property that the trading volume measure is increasing in belief dispersion follows from the partial derivative of (18) with respect to v_t, or equivalently v^2. To compute this partial derivative we rewrite (18) compactly as

$$V_t = \frac{1}{2\sigma}\left[Z_t^+\boldsymbol{\phi}\left(\frac{1}{2}\frac{X_t^2}{\sigma^2}\frac{v_T^2}{v_t^2}Z_t^-\right) + Z_t^-\boldsymbol{\phi}\left(\frac{1}{2}\frac{X_t^2}{\sigma^2}\frac{v_T^2}{v_t^2}Z_t^+\right)\right], \quad \text{(A.51)}$$

where we defined the positive deterministic processes

$$Z_t^+ \equiv \sqrt{\left(\frac{\sigma^2}{X_t^2}\frac{v_t^2}{v_T^2}\right)^2(X_t^2+4)} + \frac{\sigma^2}{X_t}\frac{v_t^2}{v_T^2}, \qquad Z_t^- \equiv \sqrt{\left(\frac{\sigma^2}{X_t^2}\frac{v_t^2}{v_T^2}\right)^2(X_t^2+4)} - \frac{\sigma^2}{X_t}\frac{v_t^2}{v_T^2},$$

with $0 < Z_t^- < Z_t^+$, and

$$\frac{\partial}{\partial v^2}Z_t^+ - \frac{\partial}{\partial v^2}Z_t^- = 2\frac{\partial}{\partial v^2}\left(\frac{\sigma^2}{X_t}\frac{v_t^2}{v_T^2}\right). \quad \text{(A.52)}$$

Substituting

$$\begin{aligned}
\frac{\partial}{\partial v^2}\boldsymbol{\phi}\left(\frac{1}{2}\frac{X_t^2}{\sigma^2}\frac{v_T^2}{v_t^2}Z_t^-\right) &= -\frac{1}{2}\frac{X_t^2}{\sigma^2}\frac{v_T^2}{v_t^2}Z_t^-\boldsymbol{\phi}\left(\frac{1}{2}\frac{X_t^2}{\sigma^2}\frac{v_T^2}{v_t^2}Z_t^-\right)\frac{\partial}{\partial v^2}\left(\frac{1}{2}\frac{X_t^2}{\sigma^2}\frac{v_T^2}{v_t^2}Z_t^-\right), \\
\frac{\partial}{\partial v^2}\boldsymbol{\phi}\left(\frac{1}{2}\frac{X_t^2}{\sigma^2}\frac{v_T^2}{v_t^2}Z_t^+\right) &= -\frac{1}{2}\frac{X_t^2}{\sigma^2}\frac{v_T^2}{v_t^2}Z_t^+\boldsymbol{\phi}\left(\frac{1}{2}\frac{X_t^2}{\sigma^2}\frac{v_T^2}{v_t^2}Z_t^+\right)\frac{\partial}{\partial v^2}\left(\frac{1}{2}\frac{X_t^2}{\sigma^2}\frac{v_T^2}{v_t^2}Z_t^+\right),
\end{aligned}$$

with

$$\begin{aligned}
\frac{\partial}{\partial v^2}\left(\frac{1}{2}\frac{X_t^2}{\sigma^2}\frac{v_T^2}{v_t^2}Z_t^-\right) &= \frac{1}{2}\left[\left(\frac{X_t^2}{\sigma^2}\frac{v_T^2}{v_t^2}\right)\frac{\partial Z_t^-}{\partial v^2} + Z_t^-\frac{\partial}{\partial v^2}\left(\frac{X_t^2}{\sigma^2}\frac{v_T^2}{v_t^2}\right)\right], \\
\frac{\partial}{\partial v^2}\left(\frac{1}{2}\frac{X_t^2}{\sigma^2}\frac{v_T^2}{v_t^2}Z_t^+\right) &= \frac{1}{2}\left[\left(\frac{X_t^2}{\sigma^2}\frac{v_T^2}{v_t^2}\right)\frac{\partial Z_t^+}{\partial v^2} + Z_t^+\frac{\partial}{\partial v^2}\left(\frac{X_t^2}{\sigma^2}\frac{v_T^2}{v_t^2}\right)\right],
\end{aligned}$$

into the partial derivative of (A.51) with respect to v^2, and using (A.52) and the equality

$$Z_t^+Z_t^- = 4\left(\frac{\sigma^2}{X_t^2}\frac{v_t^2}{v_T^2}\right)^2,$$

yields the required partial derivative

$$\begin{aligned}
\frac{\partial}{\partial v^2}V_t &= \frac{1}{2\sigma}\left[2\frac{\partial}{\partial v^2}\left(\frac{\sigma^2}{X_t}\frac{v_t^2}{v_T^2}\right) - Z_t^-\frac{\sigma^2}{X_t^2}\frac{v_t^2}{v_T^2}\frac{\partial}{\partial v^2}\left(\frac{X_t^2}{\sigma^2}\frac{v_T^2}{v_t^2}\right)\right]\boldsymbol{\phi}\left(\frac{1}{2}\frac{X_t^2}{\sigma^2}\frac{v_T^2}{v_t^2}Z_t^-\right) \\
&\quad + \frac{1}{2\sigma}\left[-2\frac{\partial}{\partial v^2}\left(\frac{\sigma^2}{X_t}\frac{v_t^2}{v_T^2}\right) - Z_t^+\frac{\sigma^2}{X_t^2}\frac{v_t^2}{v_T^2}\frac{\partial}{\partial v^2}\left(\frac{X_t^2}{\sigma^2}\frac{v_T^2}{v_t^2}\right)\right]\boldsymbol{\phi}\left(\frac{1}{2}\frac{X_t^2}{\sigma^2}\frac{v_T^2}{v_t^2}Z_t^+\right). \quad \text{(A.53)}
\end{aligned}$$

However, (A.53) is always positive because

$$\frac{\partial}{\partial v^2}\left(\frac{X_t^2}{\sigma^2}\frac{v_T^2}{v_t^2}\right)<0 \qquad \text{and} \qquad \frac{\partial}{\partial v^2}\left(\frac{\sigma^2}{X_t}\frac{v_t^2}{v_T^2}\right)>0,$$

which implies that the first square bracket term in (A.53) is positive, and if the second square bracket terms is also positive then it is easy to see that (A.53) is positive. However, if the second square bracket term is negative then we use the inequality

$$2\frac{\partial}{\partial v^2}\left(\frac{\sigma^2}{X_t}\frac{v_t^2}{v_T^2}\right)-Z_t^-\frac{\sigma^2}{X_t^2}\frac{v_t^2}{v_T^2}\frac{\partial}{\partial v^2}\left(\frac{X_t^2}{\sigma^2}\frac{v_T^2}{v_t^2}\right)>\left|-2\frac{\partial}{\partial v^2}\left(\frac{\sigma^2}{X_t}\frac{v_t^2}{v_T^2}\right)-Z_t^+\frac{\sigma^2}{X_t^2}\frac{v_t^2}{v_T^2}\frac{\partial}{\partial v^2}\left(\frac{X_t^2}{\sigma^2}\frac{v_T^2}{v_t^2}\right)\right|,$$

and the fact that

$$0<\boldsymbol{\phi}\left(\frac{1}{2}\frac{X_t^2}{\sigma^2}\frac{v_T^2}{v_t^2}Z_t^+\right)<\boldsymbol{\phi}\left(\frac{1}{2}\frac{X_t^2}{\sigma^2}\frac{v_T^2}{v_t^2}Z_t^-\right), \tag{A.54}$$

to show that the first line in (A.53) dominates the second line, and therefore (A.53) is positive.

Property that the trading volume measure is positively related to the stock volatility follows from the fact that an increase in belief dispersion leads to both a higher trading volume measure and a stock volatility. □

Lemma A1. *Let the processes M, α and β be defined as in* (A.9)–(A.10)*. Then for all numbers a and b we have*

$$\begin{aligned}\mathbb{E}_t\left[D_T^aM_T^b\right] &= D_t^aM_t^b\left(\frac{\beta_T}{\beta_t}\right)^b e^{a\left(\mu-\frac{1}{2}\sigma^2\right)(T-t)}e^{-\frac{b}{2}\frac{\alpha_t^2}{\beta_t^2}\left(1-\frac{\beta_T^2}{\beta_t^2}\left[1-\frac{b}{\gamma}\left(1-\frac{\beta_T^2}{\beta_t^2}\right)\right]^{-1}\right)}\\ &\quad\times\left[1-\frac{b}{\gamma}\left(1-\frac{\beta_T^2}{\beta_t^2}\right)\right]^{-\frac{1}{2}}e^{\frac{1}{2}\left[1-\frac{b}{\gamma}\left(1-\frac{\beta_T^2}{\beta_t^2}\right)\right]^{-1}\left(\frac{2ab\alpha_t}{\gamma}\frac{\beta_T^2}{\beta_t^2}+a^2\sigma^2\right)(T-t)},\end{aligned} \tag{A.55}$$

provided $1-\frac{b}{\gamma}\left(1-\frac{\beta_T^2}{\beta_t^2}\right)>0$.

Proof of Lemma A1. By (A.9), we have

$$M_T=M_t\left(\frac{\beta_T}{\beta_t}\right)e^{-\frac{1}{2}\frac{\alpha_t^2}{\beta_t^2}+\frac{1}{2}\frac{\alpha_T^2}{\beta_T^2}}, \tag{A.56}$$

and (A.10) gives

$$\frac{\alpha_T^2}{\beta_T^2}=\beta_T^2\left(\frac{\alpha_o^2}{\beta_o^4}+2\frac{\alpha_o}{\beta_o^2}\frac{\omega_T}{\gamma\sigma}+\frac{\omega_T^2}{\gamma^2\sigma^2}\right)=\frac{\alpha_t^2}{\beta_t^2}\frac{\beta_T^2}{\beta_t^2}+2\frac{\alpha_t}{\gamma\sigma}\frac{\beta_T^2}{\beta_t^2}\left(\omega_T-\omega_t\right)+\frac{\beta_T^2}{\gamma^2\sigma^2}\left(\omega_T-\omega_t\right)^2. \tag{A.57}$$

Substituting (A.57) into (A.56) and using the lognormality of D_T, we obtain

$$\mathbb{E}_t\left[D_T^aM_T^b\right]=D_t^ae^{a\left(\mu-\frac{1}{2}\sigma^2\right)(T-t)}M_t^b\left(\frac{\beta_T}{\beta_t}\right)^b e^{-\frac{b}{2}\frac{\alpha_t^2}{\beta_t^2}\left(1-\frac{\beta_T^2}{\beta_t^2}\right)}\mathbb{E}_t\left[e^{\left(\frac{b\alpha_t}{\gamma\sigma}\frac{\beta_T^2}{\beta_t^2}+a\sigma\right)(\omega_T-\omega_t)+\frac{1}{2}\frac{b\beta_T^2}{\gamma^2\sigma^2}(\omega_T-\omega_t)^2}\right]. \tag{A.58}$$

Since conditional on time-t information $\omega_T - \omega_t \sim \mathcal{N}(0, T-t)$, we have

$$\mathbb{E}_t\left[e^{\left(\frac{b\alpha_t}{\gamma\sigma}\frac{\beta_T^2}{\beta_t^2}+a\sigma\right)(\omega_T-\omega_t)+\frac{1}{2}\frac{b\beta_T^2}{\gamma^2\sigma^2}(\omega_T-\omega_t)^2}\right] = \left[1-\frac{b}{\gamma}\left(1-\frac{\beta_T^2}{\beta_t^2}\right)\right]^{-\frac{1}{2}} e^{\frac{1}{2}\left[1-\frac{b}{\gamma}\left(1-\frac{\beta_T^2}{\beta_t^2}\right)\right]^{-1}\left(\frac{b\alpha_t}{\gamma\sigma}\frac{\beta_T^2}{\beta_t^2}+a\sigma\right)^2(T-t)},$$

which after substituting into (A.58) and rearranging gives (A.55). □

Lemma A2. *Let the processes $\eta(\theta)$ and M be as in* (A.7) *and* (A.9)*, respectively. Then*

$$\mathbb{E}_t\left[\eta_T(\theta)^{\frac{1}{\gamma}} D_T^{1-\gamma} M_T^{\gamma-1}\right] = \xi_t S_t K^{-1}\left(\frac{1}{\sqrt{2\pi\beta_o^2}}e^{-\frac{1}{2}\frac{(\theta-\alpha_o)^2}{\beta_o^2}}\right)^{-1}\frac{1}{\sqrt{2\pi\beta_t^2 A_t^2}}e^{-\frac{1}{2}\frac{\left(\theta-\left[\alpha_t-\left(1-\frac{1}{\gamma}\right)\beta_t^2(T-t)\right]\right)^2}{\beta_t^2 A_t^2}}, \tag{A.59}$$

where the positive deterministic process A^2 is given by

$$A_t^2 = \frac{1}{\gamma} + \left(1-\frac{1}{\gamma}\right)\frac{\beta_T^2}{\beta_t^2}. \tag{A.60}$$

Proof of Lemma A2. Using (A.3) and the lognormality of D_T, we have

$$\eta_T(\theta)^{\frac{1}{\gamma}} = e^{-\frac{\theta}{\gamma\sigma^2}\left(\mu-\frac{1}{2}\sigma^2\right)T-\frac{1}{2\gamma}\frac{\theta^2}{\sigma^2}T} D_T^{\frac{\theta}{\gamma\sigma^2}}.$$

Therefore, the required expectation becomes

$$\mathbb{E}_t\left[\eta_T(\theta)^{\frac{1}{\gamma}} D_T^{1-\gamma} M_T^{\gamma-1}\right] = e^{-\frac{\theta}{\gamma\sigma^2}\left(\mu-\frac{1}{2}\sigma^2\right)T-\frac{1}{2\gamma}\frac{\theta^2}{\sigma^2}T}\mathbb{E}_t\left[D_T^{1-\gamma+\frac{\theta}{\gamma\sigma^2}} M_T^{\gamma-1}\right]. \tag{A.61}$$

Letting $a = 1-\gamma+\frac{\theta}{\gamma\sigma^2}$ and $b = \gamma-1$ in Lemma A1 yields

$$\begin{aligned}\mathbb{E}_t\left[D_T^{1-\gamma+\frac{\theta}{\gamma\sigma^2}} M_T^{\gamma-1}\right] &= D_t^{1-\gamma+\frac{\theta}{\gamma\sigma^2}} M_t^{\gamma-1}\frac{1}{A_t}\left(\frac{\beta_T}{\beta_t}\right)^{(\gamma-1)} e^{\left(1-\gamma+\frac{\theta}{\gamma\sigma^2}\right)\left(\mu-\frac{1}{2}\sigma^2\right)(T-t)} e^{-\frac{\gamma-1}{2}\frac{\alpha_t^2}{\beta_t^2}\left(1-\frac{\beta_T^2}{\beta_t^2}\frac{1}{A_t^2}\right)} \\ &\quad\times e^{\frac{\alpha_t}{A_t^2}\left(1-\frac{1}{\gamma}\right)\left(1-\gamma+\frac{\theta}{\gamma\sigma^2}\right)\frac{\beta_T^2}{\beta_t^2}(T-t)} e^{\frac{1}{2}\frac{1}{A_t^2}\left(1-\gamma+\frac{\theta}{\gamma\sigma^2}\right)^2\sigma^2(T-t)},\end{aligned} \tag{A.62}$$

where the deterministic positive process A_t^2 as in (A.60). Substituting the equality

$$D_t^{\frac{\theta}{\gamma\sigma^2}} = e^{\frac{\theta}{\gamma\sigma^2}\left(\mu-\frac{1}{2}\sigma^2\right)t+\theta\left(\frac{\alpha_t}{\beta_t^2}-\frac{\alpha_o}{\beta_o^2}\right)},$$

and (A.62) into (A.61) and rearranging leads to

$$\begin{aligned}\mathbb{E}_t\left[\eta_T(\theta)^{\frac{1}{\gamma}} D_T^{1-\gamma} M_T^{\gamma-1}\right] &= D_t^{1-\gamma} M_t^{\gamma-1}\frac{1}{A_t}\left(\frac{\beta_T}{\beta_t}\right)^{(\gamma-1)} e^{(1-\gamma)\left(\mu-\frac{1}{2}\sigma^2\right)(T-t)} \\ &\quad\times e^{-\frac{1}{2}\frac{\theta^2}{\gamma\sigma^2}T} e^{\theta\left(\frac{\alpha_t}{\beta_t^2}-\frac{\alpha_o}{\beta_o^2}\right)} e^{-\frac{\gamma-1}{2}\frac{\alpha_t^2}{\beta_t^2}\left(1-\frac{\beta_T^2}{\beta_t^2}\frac{1}{A_t^2}\right)} \\ &\quad\times e^{\frac{\alpha_t}{A_t^2}\left(1-\frac{1}{\gamma}\right)\left(1-\gamma+\frac{\theta}{\gamma\sigma^2}\right)\frac{\beta_T^2}{\beta_t^2}(T-t)} e^{\frac{1}{2}\frac{1}{A_t^2}\left(1-\gamma+\frac{\theta}{\gamma\sigma^2}\right)^2\sigma^2(T-t)}.\end{aligned} \tag{A.63}$$

We then substitute

$$\xi_t S_t = \mathbb{E}_t\left[D_T^{1-\gamma} M_T^{\gamma}\right] = D_t^{1-\gamma} M_t^{\gamma} \left(\frac{\beta_T}{\beta_t}\right)^{\gamma-1} e^{(1-\gamma)\left(\mu-\frac{1}{2}\sigma^2\right)(T-t)} e^{(1-\gamma)\alpha_t(T-t)} e^{\frac{1}{2}(1-\gamma)^2\sigma^2\frac{\beta_t^2}{\beta_T^2}(T-t)},$$

and the equalities

$$1-\frac{\beta_T^2}{\beta_t^2}\frac{1}{A_t^2} = \frac{1}{A_t^2}\frac{\beta_T^2}{\gamma^2\sigma^2}(T-t), \qquad \text{and} \qquad \left(1-\frac{1}{\gamma}\right)\frac{\beta_T^2}{\gamma\sigma^2}(T-t) = 1-A_t^2,$$

along with (A.9) into (A.63) to obtain

$$\begin{aligned}\mathbb{E}_t\left[\eta_T(\theta)^{\frac{1}{\gamma}} D_T^{1-\gamma} M_T^{\gamma-1}\right] &= \xi_t S_t K^{-1}\left(\frac{1}{\sqrt{2\pi\beta_o^2}} e^{-\frac{1}{2}\frac{(\theta-\alpha_o)^2}{\beta_o^2}}\right)^{-1} \frac{1}{\sqrt{2\pi\beta_t^2 A_t^2}} \\ &\quad \times e^{-\frac{1}{2}\frac{(\theta-\alpha_o)^2}{\beta_o^2}} e^{\frac{1}{2}\frac{\alpha_o^2}{\beta_o^2}-\frac{1}{2}\frac{\alpha_t^2}{\beta_t^2}} e^{(\gamma-1)\alpha_t(T-t)} e^{-\frac{1}{2}(1-\gamma)^2\sigma^2\frac{\beta_t^2}{\beta_T^2}(T-t)} e^{-\frac{1}{2}\frac{\theta^2}{\gamma\sigma^2}T} e^{\theta\left(\frac{\alpha_t}{\beta_t^2}-\frac{\alpha_o}{\beta_o^2}\right)} \\ &\quad \times e^{-\frac{\gamma-1}{2}\frac{\alpha_t^2}{\beta_t^2}\frac{1}{A_t^2}\frac{\beta_T^2}{\gamma^2\sigma^2}(T-t)} e^{\frac{\alpha_t}{A_t^2}\left(1-\gamma+\frac{\theta}{\gamma\sigma^2}\right)\frac{\gamma\sigma^2}{\beta_t^2}\left(1-A_t^2\right)} e^{\frac{1}{2}\frac{1}{A_t^2}\left(1-\gamma+\frac{\theta}{\gamma\sigma^2}\right)^2\sigma^2(T-t)}.\end{aligned}$$

We note that the last two rows in the above equation is equal to

$$e^{-\frac{1}{2}\frac{\left(\theta-\left[\alpha_t-\left(1-\frac{1}{\gamma}\right)\beta_t^2(T-t)\right]\right)^2}{\beta_t^2 A_t^2}},$$

and so we obtain (A.59). □

B Appendix: Parameter Values

In this Appendix, we determine the parameter values employed in our Figures by matching the model implied belief dispersion moments to the corresponding summary statistics in Yu (2011), who reports for a typical stock, a time-average dispersion value of 3.23%, a standard deviation of 0.38%, minimum of 2.70% and maximum of 4.42% for the sample period of 1981-2005.[29] We take our stock as the typical stock with an average belief dispersion, and match its implied time-average and minimum dispersion values to the corresponding reported ones for the typical stock, 3.23% and 2.70% respectively.[30] Towards this, we begin by deriving the time-average squared dispersion, denoted by $\dot{v}^2$, in terms of its initial dispersion v and the quantity $\gamma\sigma^2/T$ in closed-form as

$$\dot{v}^2 = \frac{1}{T}\int_0^T v_t^2 dt = \frac{1}{T}\int_0^T \frac{v^2\sigma^2}{\sigma^2 + \frac{1}{\gamma}v^2 t}dt = \frac{\gamma\sigma^2}{T}\ln\left(1 + \frac{v^2}{\gamma\sigma^2/T}\right), \tag{B.1}$$

and match it to the reported squared average dispersion, $\dot{v}^2 = (3.23\%)^2$. We then rearrange the dispersion equation in (10) to obtain an expression for the minimum dispersion v_T, again in terms of v an $\gamma\sigma^2/T$ as

$$\frac{1}{v_T^2} = \frac{1}{v^2} + \frac{1}{\gamma\sigma^2/T}. \tag{B.2}$$

Matching $v_T = 2.70\%$ in (B.2) and substituting it into (B.1) and numerically solving the non-linear equation with one unknown yields the relation $\gamma\sigma^2/T = 0.00136285$. Next, we set the relative risk aversion coefficient of investors as $\gamma = 2$ and the horizon $T = 10$ years, and substitute these into the previous relation to back out $\sigma = 8.25\%$.[31] The standard deviation of investor types, $\tilde{v}$, is then obtained by first backing out the initial dispersion v from (B.2) and then using the relation between v and $\tilde{v}$ (11), yielding $\tilde{v} = 3.39\%$. To plot the effects of higher belief dispersion from its average value, the standard deviation of investor types is

[29] We use the summary statistics in Yu (2011) because the dispersion measure he constructs for the typical stock is the value-weighted average of individual stock dispersion levels reported in the I/B/E/S summary database, which in turn is constructed as the standard deviation of analyst forecasts about the long-term earnings growth rates. This construction is closest to our dispersion measure as also discussed in footnote 6 in Section 3.

[30] For simplicity, we obtain the time-average dispersion by taking the square root of the time-average squared dispersion, $(1/T)\int_0^T v_t^2 dt$ rather than the time-average dispersion $(1/T)\int_0^T v_t dt$, which differs from the former due to Jensen's inequality. However, the quantitative differences between these two measures are insignificant in our calibration since the reported belief dispersion is only a few percentage points in the data. On the other hand, using the minimum rather than the maximum dispersion value for calibration ensures that the dispersion values in our model remain within the corresponding reported range in Yu (2011) which is not guaranteed for the alternative choice of maximum dispersion.

[31] Other works in this literature use similar relative risk aversion coefficient values in their calibration, including Buraschi, Trojani, and Vedolin (2013) ($\gamma = 2$), and Dumas, Kurshev, and Uppal (2009), Dumas, Lewis, and Osambela (2017) ($\gamma = 3$). Our choice of the horizon value $T = 10$ is chosen so that we obtain plausible decay rate for the belief dispersion in our model.

Name	Symbol	Value
Mean of investor types	$\tilde{m}$	0
Standard deviation of investor types	$\tilde{v}$	3.39%
Mean growth rate of the expected payoff	μ	14.23%
Uncertainty about the payoff	σ	8.25%
Investors' relative risk aversion coefficient	γ	2
Horizon	T	10
Current time	t	4.37

Table 1: **Parameter values.** This table reports the parameter values used in our main model. The derivation of these values is presented in the text.

set so that the time-average dispersion $\dot{v}$ is one standard deviation higher than its average value, $3.23\% + 0.38\% = 3.61\%$, yielding the value 3.94% for the standard deviation of investor types. The mean of investor types $\tilde{m}$ is taken to be 0 to give equal initial weights to optimistic and pessimistic views on them. We set the true mean growth rate of the expected payoff as the reported average forecast of the long-term earnings-per-share growth rate of 14.23% in Yu (2011).[32] In addition to the model parameters above, we also choose the current time t to evaluate the effects of belief dispersion in our model so that at that time the belief dispersion is equal to its time-average value, $v_t = \dot{v} = 3.23\%$. Backing out t from the dispersion expression in (10) yields $t = 4.37$, which is roughly the mid-point given the horizon value $T = 10$ in our calibration as often used in other works (e.g., Pástor and Veronesi (2012)). This procedure yields the parameter values in Table 1.[33] We would like to highlight that the behavior of our equilibrium quantities as depicted in our Figures is typical and does not vary much with alternative plausible values of parameters.

[32]Even though the value for μ may appear large, our plots do not vary much for smaller alternative values for μ since our key quantities, the average bias and dispersion of beliefs are insensitive to it.

[33]The parameter values for the multiple stocks economy of Section 6 can be determined in a similar fashion. For instance, one can set stock 1 as the typical stock with an average belief dispersion, and stock 2 as an otherwise identical stock but with a (one standard deviation) higher dispersion, as discussed above.