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Source: *The Review of Financial Studies*, 1992, Vol. 5, No. 3 (1992), pp. 387-409
Published by: Oxford University Press. Sponsor: The Society for Financial Studies.
Stable URL: https://www.jstor.org/stable/2962132

Insider Trading in Continuous Time

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The continuous-time version of Kyle's (1985) model of asset pricing with asymmetric information is studied. It is shown that there is a unique equilibrium pricing rule within a certain class. This pricing rule is obtained in closed form for general distributions of the asset value. A particular example is a lognormal distribution, for which the equilibrium price process is a geometric Brownian motion. General trading strategies are allowed. In equilibrium, the informed agent, who is risk neutral, has many optima, but he does not correlate his trades locally with the noise trades nor does he submit discrete orders.

In the Kyle (1985) model of asset pricing with asymmetric information, traders submit order quantities to risk-neutral market makers, who set prices competitively and buy or sell for their own accounts to clear the market. Excluding market makers, traders are of two types: informed or noise traders. There is a single risk-neutral informed trader, who rationally anticipates the effect of his orders on the price. The presence of noise traders makes it impossible for the uninformed to exactly invert the price and infer the informed trader's signal. Thus, markets are semi-

I am grateful to Fischer Black, David P. Brown, Eric Hughson, Pete Kyle, Chris Lamoureux, Hal Pedersen, Avanidhar Subrahmanyam, Jaeyoung Sung, and especially Phil Dybvig for helpful comments and discussion. I also thank the editor, Chester Spatt, and an anonymous referee for comments that led to improvements in the exposition. I am grateful for research support from Batterymarch Financial Management. The article was originally entitled, "Continuous Insider Trading and the Distribution of Asset Prices." Address correspondence to Kerry Back, Olin School of Business, Washington University in St. Louis, St. Louis, MO 63130.

The Review of Financial Studies 1992 Volume 5, number 3, pp. 387–409

strong, but not strong form efficient. This model has been widely used in the study of market microstructure.

The purpose of this article is to formalize and extend the continuous-time version of the Kyle model. As is the case with other models, the continuous-time version of this model is more tractable in some ways than the discrete-time version. For example, I will solve in closed form for the equilibrium pricing rule of market makers, for a general distribution of the asset value. A particular example is a lognormal distribution, for which the equilibrium price process is a geometric Brownian motion. In contrast, the discrete-time version has been solved only when the asset value is elliptically distributed [Foster and Viswanathan (1990)].

An important aspect of the continuous-time model is that the informed trader can infer the flow of noise trades without directly observing them, simply by monitoring prices continuously. In contrast, in a model with a finite number of trading opportunities, it would be advantageous for the informed trader to be able to observe contemporaneous noise trades before submitting his own orders. Rochet and Vila (1991) study a one-shot model in which the informed trader can do this. Their results differ from what one obtains in the one-shot Kyle model, in which conditioning on noise trades is not allowed. For asset value distributions with bounded support, Rochet and Vila show there is a unique equilibrium. This equilibrium satisfies a "no expected trade theorem"—conditional on the total order, the market makers' expectation of the informed order is always zero in equilibrium. In the Kyle model, uniqueness has been established only within the linear class and for elliptical distributions, and, in contrast to the no-expected-trade theorem, the expected informed order is proportional to the total order in equilibrium. The continuous-time model studied here, while it is a limit of the discrete-time Kyle model [see Kyle (1985, sec. 5)], also seems to have some of the flavor of the Rochet–Vila model. This applies to both the assumptions and conclusions. This suggests that the many-period versions of the two models may be very similar. This seems very reasonable. Rochet and Vila interpret conditioning on noise trades as representing limit orders, and limit orders should be less important when there are many trading opportunities.

To make it clear that there is no advantage in continuous time to observing the noise trades directly, I will give the informed agent the option of correlating his trades locally with the noise trades. For example, denoting the noise trades at time t by dZ_t and the informed trader's order by dX_t, we could have $dX_t = -dZ_t$. This can be interpreted as the informed trader being on the floor of the exchange and accepting orders as they arrive, instead of letting them go to the specialist.[1] I will show that it is strictly suboptimal to correlate locally with the noise trades. To do so would require submitting orders that are too large (of order $\sqrt{dt}$) relative to the optimal order size for the informed trader.

The main results of the article are as follows. There is an equilibrium in which the pricing rule of market makers (i) is a smooth, strictly monotone function of the cumulative order, (ii) satisfies a certain finite-variance condition, and (iii) is such that the Bellman equation characterizes the informed trader's optima. This equilibrium, which will be obtained in closed form, is the unique equilibrium satisfying (i)–(iii).[2]

The key observation underlying the result that there is a unique equilibrium in the class considered is that in any such equilibrium the informed trader's expectation of the price change would be zero if he were to refrain from trading. More generally, noise trades alone will not create predictable shifts in the location or slope of the residual supply curve faced by the informed trader. In other words, any equilibrium pricing rule in the class considered must have a certain unbiasedness property. Restricting attention to linear equilibriums in discrete-time models imposes this unbiasedness property exogenously, but such equilibriums will not exist for general distributions of the asset value. However, general distributions can be handled in continuous time. This can be attributed to the fact that pricing is locally linear. One can also view this distinction between discrete- and continuous-time markets in terms of what Kyle calls the "tightness" of the market. In discrete time, the marginal cost of the asset exceeds its price, because the supply curve is upward sloping. So at any point in time a finite optimum can exist even if the price is expected to increase later as a result of noise trades (i.e., even if the pricing rule is "biased"). In contrast, in continuous time, the informed trader can act as a perfectly discriminating monopsonist, moving continuously up or down the residual supply curve (i.e., the market is infinitely tight). Hence, he could exploit predictable shifts in the supply curve.

[1] We have to make some assumption about execution in this case. If by being on the floor, the informed trader can buy at the specialist's bid and sell at the ask, then it would probably be optimal to be on the floor. This is tantamount to letting the informed trader act as a market maker. Black (1990) discusses the value of being able to act as a market maker for an informed trader (even a trader who is informed only about his own past trades). Because prices are revised continuously in this model, there is no real bid or ask. However, the model is such that if $dX_t = -dZ_t$, then transactions occur at "the midpoint of the spread," in a certain sense. Under this assumption, correlating locally with the noise trades is suboptimal.

[2] Jarrow (1990) considers a closely related model and shows by means of examples that market manipulation (arbitrage) will generally be possible when the price depends on the history of a large trader's orders, rather than just on the cumulative order. This suggests that it may not be restrictive to only consider pricing rules that depend on the cumulative order. However, even if this is the case, there still could be equilibriums in which this dependence is not smooth and strictly monotone or (ii) or (iii) are not satisfied.

The unbiasedness property is a generalization of Kyle's (1985, pp. 1328–1329) result that the slope of the residual supply curve (the reciprocal of what Kyle defines as market depth) cannot vary in a deterministic fashion. In the normal distribution model, it is a constant. More generally, it is a martingale.

The unbiasedness property of the pricing rule implies that the order process must appear to market makers to be a martingale—buy and sell orders are equally likely to arrive (this is also true in the linear equilibrium of Kyle's discrete-time model). A simplifying aspect of continuous time is that continuous martingales are completely characterized by their local variances. The local variance of the total (net) order process is the same as the local variance of the noise trades, because the informed trades are of smaller order (dt as compared to $\sqrt{dt}$). This implies that the market makers view the total order process as having the same distribution as the noise trades alone. For example, the cumulative informed and noise trades over the trading period [0, 1], which we will denote by X_1 and Z_1, respectively, are joint normal with zero means, and the beta of Z_1 on X_1 is $-\frac{1}{2}$. Hence, $X_1 + Z_1$ and Z_1 have the same distribution. The total order process is not equal in distribution to the noise trades in Kyle's discrete-time model, because the total order has a larger variance at each time. However, in the linear equilibrium of the normal-distribution model with many trading periods, the distributions of the total and noise trades are approximately equal, because, as Kyle shows, the discrete-time equilibriums converge to the continuous-time equilibrium.

The model is presented in Section 1. The analysis is contained in Section 2, and a brief conclusion can be found in Section 3.

1. The Model

There is to be a public release of information at a known date that will affect the value of an asset. The announcement date is called date 1 and the present date is called date 0. Trading of this asset and a risk-free asset is assumed to occur continuously during the interval [0, 1]. The risk-free rate is taken to be zero. The information is already possessed by a single insider. The information is represented as a signal $\tilde{v}$. We will interpret $\tilde{v}$ as the price at which the asset will trade after the release of information. Alternatively, because of the risk neutrality, one can interpret $\tilde{v}$ as being merely an unbiased signal of this price. The distribution function of $\tilde{v}$ is denoted by F. Assume the support of F is an interval, possibly the whole real line or a half line, and that F is continuous on this interval. Therefore, F^{-1} is well defined on the interval (0, 1). Assume the second moment $\int_{-\infty}^{\infty} v^2 \, dF$ is finite.

In addition to the insider, there are liquidity traders who have random, price-inelastic demands, and market makers who are risk neutral. All orders are market orders and are observed by all market makers.[3] Denote by Z_t the cumulative orders of liquidity traders through time t. The process Z is assumed to be a Brownian motion independent of $\tilde{v}$, which has mean zero and variance σ^2 (per unit of time). Let X_t denote the cumulative orders of the informed trader, and set $Y = X + Z$.

Market makers only observe the process Y, so they cannot distinguish between informed and uninformed trades. We will study equilibriums having the property that the price at any time t depends only on cumulative orders Y_t and not on the history of orders. Therefore, we assume

$$P_t = H(Y_t, t), \tag{1}$$

for some function H. Assume H is twice-continuously differentiable in y, continuous in $t \in [0, 1]$, continuously differentiable in $t \in (0, 1)$, and satisfies $EH(Z_1, 1)^2 < \infty$, and $H(\cdot, t)$ is strictly monotone for each t. Here and throughout the article—unless conditioning on particular information is explicitly indicated—the symbol E denotes expectation taken over Z and $\tilde{v}$.

The monotonicity of H implies that the insider can invert H to compute Y_t at each time t. Hence, before he submits his order at time t, he can be assumed to know $\{Z_s | 0 \leq s < t\}$. Because Z is continuous, this is equivalent to knowing $\{Z_s | 0 \leq s \leq t\}$. Let $\mathbf{F} \equiv \{\mathcal{F}_t | 0 \leq t \leq 1\}$ denote the usual augmentation [see, e.g., Dothan (1990, def. 10.1)] of the increasing family of σ-fields generated by the stochastic process ξ, where $\xi_0 \equiv \tilde{v}$ and $\xi_t \equiv Z_t (\forall t > 0)$. We will require the informed trader's strategy X to be adapted to $\mathbf{F}$, which means that the informed trader knows $\tilde{v}$ at time 0 and observes (infers) Z_t at each time t.

To motivate the formulation of the budget constraint of the informed trader, it is useful to consider a discrete-time model ($t = 1,\dots,T$). Let W denote the agent's wealth and B the investment in the risk-free asset, so $W = B + PX$. In the competitive model, one usually thinks of the price changing from P_{t-1} to P_t, generating the capital gain $X_{t-1}(P_t - P_{t-1})$, and portfolio rebalancing then occurring subject to the intertemporal budget constraint

$$B_t + P_t X_t = B_{t-1} + P_t X_{t-1}.$$

Thus, the change in wealth is

$$W_t - W_{t-1} = X_{t-1}(P_t - P_{t-1}).$$

[3] As will be explained, the informed trader will be allowed to condition his orders on the contemporaneous liquidity orders. Conditioning on liquidity orders is similar to conditioning on price, so this model also has the flavor of a limit order model.

Here we think of the agent submitting a market order $X_t - X_{t-1}$ and then the price changing from P_{t-1} to P_t. The order is executed at the price P_t. Execution of the order causes the investment in the risk-free asset to change by

$$B_t - B_{t-1} = -P_t(X_t - X_{t-1}),$$

so the change in wealth is

$$\begin{aligned} B_t - B_{t-1} + P_tX_t - P_{t-1}X_{t-1} &= -P_t(X_t - X_{t-1}) + P_tX_t - P_{t-1}X_{t-1} \\ &= X_{t-1}(P_t - P_{t-1}). \end{aligned} \tag{2}$$

The upshot is that the formula for wealth dynamics is the same in the market-order model as in the usual model. Notice that it does not matter whether the agent views his market order as causing the change in price. The budget constraint does not depend on whether the agent acts as a monopsonist or as a perfect competitor.[4]

Extending this to continuous time, it is clear that we should use the Merton (1969) and Harrison and Pliska (1981) formula for the wealth dynamics of the informed trader. The only deviation from the Harrison–Pliska model that we will make here is that we will allow only a subset of the trading strategies that Harrison and Pliska allow. The motivation is that we want to think of the stochastic differential $dX(t)$ of the order process as the market order. Formulas involving stochastic differentials are meaningful only in integrated form, and stochastic integrals can be defined with X as an integrator only if X is a semimartingale. So we will require the order strategy X to be a semimartingale and define the wealth dynamics by

$$dW_t = X_{t-}\, dP_t. \tag{3}$$

Here, and throughout the article, the symbol X_{t-} denotes the left limit $\lim_{s\uparrow t} X_s$. The use of the left limit is an obvious extension of (2). Given the smoothness assumptions on H, the process $P_t = H(Y_t, t)$ will also be a semimartingale [by Itô's formula; see Dellacherie and Meyer (1982, VIII.25)]. Moreover, the left-continuous process X_{t-} is predictable and locally bounded, so the stochastic integral $\int X_{t-}\, dP_t$ exists. Harrison and Pliska allow as a trading strategy [i.e., as an integrand in (3)] any predictable process for which the stochastic integral exists. Thus, the class of strategies allowed here is a subclass of the class allowed by Harrison and Pliska.

The requirement that X be a semimartingale means that it is right continuous, the left limits X_{t-} exist, and it can be written as $X = D - S + M$, where D and S are positive, increasing, right-continuous

[4] Another point worthy of note is that being able to anticipate the price change before submitting a market order does not necessarily lead to arbitrage, because the capital gain $X_{t-1}(P_t - P_{t-1})$ is still calculated on the basis of X_{t-1}, which is the position held before the price change.

processes and M is a local martingale.[5] The right-continuity is a normalization, which means that we are taking X_t to include any jump $\Delta X_t \equiv X_t - X_{t-}$ made at time t. Jumps can be interpreted as discrete orders. We do not expect to see any discrete orders in equilibrium, because they can be identified as coming from the informed trader. Pricing rules of the form (1) respond to discrete orders in exactly the same way as if they had been submitted in infinitesimal pieces. Given this pricing, we will see that discrete orders are indeed suboptimal.[6] When $M = 0$, as will be true in equilibrium, the process $X = D - S$ is naturally interpreted as the difference of purchases and sales.

We are allowing for the possibility that there will be a jump in the price at time 1 after the announcement is made. Including the capital gain from such a jump, the formula (3) implies the final wealth of the informed trader is

$$W_1 = (\tilde{v} - P_1)X_1 + \int_{[0,1]} X_{t-}\, dP_t. \tag{4}$$

Here we are, without loss of generality, taking $W_0 = 0$. It is necessary to explicitly include the endpoints in the region of integration, because of the possibility of jumps.[7]

The technical importance of the assumption that X is a semimartingale is that it allows us to integrate by parts to reformulate the wealth equation (4). This yields a formula that is a direct generalization of Kyle's. Specifically, (4) is equivalent to

$$W_1 = \int_{[0,1]} (\tilde{v} - P_{t-})\, dX_t - [P, X]_1, \tag{5}$$

where $[P, X]$ is the "optional quadratic variation" process [Dellacherie and Meyer (1982, VIII.18)]. The differential of this process corresponds to what one usually writes as $dP\,dX$.[8] The formula (5) is the one we will use in the remainder of the article.

[5] One is free to assume M is a martingale, because a martingale is a special case of a local martingale.

[6] Three points are worthy of note here. First, our equilibrium concept will not constrain off-equilibrium beliefs, so we are free to specify beliefs and hence prices following discrete orders in any way we desire. Second, we are not claiming that (1) is the only possible rational pricing rule that will discourage discrete orders. Third, there are pricing rules that will encourage discrete orders, which in fact will lead to arbitrage. For example, suppose the price change ΔP in response to a discrete order ΔX is of the form $\Delta P/P = \lambda \Delta X$, for a constant λ. To see how this can be manipulated, suppose $\lambda = .01$, the true value is \$20, and the price is \$10. Selling 10 shares will cause the price to be reduced to \$9 and generate \$90 in revenue. Immediately buying 100/9 shares will move the price back to \$10 and cost \$1000/9. The net cost of this pair of trades is \$190/9. However, the net trade is +10/9 shares, each of which is worth \$20. Thus, the net profit is \$10/9. This can be repeated indefinitely, leading to infinite profits.

[7] Without loss of generality, we take the intitial position X_{0-} to be zero. We also adopt the convention that P_{0-} equals the unconditional expectation of $\tilde{v}$.

[8] See Dothan (1990, especially chap. 5) for the definition of this process and for an explanation of this point.

To see the connection between (5) and Kyle's formula, suppose first that X has differentiable paths. This will be true in equilibrium. In this case, $[P, X] \equiv 0$. Moreover, P will also be continuous, by virtue of (1) and the continuity of H. Hence, (5) specializes to

$$W_1 = \int_0^1 (\tilde{v} - P_t)\, dX_t, \tag{6}$$

which is Kyle's formula. This can be interpreted as the value of the final position ($\tilde{v}X_1$) less the cost of acquiring it ($\int_0^1 P_t\, dX_t$). This formula for the cost of acquiring the position is analogous to the usual formula for the cost of a perfectly discriminating monopsonist. Kyle motivates it by stating that, since the market order is executed at the post-order price, the cost of the order is $(P_t + dP_t)\, dX_t$. Kyle notes that, by the usual rules for multiplication of stochastic differentials, $dP\, dX$ is zero when X is differentiable, which yields (6) as the formula for final wealth. More generally, it seems that we should interpret the cost of a market order as $(P_{t-} + dP_t)\, dX_t$ (i.e., as $P_{t-}\, dX_t + d[P, X]_t$). This definition agrees with (5).

We will show that, in equilibrium,

$$dP_t = H_y(Y_t, t)\, dY_t,$$

where the subscript denotes the partial derivative. Thus, price changes are locally proportional to order sizes. The transaction price $P_{t-} + dP_t$ therefore depends on the size and magnitude of the order dY_t and is centered on P_{t-}. If the informed traders' order exactly offsets the noise trades (i.e., if $dX_t = -dZ_t$) then dP_t will be zero, so the transaction price will be P_{t-}, the "midpoint of the spread." In this sense, the informed trader and noise traders receive equally good execution.

A technical problem is that, as in competitive models [see Harrison and Kreps (1979)], we must exclude doubling strategies by the informed trader, or else the model will be degenerate. A doubling strategy in this model is to repeatedly double the following bet until it is won: buy the asset in the hope that noise traders will subsequently buy it and drive the price up. It is sufficient to require that

$$E \int_0^1 H(X_{t-} + Z_t, t)^2\, dt < \infty. \tag{7}$$

The constraint (7) is related to the integrability condition used to rule out doubling strategies in the competitive model [see Duffie and Huang (1985) or Dybvig and Huang (1988)].

The role played by the constraint (7) is to guarantee that the process $\int_0^t P_{s-}\, dZ_s$ is a martingale. To understand this, it is useful to consider a model with discrete trades, like Glosten and Milgrom (1985). In that model, market makers make money from noise traders because of the bid–ask spread. This is compensation for their expected losses to informed traders. The only reason noise traders lose money on average is that they are always trading on the wrong side of the spread. The requirement here that $\int_0^t P_{s-}\, dZ_s$ be a martingale means that noise traders would not lose money on average if they could always trade at the midpoint of the spread.[9] This is obviously a feature we want the model to have.

We will implement (7) by imposing separate constraints on the pricing rule H and order strategy X. As with the other constraints imposed on H, this one will not be binding in equilibrium, but it does limit the scope of the uniqueness of equilibrium result. All the assumptions on H are captured in the following definition: let $\mathcal{H}$ denote the class of continuous functions $H : \Re \times [0, 1] \to \Re$ that are twice-continuously differentiable in y and continuously differentiable in t on $\Re \times (0, 1)$ and for which $H(\cdot, t)$ is strictly monotone for each $t \in [0, 1]$ and

$$EH(Z_1, 1)^2 < \infty \qquad \text{and} \qquad E\int_0^1 H(Z_t, t)^2\, dt < \infty. \tag{8}$$

A *pricing rule* is an element of $\mathcal{H}$. Let $\mathcal{X}$ denote the class of semimartingales X adapted to $\mathbf{F}$ such that

$$(\forall H \in \mathcal{H}) \qquad E\int_0^1 H(X_{t-} + Z_t, t)^2\, dt < \infty. \tag{9}$$

A *trading strategy* is an element of $\mathcal{X}$. The continuity of each $H \in \mathcal{H}$ implies that (9) leaves the density function of $X_{t-} + Z_t$ on any bounded set completely unrestricted (the distribution can even have mass points). A sufficient condition for X to satisfy (9) is that the ratio of the density function of $X_{t-} + Z_t$ to the density function of Z_t be bounded uniformly in t on $(-\infty, -n) \cup (n, \infty)$, for some n.

Given a trading strategy X, a pricing rule is *rational* if it satisfies

$$H(Y_t, t) = E[\tilde{v} \mid (Y_s)_{s \le t}]. \tag{10}$$

Given a pricing rule H, a trading strategy is *optimal* if it maximizes

$$E\left\{\int_{[0,1]} (\tilde{v} - P_{t-})\, dX_t - [P, X]_1\right\} \tag{11}$$

[9] Of course, they cannot in general trade at the midpoint of the spread. In accordance with the definition of insider profits, the losses of noise traders are defined as $\int_{[0,1]} (P_{t-} - \tilde{v})\, dZ_t + [P, Z]_1$. Given (7), the expected losses are $E\{[P, Z]_1\}$. The variable $[P, Z]_1$ is interpreted as the sum over time of the "bid–ask spread costs" $dP_t\, dZ_t$. It will be positive. By Itô's formula, $[P, Z]_1 = \sigma^2 \int_0^1 H_y(Y_{t-}, t)\, dt$, and the partial derivative H_y is strictly positive by assumption.

on $\mathcal{X}$. An *equilibrium* is a pair (H, X) such that H is a rational pricing rule, given X, and X is an optimal trading strategy, given H. If (H, X) is an equilibrium for any trading strategy X, then H is an *equilibrium pricing rule.*

2. Equilibrium

The purpose of this section is to explain and prove the following theorems. Recall that F denotes the distribution function of $\tilde{v}$. Let V denote the support of $\tilde{v}$ excluding the endpoints, if there are any (the endpoints have zero probability because of the continuity of F). Let N denote the normal $(0, \sigma^2)$ distribution function.

Theorem 1. *Define*

$$H(y, t) = Eh(y + Z_1 - Z_t), \tag{12}$$

where $h = F^{-1} \circ N$. For each $v \in V$, define

$$X_t = (1 - t) \int_0^t \frac{h^{-1}(v) - Z_s}{(1 - s)^2}\, ds; \tag{13}$$

then (H, X) is an equilibrium.

Theorem 2. *The pricing rule (12) is the unique equilibrium pricing rule H for which there exists a nonnegative, smooth function $J(v, y, t)$ on $V \times \Re \times [0, 1]$ satisfying the Bellman equation*

$$\max_{\theta \in \Re} \left\{ J_t + J_y\, \theta + \frac{1}{2}\sigma^2 J_{yy} + (v - H)\theta \right\} = 0 \quad \text{on } V \times \Re \times (0, 1), \tag{14}$$

and boundary condition

$$J(v, y, 1) > J(v, h^{-1}(v), 1) = 0 \qquad [\forall v \in V, \quad \forall y \neq h^{-1}(v)], \tag{15}$$

where $h(\cdot) \equiv H(\cdot, 1)$.

Theorem 3. *Let (H, X) be an equilibrium. Suppose H is such that there exists a smooth solution J to the Bellman equation (14) and boundary condition (15). Then*

$$dP_t = H_y(Y_t, t)\, dY_t, \tag{16}$$

and the process Y is distributed as a Brownian motion with zero drift and variance σ^2, given the market makers' information (i.e., on the filtration generated by Y). The process $H(Z_t, t)$ is a martingale given *the informed trader's information (i.e., on the filtration **F**). If F has a density function and $EH_y(Z_1, 1) < \infty$, then the process $H_y(Z_t, t)$ is a martingale given the informed trader's information, and the process $H_y(Y_t, t)$ is a martingale given the market makers' information.*

Before explaining the Bellman equation and boundary condition, we will consider two examples.

Example 1. Assume $\tilde{v}$ is normally distributed with mean α and variance ϕ^2. Set $h(y) = F^{-1}(N(y))$. Since $F(h(y)) = N^*((h(y) - \alpha)/\phi)$ and $N(y) = N^*(y/\sigma)$, where N^* denotes the standard normal distribution function, the definition $F(h(y)) = N(y)$ implies $h(y) = \alpha + \phi y/\sigma$. Thus, $H(y, t) = \alpha + \lambda y$, $\forall(y, t)$, where $\lambda \equiv \phi/\sigma$. From Theorem 3, we see that $dP_t = \lambda\, dY_t$, and Y is viewed as a Brownian motion by the market makers. Thus, P is a Brownian motion given the market makers' information, or equivalently, given public information (the filtration generated by P). This is the equilibrium pricing rule described by Kyle. The insider's strategy in Theorem 1 is

$$dX_t = \frac{h^{-1}(\tilde{v}) - Y_t}{1 - t}\, dt = \frac{\tilde{v} - P_t}{\lambda(1 - t)}\, dt,$$

which is also as in Kyle.

Example 2. Assume $\log \tilde{v}$ is normally distributed with mean α and variance ϕ^2. Set $h(y) = F^{-1}(N(y))$. Now we have $F(h(y)) = N^*((\log h(y) - \alpha)/\phi)$, so $h(y) = \exp(\alpha + \lambda y)$, where as before $\lambda = \phi/\sigma$. Thus, $H(y, t) = \exp(\alpha + \lambda y + \phi^2(1 - t)/2)$. If the insider were to place a discrete order ΔX_t, then the jump in price would be $\Delta P_t = P_{t-}(\exp(\lambda\Delta X_t) - 1)$. However, in equilibrium, there are no discrete orders and no martingale component in the insider's strategy. We have $H_y(Y_t, t) = \lambda P_t$, so from Theorem 3,

$$\frac{dP_t}{P_t} = \lambda\, dY_t.$$

Because Y is a Brownian motion given the market makers' information, P is a geometric Brownian motion. Notice that λ is the sensitivity of the price to orders measured in dollar terms (i.e., as $P_t\, dY_t$). According to Kyle's definition, $1/\lambda P_t$ is the market depth, but it would also be reasonable to interpret $1/\lambda$ as the depth. The price-response coefficient $H_y(Y_t, t) = \lambda P_t$ is a martingale given the market makers' information. If the informed trader refrains from trading (unbeknownst to the market makers, of course), then the price-response coefficient will evolve as the martingale $\lambda H(Z_t, t)$. It may be interesting to compare the equilibrium pricing rule here to the example of arbitrage in note 6.

Now we turn to the Bellman equation and boundary condition. The meaning of the term "smooth" in this article for a function J on $V \times \Re \times [0, 1]$ will be that, $\forall v \in V$, $J(v, \cdot)$ and $J_y(v, \cdot)$ are continuous on $\Re \times (0, 1]$, and $J_{yy}(v, \cdot)$ and $J_t(v, \cdot)$ are continuous on $\Re \times (0, 1)$. The subscripts here and in (14) denote partial derivatives. The arguments of J and H have been omitted in (14) for convenience.

The formulation of the Bellman equation is based on an assumption that the order rate $\theta_t \equiv dX_t/dt$ exists, but it is also useful in general. When the order rate exists, the objective function (11) specializes to

$$E \int_0^1 [\tilde{v} - H(Y_t, t)]\theta_t \, dt.$$

Given the path independence of the pricing rule, it is natural to use $\tilde{v}$ and Y_t as the state variables for the informed trader's optimization problem. The dynamics of Y when the order rate exists are given by

$$dY_t = \theta_t \, dt + dZ_t.$$

Hence, by Itô's formula,

$$\begin{aligned} dJ(\tilde{v}, Y_t, t) = {} & [J_t(\tilde{v}, Y_t, t) + J_y(\tilde{v}, Y_t, t)\theta_t + \tfrac{1}{2}\sigma^2 J_{yy}(\tilde{v}, Y_t, t)] \, dt \\ & + J_y(\tilde{v}, Y_t, t) \, dZ_t. \end{aligned}$$

The Bellman equation is the statement that the instantaneous profit

$$[\tilde{v} - H(Y_t, t)]\theta_t \, dt$$

is exactly offset by the expected change in J when an "optimal" policy is followed, and the instantaneous profit is not sufficient to offset the expected change in J when a "suboptimal" policy is followed. The terms "optimal" and "suboptimal" are used here only to indicate whether the maximum in (14) is attained. We are not claiming at this point that an optimal trading strategy as defined in the previous section necessarily attains the maximum in (14). Nor are we claiming that the actual value function solves (14) and (15). We are simply viewing (14) and (15) as a functional equation to be solved for some function J.

The boundary condition (15) is a little unusual in that one might expect J to be identically zero at time 1. The interpretation of (15) is that J is defined by continuity at time 1, and the remaining value $J(\tilde{v}, Y_t, t)$ at times t close to 1 is near zero if and only if Y_t is close to $h^{-1}(\tilde{v})$ (i.e., P_t is close to $\tilde{v}$).

The outline of the proofs is as follows. With Lemma 1, I will construct a solution of the Bellman equation and boundary condition for a class of pricing rules that includes (12). In Lemma 2, I will characterize the optima for the informed trader given a pricing rule in this class. It is essentially a "verification theorem," showing that the optima are characterized by the Bellman equation. In Lemma 3, I characterize the distribution of Y, given the strategy (13) for the informed trader. It follows from Lemmas 2 and 3 that (13) is an optimal strategy, when pricing is based on (12). Theorem 1 is then proven by showing that the function H defined in (12) belongs to $\mathcal{H}$ and is rational.

Lemma 1 shows that, for the pricing rule (12), there is a solution to the Bellman equation and boundary condition. For the uniqueness part of Theorem 2 and in Theorem 3, we assume there is a solution to the Bellman equation and boundary condition. Under this assumption, Lemma 4 shows that the price would be a martingale if the informed agent refrained from trading, and Lemma 5 shows that the order process must be a martingale in equilibrium. Theorems 2 and 3 follow directly.

The key to understanding the Bellman equation is to observe that the maximand is linear in the choice variable θ. Because the choice variable is unconstrained (i.e., one can buy or sell at arbitrary rates), there can be a finite maximum only if its coefficient is zero. Setting this coefficient equal to zero, and then setting the sum of the remaining terms equal to zero, gives the following:

$$J_y(v, y, t) = H(y, t) - v \qquad (\forall (v, y, t) \in V \times \Re \times (0, 1]), \tag{17}$$

$$J_t(v, y, t) + \tfrac{1}{2}\sigma^2 J_{yy}(v, y, t) = 0 \qquad [\forall (v, y, t) \in V \times \Re \times (0, 1)]. \tag{18}$$

These relations follow directly from (14) only for $t \in (0, 1)$, but we can include the endpoint $t = 1$ in (17) because of the continuity of J_y and H. If J is nonnegative and smooth and satisfies (18), then

$$J(v, y, t) = E[J(v, y + Z_s - Z_t, s)] \qquad [\forall (v, y) \in V \times \Re, \quad \forall 0 < t < s \leq 1], \tag{19}$$

where we are taking the expectation over Z, regarding v as a constant (see Karatzas and Shreve [1987, theorem 4.3.6, exercise 3.8 (ii)]). The evident interpretation of (19) is that the maximum value attainable at time t can actually be attained by not trading at all until some later time s (at which time we will have $Y_s = Y_t + Z_s - Z_t$) and then trading optimally from time s on. While all of this is very conjectural at this point, because we have not yet established a connection between the Bellman equation and the value function, it does motivate the construction in Lemma 1. In this lemma, we try to calculate the value by waiting until the "last instant" and then trading. The profit from this limit strategy is calculated by moving up or down the residual supply curve at time 1 to the point $p = v$.

Lemma 1. *Let h be a strictly monotone function that satisfies $E|h(Z_1)| < \infty$. Suppose the pricing rule is*

$$H(y, t) = Eh(y + Z_1 - Z_t). \tag{20}$$

Define

$$j(v, y) = \int_y^{h^{-1}(v)} (v - h(x))\ dx, \tag{21}$$

and

$$J(v, y, t) = E[j(v, y + Z_1 - Z_t)], \tag{22}$$

where we are taking the expectation over Z, regarding v as a constant. Assume $J(v, 0, 0) < \infty$ ($\forall v \in V$). Then J is a smooth solution of (14) and (15).

Proof. See the Appendix. ■

Now we will show that any solution of the Bellman equation and boundary condition is necessarily the value function for the informed trader's optimization problem. This yields a characterization of the informed trader's optima.

Lemma 2. *Let H be an arbitrary pricing rule. Suppose a nonnegative, smooth solution J of (14) and (15) exists. Then for any trading strategy X, the expected profit (11) is no larger than $EJ(\tilde{v}, 0, 0)$. Any trading strategy $X \equiv D - S + M$ which has continuous paths, for which $M \equiv 0$, and which implies $H(Y_1, 1) = \tilde{v}$ almost surely (a.s.), gives an expected profit equal to $EJ(\tilde{v}, 0, 0)$ and is therefore an optimal strategy. If X is any trading strategy that includes discrete orders, or has a nonzero local martingale part, or does not imply $H(Y_1, 1) = \tilde{v}$ a.s., then the expected profit from X is strictly less than $EJ(\tilde{v}, 0, 0)$.*

Proof. See the Appendix. ■

So the necessary and sufficient conditions for optimality, when the Bellman equation and boundary condition can be solved, are that there be no discrete orders (which impose too much price pressure), no local correlation with the noise trades (again, because this involves trades that are too large[10]) and no jump in the price following the announcement. If the market has not fully incorporated the information prior to the announcement [i.e., if $P_1 \equiv H(Y_1, 1) \neq \tilde{v}$], then

[10] Similarly, it is suboptimal to include a martingale component that is uncorrelated with the noise trades. The arguments are exactly the same when this type of strategy is allowed. It is the infinite-variation property of continuous-time martingales that leads to nonzero "bid–ask spread costs" $dP\,dX$ and renders them undesirable.

it is clear that profitable trades were forgone by the informed trader, which is inconsistent with equilibrium.

The key remaining step in demonstrating the optimality of the trading strategy (13) is to show that there will be no market response to the announcement when the strategy is followed. This is implied by the following result, which is also instrumental for demonstrating the rationality of (12).

Lemma 3. *Assume the informed trader follows the strategy (13), where h is defined in Theorem 1. Then, on the filtration $\mathbf{F}$, the process Y is a Brownian bridge with instantaneous variance σ^2, terminating at $h^{-1}(\tilde{v})$. On the filtration generated by Y, the process Y is a Brownian motion with zero drift and instantaneous variance σ^2.*

Proof. Note that

$$dX_t = \frac{h^{-1}(\tilde{v}) - Z_t - X_t}{1 - t}\,dt. \tag{23}$$

Thus, the sum of the informed and noise trades is

$$dY_t = \frac{h^{-1}(\tilde{v}) - Y_t}{1 - t}\,dt + dZ_t.$$

It follows that, on $\mathbf{F}$, Y is a Brownian bridge, with variance σ^2, beginning at 0 and ending at $h^{-1}(\tilde{v})$ [Karatzas and Shreve (1987, p. 358)].

The finite-dimensional distributions of a Brownian bridge are the same as a Brownian motion conditional on the terminal value being known [Karatzas and Shreve (1987, problem 5.6.11)]. The terminal value here is the random variable $h^{-1}(\tilde{v}) \equiv N^{-1}(F(\tilde{v}))$, which is normally distributed with mean zero and variance σ^2 and is independent of Z. Hence, the finite-dimensional distributions of Y, unconditional on $\tilde{v}$ or Z, are the finite-dimensional distributions of a Brownian motion. ■

Proof of Theorem 1. To demonstrate the rationality of the pricing rule (12), given the trading strategy (13), we will explicitly indicate the conditional expectation at time t given the market makers' information (the filtration generated by Y) by $E^M[\cdot]$ and the conditional expectation given the informed trader's information (the filtration $\mathbf{F}$) by $E^I[\cdot]$. We can write the definition (12) as

$$H(y, t) = E^I[H(Z_1, 1) \mid Z_t = y],$$

where $H(\cdot, 1) = F^{-1}(N(\cdot))$. Lemma 3 shows that the distribution of Z with respect to the informed trader's information is the same as the distribution of Y with respect to the market makers' information.

Hence,

$$H(y, t) = E^M[H(Y_1, 1) \mid Y_t = y] = E^M[H(Y_1, 1) \mid (Y_s)_{s \le t}],$$

where $(Y_s)_{s \le t}$ denotes any history with $Y_t = y$. We are using the Markov property of a Brownian motion here. Lemma 3 also establishes that $H(Y_1, 1) = \tilde{v}$ a.s. Making this substitution, the above reduces to the definition of rationality.

It remains only to verify the regularity conditions used in Lemma 1 and in the definitions of $\mathcal{H}$ and $\mathcal{X}$. The fact that $X \in \mathcal{X}$ follows from the fact that the unconditional distribution of $X_t + Z_t$ is the same as that of Z_t. The smoothness and strict monotonicity of H follow directly from the definition (12) [for the smoothness, see Karatzas and Shreve (1987, problem 4.3.1)]. The process $H(Z_t, t)$ is a martingale and $H(Z_1, 1)$ has the same distribution as $\tilde{v}$ by the definition of h. Hence,

$$EH(Z_t, t)^2 \le EH(Z_1, 1)^2 = E\tilde{v}^2 < \infty,$$

which implies that $H \in \mathcal{H}$. This in turn implies the assumption on h used in Lemma 1. ■

The first step in proving Theorems 2 and 3 is to derive the unbiasedness property mentioned in the introduction (i.e., to show that the informed trader's expected price change is zero when he does not trade). This property is consistent with the interpretation we gave for (19), because the existence of a predictable component to the price change during an interval $[t, s]$ when the informed trader did not trade would render it strictly optimal to trade during that interval. The unbiasedness property leads directly to the result that price changes are locally proportional to order sizes in equilibrium (i.e., $dH = H_y\, dY$).

Lemma 4. *Let H be an arbitrary pricing rule. Assume there exists a smooth solution J to (14) and (15). Then the process $H(Z_t, t)$ is a martingale on the filtration **F**. If $X = D - S + M$ is any trading strategy that has continuous paths and for which $M \equiv 0$, then, for all t,*

$$H(Y_t, t) = H(0, 0) + \int_0^t H_y(Y_s, s)\, dY_s. \tag{24}$$

Proof. The martingale property follows from the martingale property of J [i.e., (19)], after differentiating (19) and using (17). The technical details will be supplied in the Appendix.

Assuming $H(Z_t, t)$ is a martingale, H must satisfy the partial differential equation

$$H_t + \tfrac{1}{2}\sigma^2 H_{yy} = 0$$

on $\Re \times (0, 1)$ [Karatzas and Shreve (1987, p. 254)]. Now applying Itô's formula to $H(Y_t, t)$ and making this substitution yield (24). We have used here the fact that the quadratic variation process of Y is the same as that of Z, namely $\sigma^2 t$. ■

Lemma 5. *Let (H, X) be an equilibrium. Assume there exists a smooth solution J to (14) and (15). Then, on the filtration it generates, the process Y must be a Brownian motion with zero drift and variance σ^2.*

Proof. The formula (24) implies

$$Y_t = Y_0 + \int_0^t \frac{1}{H_y(Y_s, s)}\, dH(Y_s, s).$$

Because $H(Y_t, t)$ is a martingale on the filtration generated by Y and $H_y(Y_t, t)$ is strictly positive, with continuous paths, the process Y must be a local martingale on this filtration. The quadratic variation process of Y is $\sigma^2 t$. Any continuous local martingale with this quadratic variation process is a Brownian motion, by Lévy's theorem [Karatzas and Shreve (1987, theorem 3.16)]. ■

Proof of Theorem 2. We have shown in Lemma 1 that there exists a solution to the Bellman equation and boundary condition when the equilibrium pricing rule (12) is used. For the uniqueness, suppose H is any equilibrium pricing rule for which there exists a solution to the Bellman equation and boundary condition. The martingale property of $H(Z_t, t)$ established in Lemma 4 implies that $H(y, t) = E[h(y + Z_1 - Z_t)]$, where $h(\cdot) = H(\cdot, 1)$. We have from Lemma 2 that, in equilibrium, $h(Y_1) = \tilde{v}$ a.s., so $Y_1 = h^{-1}(\tilde{v})$ a.s. Hence, for any scalar a, the probability, given the market makers' information at time 0, that $Y_1 \leq a$ is $F(h(a))$. According to Lemma 5, the distribution function of Y_1, given the market makers' information at time 0, is N. Therefore $N = F \circ h$, implying $h = F^{-1} \circ N$. ■

Proof of Theorem 3. Lemmas 4 and 5 contain everything except for the price-response coefficient being a martingale. The process $H(Y_t, t)$ being a martingale on the filtration generated by Y is equivalent to the process $H_y(Z_t, t)$ being a martingale on $\mathbf{F}$, given the equality of the distributions of Y and Z on these respective filtrations. These martingale properties follow from the martingale property of $H(Z_t,$

t), written as

$$H(y, t) = EH(y + Z_1 - Z_t, 1). \tag{25}$$

All we need to do is to differentiate both sides with respect to y, differentiating under the expectation operator on the right-hand side. The proof that this interchange of differentiation and expectation is possible is deferred to the Appendix. ■

3. Conclusion

The key aspect of the continuous-time model is that the informed trader can move continuously up or down the residual supply curve. This flexibility on the part of the insider, combined with risk neutrality, helps to pin down the equilibrium beliefs of market makers. In equilibrium, the insider has many optima, because there is no expected cost in moving up and then back down the supply curve, or vice versa, or simply delaying trading. This reflects the infinite tightness of the market and the fact that noise trades do not shift the residual supply curve in predictable ways. The uniqueness of equilibrium and the multiplicity of optima in equilibrium is analogous to the competitive model. In a competitive equilibrium with a risk-neutral agent and a fixed risk-free rate, expected returns on all assets are uniquely determined, but any portfolio is optimal for the risk-neutral agent. The situation is very different when agents are risk averse. It is important to determine to what extent the results of this article are robust to risk aversion.

The model was solved in this article without recourse to the filtering technology used by Kyle. This permitted the analysis of general asset value distributions. The solution method is extended in Back (1992a) to study the effect of asymmetric information in options markets and in Back (1992b) to study the effect of time-varying noise trading. Hopefully, it will also prove useful for extending the model in other ways.

Appendix

Proof of Lemma 1

We will fix a $v \in V$ and omit writing it as an argument of j and J. Obviously, $J(\cdot, 1) = j(\cdot)$ is continuous, nonnegative, and satisfies the boundary condition (15). The function J is twice continuously differentiable in y and continuously differentiable in t on $\Re \times (0, 1)$ and satisfies (18) by Karatzas and Shreve (1987, p. 254). We want to show that the derivative of the right-hand side of (22) can be taken under the expectation operator. This is true if for each y, there exists an $\epsilon > 0$ such that the family of random variables

$$\{[j_y(y' + Z_1 - Z_t)|: |y' - y| < \epsilon\}$$

is uniformly integrable. We have $j_y(y, t) = h(y, t) - v$. Because h is monotone, $|h(y' + Z_1 - Z_t)|$ is no larger than the maximum of $|h(y - \epsilon + Z_1 - Z_t)|$ and $|h(y + \epsilon + Z_1 - Z_t)|$. For (almost) any $y \in \Re$, each of these random variables is integrable, because

$$Eh(y \pm \epsilon + Z_1 - Z_t) = E[h(Z_1)|Z_t = y \pm \epsilon] < \infty.$$

Hence, the maximum is integrable, and, consequently, the above family of random variables is uniformly integrable. Taking the derivative under the expectation operator yields

$$\begin{aligned} J_y(y, t) &= E[j_y(y + Z_1 - Z_t)] = E[h(y + Z_1 - Z_t)] - v \\ &= H(y, t) - v. \end{aligned} \tag{A1}$$

This holds for all $(v, y, t) \in V \times \Re \times (0, 1)$. Continuity of J and J_y at $t = 1$ follows from the martingale properties (19) and (A1), using Karatzas and Shreve (1987, problem 4.3.2). ■

Proof of Lemma 2

We will work with the stochastic processes X, Z, and $J(\tilde{v}, Y_t, t)$ on the filtration $\mathbf{F}$. We will omit writing the random variable $\tilde{v}$ as an argument of J.

Itô's formula [Dellacherie and Meyer (1982, p. 335)] states that

$$\begin{aligned} J(Y_1, 1) = J(Y_{0-}, 0) &+ \int_{[0,1]} J_y(Y_{t-}, t)\, dY_t + \int_0^1 J_t(Y_{t-}, t)\, dt \\ &+ \frac{1}{2}\int_0^1 J_{yy}(Y_{t-}, t)\, d[Y^c, Y^c]_t \\ &+ \sum_{0 \le t \le 1} \Delta J(Y_t, t) - \sum_{0 \le t \le 1} J_y(Y_{t-}, t)\Delta Y_t. \end{aligned}$$

By construction, $Y_{0-} = 0$. We have

$$\begin{aligned} [Y^c, Y^c]_t &= [X^c, X^c]_t + 2[X^c, Z]_t + [Z, Z]_t \\ &= [X^c, X^c]_t + 2[X^c, Z]_t + \sigma^2 t \qquad (\forall t). \end{aligned}$$

Also, $\Delta Y \equiv \Delta X$. Therefore, substituting (17) and (18) yields

$$\begin{aligned} J(Y_1, 1) = J(0, 0) &+ \int_{[0, 1]} (P_t - \tilde{v})\, dY_t \\ &+ \frac{1}{2}\int_0^1 J_{yy}(Y_{t-}, t)\, d[X^c, X^c]_t \end{aligned}$$

$$+ \int_0^1 J_{yy}(Y_{t-}, t)\, d[X^c, Z]_t$$
$$+ \sum_{0 \le t \le 1} \Delta J(Y_t, t) - \sum_{0 \le t \le 1} (P_{t-} - \tilde{v})\Delta X_t.$$

Substituting $dX + dZ = dY$, subtracting $[P, X]_1$ from both sides, and rearranging terms a bit give

$$\begin{aligned}
&\int_{[0,1]} (\tilde{v} - P_{t-})\, dX_t - [P, X]_1 - J(0, 0) \\
&\quad = -J(Y_1, 1) + \int_0^1 (P_{t-} - \tilde{v})\, dZ_t \\
&\qquad + \frac{1}{2}\int_0^1 J_{yy}(Y_{t-}, t)\, d[X^c, X^c]_t \\
&\qquad + \int_0^1 J_{yy}(Y_{t-}, t)\, d[X^c, Z]_t \\
&\qquad + \sum_{0 \le t \le 1} \Delta J(Y_t, t) - \sum_{0 \le t \le 1} (P_{t-} - v)\Delta X_t - [P, X]_1. \qquad \text{(A2)}
\end{aligned}$$

We want to show that the expectation of the left-hand side is nonpositive and that this expectation equals zero iff $X^c \equiv 0$, $\Delta X \equiv 0$, and $P_1 = \tilde{v}$ a.s. When $\Delta X \equiv 0$, then the martingale M is continuous and hence equal to X^c. Thus, the conditions $\Delta X \equiv 0$ and $X^c = 0$ are collectively equivalent to X being continuous and having no martingale part. This will complete the proof.

We need to evaluate the right-hand side. Recall that

$$[P, X]_1 \equiv [P^c, X^c]_1 + \sum_{0 \le t \le 1} \Delta P_t \Delta X_t.$$

By Itô's formula, the continuous local martingale part of P is $\int H_y(Y_{t-}, t)\, dY_t^c$. Using (17), we obtain

$$\begin{aligned}
[P^c, X^c]_1 &= \int_0^1 H_y(Y_{t-}, t)\, d[Y^c, X^c]_t \\
&= \int_0^1 J_{yy}(Y_{t-}, t)\, d[X^c, X^c]_t + \int_0^1 J_{yy}(Y_{t-}, t)\, d[X^c, Z]_t.
\end{aligned}$$

Using (17) again, we have

$$(P_{t-} - \tilde{v})\Delta X_t + \Delta P_t \Delta X_t = (P_t - \tilde{v})\Delta X_t = J_y(Y_t, t)\Delta X_t.$$

Therefore, substituting for $[P, X]_1$ in the right-hand side of (A2), it simplifies to

$$-J(Y_1, 1) + \int_0^1 (P_{t-} - \tilde{v})\, dZ_t - \frac{1}{2}\int_0^1 J_{yy}(Y_{t-}, t)\, d[X^c, X^c]_t$$
$$+ \sum_{0 \le t \le 1} \Delta J(Y_t, t) - \sum_{0 \le t \le 1} J_y(Y_t, t)\Delta X_t. \tag{A3}$$

The lemma follows from the following facts, which will be established:

$$-J(Y_1, 1) \le 0 \text{ with equality iff } P_1 = \tilde{v}, \tag{A4}$$

$$E\int_0^1 (P_{t-} - \tilde{v})\, dZ_s = 0, \tag{A5}$$

$$-\frac{1}{2}\int_0^1 J_{yy}(Y_{t-}, t)\, d[X^c, X^c]_t \le 0 \text{ with equality iff } X^c \equiv 0, \tag{A6}$$

$$\sum_{0 \le t \le 1} [\Delta J(Y_t, t) - J_y(Y_t, t)\Delta X_t] \le 0 \text{ with equality iff } \Delta X \equiv 0. \tag{A7}$$

Condition (A4) is just the boundary condition (15). Condition (A5) follows from the fact that we have ruled out "doubling strategies." It follows from (17) and the monotonicity of H that $J_{yy} > 0$. The measure $d[X^c, X^c]$ is positive unless $X^c \equiv 0$. This implies (A6). Finally, $J_{yy} > 0$ (convexity) implies (A7) ∎

Proof of Lemma 4

We can rewrite (19) as

$$J(v, y, t) = E[J(v, y + Z_1 - Z_t, 1)]$$
$$(\forall (v, y, t) \in \mathrm{V} \times \Re \times (0, 1]), \tag{A8}$$

where the expectation is taken over Z, v being regarded as a constant. We need to differentiate the right-hand side of (A8) with respect to y under the expectation operator. The proof that this can be done is exactly the same as in the proof of Lemma 1. Differentiation yields

$$J_y(v, y, t) = E[J_y(v, y + Z_1 - Z_t)] \qquad (\forall (v, y, t) \in V \times \Re \times (0, 1]).$$

In view of (17), this implies

$$H(y, t) = E[H(y + Z_1 - Z_t, 1)] \qquad (\forall (y, t) \in \Re \times [0, 1]).$$

We have included the endpoint $t = 0$ here because both sides are continuous at $t = 0$. [To see the continuity of the right-hand side, use Lebesgue's convergence theorem and (8), writing the integral as $\int h(z)p(y, z, \sigma^2(1 - t))\, dz$, where $p(\mu, \cdot, \phi)$ denotes the normal distribution function with mean μ and variance ϕ.] This implies that the process $H(Z_t, t)$ is a martingale on the filtration $\mathbf{F}$. ∎

Proof of Theorem 3

Denote $H(\cdot, 1)$ by $h(\cdot)$. We have from Theorem 2 that $h = F^{-1} \circ N$. Hence,

$$h_y(y) = \frac{n(y)}{f(h(y))},$$

where n denotes the normal $(0, \sigma^2)$ density function, and the assumption of the theorem implies the random variable

$$\frac{n(Z_1)}{f(h(Z_1))}$$

is integrable. It follows that

$$E\left[\frac{n(y + Z_1 - Z_t)}{f(h(y + Z_1 - Z_t))}\right] = E\left[\frac{n(Z_1)}{f(h(Z_1))} \,\middle|\, Z_t = y\right] < \infty,$$

for almost all $y \in \Re$. For any ϵ and any $|y' - y| < \epsilon$, the random variable

$$h_y(y' + Z_1 - Z_t) = \frac{n(y' + Z_1 - Z_t)}{f(h(y' + Z_1 - Z_t))} \tag{A9}$$

is dominated a.s. by the larger of the four random variables

$$\frac{n(y \pm \epsilon + Z_1 - Z_t)}{f(h(y \pm \epsilon + Z_1 - Z_t))}.$$

This follows from the monotonicity of h. Therefore, the random variables (A9) for $|y' - y| < \epsilon$ are uniformly integrable. This implies that we can interchange differentiation and expectation in (25) as desired. ■

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