4. Determine how large drops must be beyond the critical radius before solute (Raoult) effects are negligibly small relative to the curvature (Kelvin) effect.

5. Consider a parcel of air at $T = -5°C$ and $p = 800$ hPa. Assume that a slight supersaturation exists with $\mathcal{H} = 100.5\%$ (with respect to liquid).

a) Compute how long it would take to grow a cloud drop from an initial radius of 1 μm to a drop radius of 10 μm, 100 μm, and 1000 μm.

b) Compute how long it would take to grow a spherical ice ball from an initial radius of 1 μm to a radius of 10 μm, 100 μm, and 1000 μm.

6. Derive expression for $\mathcal{A}_2$ in the following equation:

$$\frac{d\mathcal{S}}{dt} = \mathcal{A}_1 \frac{dz}{dt} - \mathcal{A}_2 \frac{dq_l}{dt}$$

7. An analytic expression of the following form has been used to describe drop size spectra:

$$n(r) = Ar^2 \exp(-Br)$$

where A and B are parameters. For a drop size spectrum represented by this relationship, determine the following:

a) the total drop concentration per volume of air:

$$N = \int_0^\infty n(r)\, dr$$

b) the mean drop radius:

$$\bar{r} = \frac{1}{N}\int_0^\infty r\, n(r)\, dr$$

c) the coefficients A and B for $N = 200$ cm^{-3} and $\bar{r} = 10\ \mu$m;

d) the liquid water mixing ratio, w_l:

$$w_l = \frac{\rho_l}{\rho_a}\frac{4}{3}p \int_0^\infty r\, n(r)\, dr$$

where ρ_l is the density of water and ρ_a is the density of air.

Chapter 6 Thermodynamic Transformations of Moist Air

In this chapter we consider the thermodynamic processes that result in the formation and dissipation of clouds. Based on microphysical considerations, we found in Chapter 5 that the liquid phase is nucleated at relative humidities only slightly greater than 100%. For simplicity, we assume here that clouds form in the atmosphere when the water vapor reaches its saturation value and $\mathcal{H} = 100\%$.

In a closed system consisting of moist air, the water vapor mixing ratio remains constant through the course of thermodynamic transformations as long as condensation does not occur. However, vapor pressure and relative humidity do not remain the same during such transformations. For example, in an adiabatic expansion the vapor pressure decreases, since it remains proportional to atmospheric pressure.

The relative humidity was defined in Section 4.4 as

$$\mathcal{H} \approx \frac{w_v}{w_s(T)}$$

where w_v is the water vapor mixing ratio and w_s is the saturation mixing ratio. For initially unsaturated air to become saturated, the relative humidity must increase. An increase in relative humidity can be accomplished by increasing the amount of water vapor in the air (i.e., increasing w_v), and/or by cooling the air, which decreases $w_s(T)$. The amount of water vapor in the air can increase by evaporation of water from a surface or via evaporation of rain falling through unsaturated air. The temperature of the atmosphere can decrease by isobaric cooling (e.g., radiative cooling) or by adiabatic cooling of rising air. An additional mechanism that can increase the relative humidity is the mixing of two unsaturated parcels of air.

In this chapter, we begin by writing the combined first and second laws of thermodynamics for a system that consists of moist air plus condensed water. To understand the changes in thermodynamic state associated with the formation and dissipation of clouds, we apply the combined first and second laws to the following idealized thermodynamic reference processes associated with phase changes of water:

- isobaric cooling;
- adiabatic isobaric processes;
- adiabatic expansion;
- adiabatic isobaric freezing.

Although real clouds nearly always involve more than one of these reference processes in their formation, consideration of the individual processes provides a convenient framework for understanding mechanisms that cause clouds to form and dissipate.

6.1 Combined First and Second Laws

To understand thermodynamic processes in moist and cloudy air, consider the combined first and second laws for a system that consists of two components (dry air and water) and two phases (gas and liquid). For the present, we ignore surface and solute effects in the condensed phase. Following Section 4.3, the combined first and second laws are written as

$$dU = T\,d\eta - p\,dV + \mu_d\,dn_d + \mu_v\,dn_v + \mu_l\,dn_l$$

$$dH = T\,d\eta + V\,dp + \mu_d\,dn_d + \mu_v\,dn_v + \mu_l\,dn_l$$

$$dG = -\eta\,dT + V\,dp + \mu_d\,dn_d + \mu_v\,dn_v + \mu_l\,dn_l$$

where the subscripts d, v, and l refer to dry air, water vapor, and liquid water, respectively.

The exact differential of the enthalpy, dH, where $H = H(T, p, m_d, m_v, m_l)$, can be expanded as follows:

$$dH = \left(\frac{\partial H}{\partial T}\right) dT + \left(\frac{\partial H}{\partial p}\right) dp + \left(\frac{\partial H}{\partial m_d}\right) dm_d + \left(\frac{\partial H}{\partial m_v}\right) dm_v + \left(\frac{\partial H}{\partial m_l}\right) dm_l$$

If the system is closed, then $dm_d = 0$ and $dm_v = -\,dm_l$, and therefore

$$dH = \left(\frac{\partial H}{\partial T}\right) dT + \left(\frac{\partial H}{\partial p}\right) dp + \left[\left(\frac{\partial H}{\partial m_v}\right) - \left(\frac{\partial H}{\partial m_l}\right)\right] dm_v \tag{6.1a}$$

Since $(h_v - h_l) = L_{lv}$ (Section 4.3), we have

$$dH = \left(\frac{\partial H}{\partial T}\right) dT + \left(\frac{\partial H}{\partial p}\right) dp + L_{lv}\,dm_v \tag{6.1b}$$

To evaluate $\partial H/\partial T$ and $\partial H/\partial p$, consider the total enthalpy as the sum of the individual contributions from the dry air, water vapor, and liquid water, so that $H = m_d h_d + m_v h_v + m_l h_l$. We can then write

$$\frac{\partial H}{\partial T} = m_d\,c_{pd} + m_v\,c_{pv} + m_l\,c_l \tag{6.2a}$$

Recall that in Section 2.9 we established that there is little difference between the specific heats of liquid water at constant pressure and volume, so henceforth we do not distinguish between them. In Section 2.3, we found that $\partial H/\partial p = 0$ for an ideal gas. For liquid water, $\partial H/\partial p \neq 0$, but the value is small and thus neglected here. We can therefore write (6.1) as

$$dH = \left(m_d\,c_{pd} + m_v\,c_{pv} + m_l\,c_l\right) dT + L_{lv}\,dm_v \tag{6.2b}$$

In the atmosphere, the mass of water vapor is only a few percent of the mass of dry air (Section 1.1), and the mass of condensed water is a small fraction of the mass of water vapor. Thus $m_d >> m_v >> m_l$ and we can approximate (6.2b) by

$$dH \approx m_d\,c_{pd}\,dT + L_{lv}\,dm_v \tag{6.3}$$

The enthalpy of a system consisting of moist air and a liquid water cloud is not only a function of temperature (as was the ideal gas), but also a function of the latent heat associated with the phase change. In intensive form, we have

$$dh \approx c_{pd}\,dT + L_{lv}\,dw_v \tag{6.4}$$

In a similar manner, we can write an equation for internal energy[1] as

$$du = \left(c_{vd} + w_v\,c_{vv} + w_l\,c_l\right) dT + L_{lv}\,dw_v \tag{6.5}$$

and an approximate form as

$$du \approx c_{vd}\,dT + L_{lv}\,dw_v \tag{6.6}$$

where w_l is the liquid water mixing ratio introduced in (5.28).

[1] Mixing ratio is used here instead of specific humidity to avoid confusion of the notation q (specific humidity) with q (heat). Note that a liquid water specific humidity, q_l, can be defined analogously to the liquid water mixing ratio, w_l.

Depending on how the thermodynamic system is defined, the term $L_{lv}dw_v$ may be included as part of the enthalpy, or it may constitute an external heat source. For a closed system, we can write

$$dq = c_{pd}\,dT + L_{lv}\,dw_v - v\,dp \tag{6.7a}$$

and for an adiabatic process,

$$0 = c_{pd}\,dT + L_{lv}\,dw_v - v\,dp \tag{6.7b}$$

Now consider a system that consists of moist air, with an external heat source associated with evaporation from a water source (such as moist air over a lake). The first law of thermodynamics can be written as

$$dq = dh - v\,dp$$

where $dh = c_{pd}\,dT$ and $dq = L_{lv}\,dw_l = -L_{lv}\,dw_v$. We can then write

$$-L_{lv}\,dw_v = c_{pd}\,dT - v\,dp \tag{6.8}$$

Note that (6.7) and (6.8) are mathematically equivalent; however, in (6.7b) the term $L_{lv}\,dw_v$ is part of the enthalpy, while in (6.8) the term $L_{lv}\,dw_v$ is a heat source. This example illustrates the care that must be taken to interpret correctly the thermodynamic equation in the context in which the system is defined.

The combined first and second law for a system consisting of moist air and a liquid water cloud can be written using (4.7) and (2.33) as

$$T\,d\eta = dH - V\,dp - \sum_j \mu_j\,dn_j \tag{6.9}$$

Including only the liquid–vapor phase change, we can incorporate (6.2) into (6.9) and write

$$T\,d\eta = \left(m_d\,c_{pd} + m_v\,c_{pv} + m_l\,c_l\right)dT + L_{lv}\,dm_v - V\,dp - \mu_v\,dm_v - \mu_l\,dm_l \tag{6.10}$$

If the system is closed, then $dm_d = 0$ and $dm_v = -\,dm_l$, and analogously to (6.1b) we can write (6.10) in intensive form as

$$d\eta = \left(c_{pd} + w_v c_{pv} + w_l c_l\right) d(\ln T) - R_d\,d(\ln p_d) - w_v R_v\,d(\ln e) + \frac{L_{lv} + A_{lv}}{T}\,dw_v \tag{6.11}$$

In (6.11) we have separated the expansion work term into components (neglecting the expansion work of liquid water). The *affinity for vaporization*, A_{lv}, is defined (following Dutton, 1986) as $A_{lv} = \mu_l - \mu_v$, which can be evaluted following (5.10). If the liquid and vapor phases are in equilibrium ($\mu_v = \mu_l$), then $A_{lv} = 0$. In subsaturated or supersaturated conditions, the affinity term can be of the order of several percent of the latent heat of vaporization. Using the first and second latent heat equations (4.19) and (4.29), we can write (6.11) as

$$d\eta = \left(c_{pd} + w_t c_l\right) d(\ln T) - R_d\,d(\ln p_d) + d\left(\frac{L_{lv} w_v}{T}\right) + w_v\,d\left(\frac{A_{lv}}{T}\right) \tag{6.12}$$

where w_t is the *total water mixing ratio* ($w_t = w_v + w_l$).

Analogous arguments can be used to incorporate the ice phase into the entropy equation. The complete thermodynamic equation for moist air and clouds that includes all three phases of water is written as

$$\begin{aligned} d\eta = {} & \left(c_{pd} + w_t c_l\right) d(\ln T) - R_d\,d(\ln p_d) + w_v\,d\left(\frac{A_{lv}}{T}\right) + d\left(\frac{L_{lv} w_v}{T}\right) \\ & - w_i\,d\left(\frac{A_{il}}{T}\right) - d\left(\frac{L_{il} w_i}{T}\right) \end{aligned} \tag{6.13}$$

where the total water mixing ratio, w_t, in (6.13) includes the *ice water mixing ratio*, w_i. The *affinity for freezing*, A_{il}, is defined analogously to that for vaporization as $A_{il} = \mu_i - \mu_l$. The affinity for freezing can reach 20% of the latent heat of fusion.

6.2 Isobaric Cooling

A thermodynamic process can be approximated as isobaric if vertical motions are small and there is only a small departure from a reference pressure. In the absence of condensation, the first law of thermodynamics for an isobaric process in moist air is written (following 2.16) as

$$dq = dh = c_p\,dT$$

where c_p can be approximated as the dry air value, or alternatively the contribution from water vapor can be incorporated following (2.65). As moist air cools, relative humidity increases: w_v remains the same, but as the temperature decreases then w_s decreases. If the cooling continues, w_s will become equal to w_v and $\mathcal{H}$ will equal unity; at this point, the air has reached saturation. Further cooling beyond saturation results in condensation.

The temperature at which saturation is reached in an isobaric cooling process is the *dew-point temperature*, which is illustrated in Figure 6.1a. The dew-point temperature, denoted by T_D, can be defined by

$$e = e_s\left(T_D\right) \tag{6.14}$$

or equivalently by

$$w_v = w_s\left(T_D\right) \tag{6.15}$$

We can determine the dew-point temperature by inverting either (6.14) or (6.15), which can be done using (4.31) and (4.36).

Analogously to the dew-point temperature, we define the *frost-point temperature* as the temperature at which ice saturation occurs. The frost-point temperature, T_F, is thus defined as

$$e = e_{si}\left(T_F\right) \tag{6.16}$$

or equivalently as

$$w_v = w_{si}\left(T_F\right) \tag{6.17}$$

In Figure 6.1b, it is seen that if the vapor pressure is initially below the triple–point pressure of water (point 1), isobaric cooling results in deposition once the frost point is reached (point 2). As described in Section 5.3, saturation with respect to ice is not sufficient to initiate the ice phase in the atmosphere. Deposition occurs at the frost point only if ice crystals already exist in the atmosphere. Since $T_F > T_D$, the formation of frost on the ground must occur by deposition rather than by freezing of condensed water vapor; grass and other structures provide a good substrate for initiating the ice phase by deposition.

Although the units of the dew-point temperature are kelvins, the dew-point temperature is a measure not of temperature but of atmospheric humidity. By examining Figure 6.1 and the Clausius–Clapeyron relationship (4.19), it is seen that

$$\frac{d(\ln e)}{dT_D} = \frac{L_{lv}}{R_v T_D^2} \tag{6.18}$$

and that e and T_D give equivalent information about the amount of water vapor in the atmosphere. A relationship between T_D and $\mathcal{H}$ can be obtained by integrating (6.18) between T and T_D:

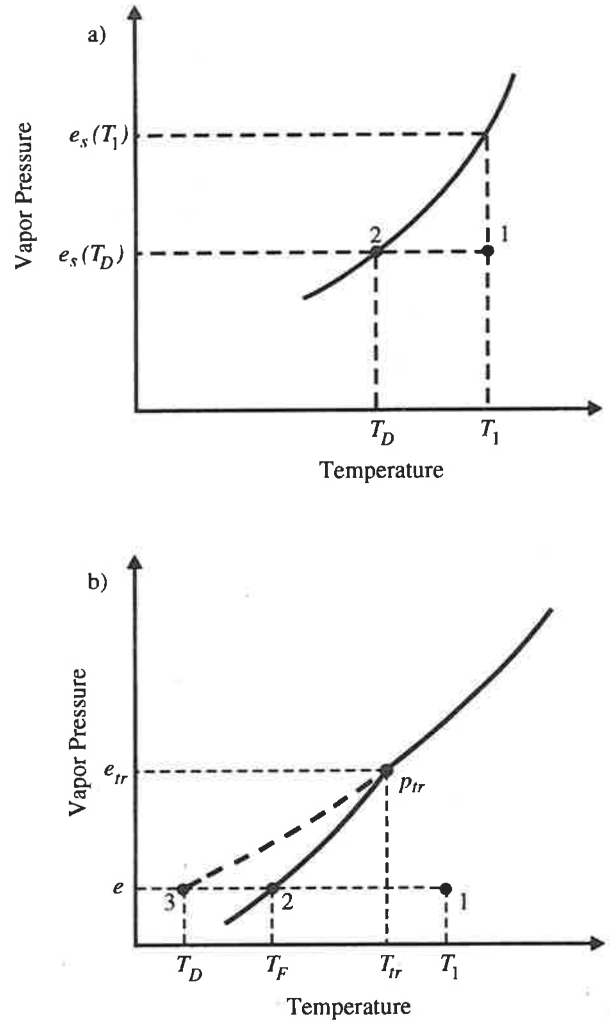


Figure 6.1 a) Relationship between temperature and vapor pressure in an isobaric cooling process. Air initially at temperature T_1 (point 1) is cooled isobarically until it reaches saturation (point 2). The temperature at point 2 defines the dew-point temperature, T_D. b) Air at T_1 (point 1) cools isobarically until it reaches saturation. If the saturation is reached with respect to ice (point 2), the temperature is called the frost point, T_F.

$$\ln\frac{e_s}{e} = -\ln\mathcal{H} = \frac{L_{lv}}{R_v}\left(\frac{1}{T_D} - \frac{1}{T}\right)$$

or equivalently

$$\mathcal{H} = \exp\left[-\frac{L_{lv}}{R_v}\left(\frac{T - T_D}{T T_D}\right)\right] \tag{6.19}$$

The term $T - T_D$ in (6.19) is called the *dew-point depression.* Figure 6.2 illustrates that dew-point depression is inversely proportional to relative humidity and that a relative humidity of 100% corresponds to a dew-point depression of zero.

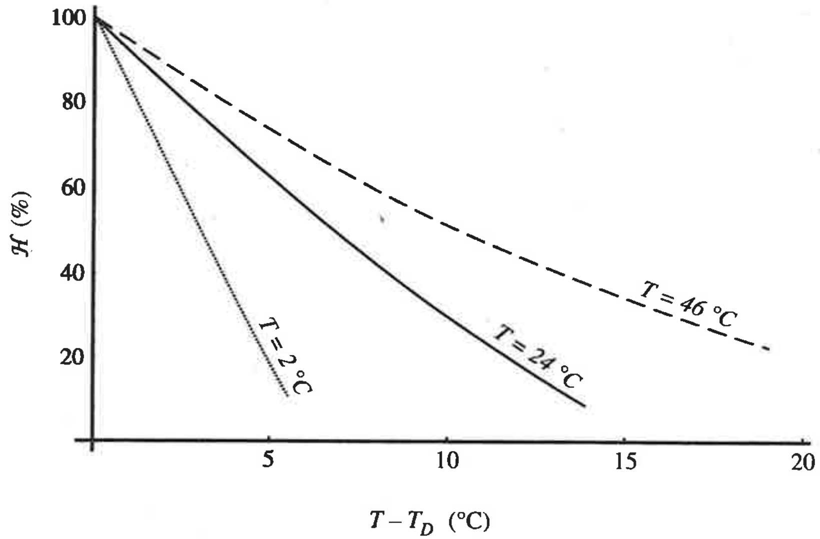


Figure 6.2 Dew-point depression. As the relative humidity increases, the difference between the ambient temperature and the dew-point temperature (i.e., the *dew-point depression*) decreases. As the ambient temperature decreases, the dew-point depression becomes less sensitive to changes in the relative humidity.

Thus, through (6.14), (6.15), and (6.19), the dew-point temperature is shown to be a humidity variable. If temperature, dew-point temperature and pressure are given, then the values of mixing ratio, relative humidity, and vapor pressure can be calculated. Analogously, the frost-point temperature can be related to all of the other humidity variables. In an isobaric process in the absence of condensation, the dew-point and frost-point temperatures are conservative; that is, they do not change during the cooling process until condensation is reached.

Once the air is cooled slightly below the dew-point temperature, condensation begins. After condensation begins, the first law of thermodynamics for an isobaric process is written following (6.4) in the approximate form

$$dq = dh = c_p\, dT + L_{lv}\, dw_v \tag{6.20}$$

Assuming that condensation occurs at saturation ($\mathcal{H} = 1$) and that the water vapor mixing ratio is equal to the saturation vapor mixing ratio $w_v = w_s$, we can write

$$w_t = w_s + w_l \tag{6.21}$$

In a closed system, w_t remains constant, so

$$dw_l = -dw_s$$

Using the approximation $w_s \approx \varepsilon\, e_s / p$ from Section 4.4 and the Clausius–Clapeyron relation (4.19), we can write

$$dw_l = -dw_s = -\varepsilon\frac{de_s}{p} = -\frac{\varepsilon L_{lv} e_s}{p R_v T^2}\, dT \tag{6.22a}$$

Incorporating (6.22a) into (6.20) and using $R_d = R_v / \varepsilon$, we obtain

$$dw_l = -\left(\frac{L_{lv} e_s}{c_p p R_d T^2 + L_{lv} e_s}\right) dq \tag{6.22b}$$

Combination of (6.22b) with (6.20) gives a relationship between dq and dT during isobaric condensation:

$$dq = -\left(c_p + \frac{L_{lv} e_s}{p R_d T^2}\right) dT \tag{6.22c}$$

Integration of (6.22b) (which is most easily done numerically, since e_s is a function of T) allows determination of the amount of isobaric cooling, Δq, required to condense an amount of liquid water, Δw_l. Analogously, integration of (6.22c) allows determination of the temperature change, ΔT, in response to the isobaric cooling, Δq. Before condensation occurs, we have $\Delta q = -c_p \Delta T$. Once condensation begins, it is seen from (6.22c) that the temperature drops much more slowly in response to the isobaric cooling, because the heat loss is partially compensated by the latent heat released during condensation.

Once condensation begins, the dew-point temperature decreases, since the water vapor mixing ratio is decreasing as the water is condensed. Relative humidity remains constant, at $\mathcal{H} = 1$.

Isobaric cooling is a primary formation mechanism for certain types of fog and stratus clouds (see Section 8.4). The equations derived in this section are equally applicable to isobaric heating. In this instance, an existing cloud or fog can be dissipated by evaporation that ensues from isobaric heating (e.g., solar radiation).

6.3 Cooling and Moistening by Evaporation of Water

Consider a system composed of unsaturated moist air plus rain falling through the air. Because the air is unsaturated, the rain will evaporate. If there are no external heat sources ($\Delta q = 0$), and the evaporation occurs isobarically ($dp = 0$), we can write an adiabatic, isobaric (or *isenthalpic*) form of the enthalpy equation (6.20) as

$$0 = dh = c_p \, dT - L_{lv} \, dw_l = c_p \, dT + L_{lv} \, dw_s \tag{6.23}$$

where c_p can be approximated as the dry–air value, or alternatively the contributions from water vapor and liquid water can be incorporated following (6.2a). Since $dh = 0$, (6.23) can be used to determine a relationship between temperature and humidity variables for isenthalpic processes in the atmosphere that involve a phase change of water.

If we allow just enough liquid water from the rain to evaporate so that the air becomes saturated, we can integrate (6.23)

$$c_p \int_T^{T_W} dT = -L_{lv} \int_{w_l}^{0} dw_s$$

where w_l represents the amount of water that must be evaporated to bring the air to saturation. During the evaporation process, latent heat is drawn from the atmosphere, and the final temperature, referred to as the *wet-bulb temperature* (T_W), is cooler than

the original temperature. Integration gives

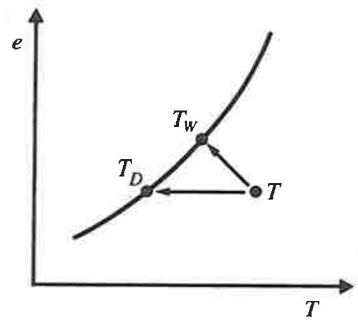
$$c_p \left(T_W - T\right) = -L_{lv}\left[w_s\left(T_W\right) - w_v\right]$$

or alternatively

$$T_W = T - \frac{L_{lv}}{c_p}\left[w_s\left(T_W\right) - w_v\right] \tag{6.24}$$

where the temperature dependence of L_{lv} has been neglected. Given w_v and T, this expression is implicit for T_W and must be solved numerically. However, if T and T_W are given, then w_v is easily determined. T_W can be measured using a *wet-bulb thermometer*, whereby a wetted muslin wick is affixed to the bulb of a thermometer. Concurrent measurement of the "dry-bulb" temperature by a normal thermometer can then provide a means of determining the water vapor mixing ratio and therefore atmospheric humidity. For this reason, (6.24) is often referred to as the *wet-bulb equation*.

The wet-bulb temperature is thus defined as the temperature to which air would cool isobarically as the result of evaporating sufficient liquid water into the air to make it saturated. As such, the wet-bulb temperature in the atmosphere is conservative with respect to evaporation of falling rain. Calculations for given values of T and w show that $T_D < T_W < T$. This can be shown graphically. Since e increases while T decreases during the approach to T_W, the Clapeyron diagram looks like:



If ice is the evaporating phase, we can determine an analogous *ice-bulb temperature*, T_I:

$$T_I = T - \frac{L_{iv}}{c_p}\left[w_{si}(T_I) - w_v\right] \tag{6.25}$$

It is easily shown that $T_I > T_W$.

6.4 Saturation by Adiabatic, Isobaric Mixing

We have seen in Sections 6.2 and 6.3 how unsaturated air can be brought to saturation by isobaric cooling and by the adiabatic, isobaric evaporation of falling rain. There is an additional isobaric process that can bring unsaturated air to saturation. Under some circumstances, the isobaric mixing of two samples of unsaturated air leads to saturation. One example of this process occurs when your breath produces a puff of cloud on a cold day.

Consider the isobaric mixing of two moist air masses, with different temperatures and humidities but at the same pressure. Condensation is assumed not to occur. For adiabatic, isobaric mixing, we can write the first law of thermodynamics from (2.16) as

$$0 = dH \approx m_1 c_{pd}\, dT_1 + m_2 c_{pd}\, dT_2$$

where dT_1 and dT_2 correspond to the temperature change of the air masses upon mixing and we have ignored the heat capacity of the water vapor in accordance with (6.4). Upon integration from an initial state where the air masses are unmixed to a final state where the air masses both have the same final temperature, T, we have

$$m_1 c_{pd}\left(T - T_1\right) + m_2 c_{pd}\left(T - T_2\right) = 0$$

Solving for T we obtain

$$T \approx \frac{m_1}{m_1 + m_2} T_1 + \frac{m_2}{m_1 + m_2} T_2$$

The total mass $m = m_1 + m_2$ remains constant during the mixing process, so the specific humidity is a mass-weighted average of q_{v1} and q_{v2}

$$q_v = \frac{m_1}{m_1 + m_2} q_{v_1} + \frac{m_2}{m_1 + m_2} q_{v_2}$$

Thus, both the temperature and specific humidity mix linearly if the heat capacity of the water vapor is neglected. Since $q_v \approx w_v$, we can also assume that the mixing ratios mix linearly. If we further assume that $w_v \approx \varepsilon e/p$, then vapor pressure mixes linearly as well.

Because of the nonlinearity of the Clausius–Clapeyron equation, adiabatic isobaric mixing results in an increase in relative humidity. This mixing process is illustrated in Figure 6.3 using a T, e diagram. If Y_1 and Y_2 are the image points for the two

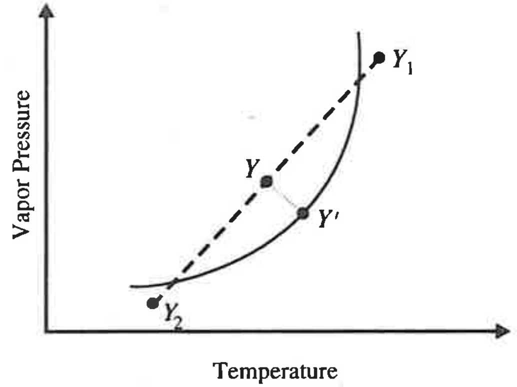


Figure 6.3 Adiabatic isobaric mixing and condensation. Two air masses with (e,T) given by points Y_1 and Y_2 mix, resulting in a single air mass with (e,T) given by point Y. Since $\mathcal{H} > 1$ at this point, water will condense, and the temperature of the air mass will increase while the vapor pressure decreases. Condensation will continue until the temperature and vapor pressure of the air mass coincide with the saturation vapor pressure curve (point Y').

air masses, the image point for the mixture lies on a straight line joining Y_1 and Y_2. If $m_1 = m_2$, then T and e for the mixture will lie midpoint on this line. Because of the exponential relationship between e_s and T, the mixing process increases the relative humidity. In the example shown in Figure 6.3, the mixing process results in the image point Y having a relative humidity that exceeds 100%, crossing the $f = 1$ line into the liquid phase (see also Figure 4.3). Water will condense and latent heat will be released, with the final equilibrium image point at Y' on the $f = 1$ line.

The slope of the line between Y and Y' can be determined from the first law of thermodynamics for an adiabatic isobaric process in which condensation occurs (6.23):

$$dh = 0 \approx c_p\, dT + L_{lv}\, dw_s$$

Using the definition of the saturated water vapor mixing ratio, $w_s = \varepsilon e_s/p$, we can write

$$0 = c_p\, dT + \frac{L_{lv}\, \varepsilon}{p}\, de$$

or

$$\frac{de}{dT} = -\frac{p c_p}{\varepsilon L_{lv}} \tag{6.26}$$

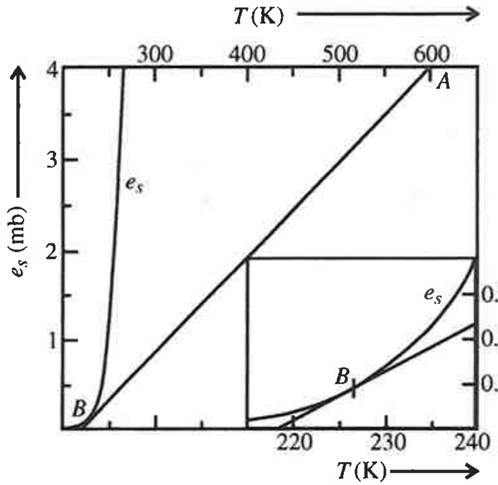


Figure 6.4 The formation of contrails by adiabatic, isobaric mixing. A jet flying at 200 mb ejects water vapor into the atmosphere at the temperature and vapor pressure represented by point A. For atmospheric temperatures less than about −47°C (226 K), the water vapor will condense, forming condensation trails. (From Ludlam, 1980.)

The value of (e, T) at Y' can be found by simultaneously solving (6.26) with the Clausius–Clapeyron equation (4.19). The amount of liquid water condensed during the mixing is

$$\Delta w_l = \frac{\varepsilon}{p}\left[e(Y) - e(Y')\right] \tag{6.27}$$

A notable example of the formation of clouds by adiabatic, isobaric mixing occurs when the exhaust gases from the combustion of fuels by an aircraft mixes with the ambient atmosphere. The trails of clouds often formed by an aircraft in flight at high altitude are referred to as condensation trails, or *contrails*. In the exhaust, the aircraft ejects heat and water vapor; the temperature of the exhaust is typically 600 K. Figure 6.4 indicates that for an aircraft flying at 200 mb, atmospheric temperatures below about −47°C will form contrails. Once contrails form, their persistence depends on the atmospheric humidity and the rate at which the exhaust trail is diffused. If the particles are ice, atmospheric humidity in excess of the ice saturation value will result in growth of the contrails.

6.5 Saturated Adiabatic Cooling

Adiabatic cooling is the most important mechanism by which moist air is brought to saturation. As described in Sections 2.1 and 2.10, adiabatic expansion in the atmosphere

occurs when a dry air mass rises due to mechanical lifting (e.g., orographic, frontal), large-scale low-level convergence, turbulent mixing, and buoyancy caused by surface heating.

Recall from Section 2.4 that the first law of thermodynamics for an adiabatic process for moist air in the absence of condensation is written as (2.19b)

$$c_p\, dT - v\, dp = 0$$

from which we derived an expression for the potential temperature (2.62)

$$\theta = T\left(\frac{1000}{p}\right)^{R/c_p}$$

and the dry adiabatic lapse rate (2.68)

$$\Gamma_d = \frac{g}{c_p} \approx 10°\text{C km}^{-1}$$

Recall that the potential temperature, θ, is conserved in reversible, dry adiabatic processes in the atmosphere.

As air expands adiabatically and cools, the relative humidity increases as the temperature and saturation mixing ratio decrease. The water vapor mixing ratio remains constant during adiabatic ascent. At some point, the relative humidity reaches 100%, and further cooling results in condensation. To determine the temperature and pressure at which saturation is reached, we logarithmically differentiate $\mathcal{H} = e/e_s$

$$d(\ln\mathcal{H}) = d(\ln e) - d(\ln e_s) \tag{6.28a}$$

Using Dalton's law of partial pressure (1.13), we have $d(\ln p) = d(\ln e)$, and we can write the first law of thermodynamics for an adiabatic process in enthalpy form (2.19b) as

$$d(\ln e) = \frac{c_p}{R_d}\, d(\ln T) \tag{6.28b}$$

Using the Clausius–Clapeyron equation (4.19), we can write

$$d(\ln e_s) = \frac{L_{lv}}{R_v T}\, d(\ln T) \tag{6.28c}$$

Incorporating (6.28b) and (6.28c) into (6.28a), we can integrate (6.28a) from the initial condition to conditions where saturation is attained, indicated by $\mathcal{H} = 1$ and $T = T_s$, where T_s is the *saturation temperature*

$$\int_{\mathcal{H}}^{1} d(\ln \mathcal{H}') = \int_{T}^{T_s} \left(\frac{c_p}{R_d} - \frac{\varepsilon L_{lv}}{R_d T} \right) d(\ln T')$$

to obtain

$$-\ln \mathcal{H} = \frac{c_p}{R_d} \ln\left(\frac{T_s}{T}\right) + \frac{\varepsilon L_{lv}}{R_d}\left(\frac{1}{T_s} - \frac{1}{T}\right) \tag{6.29}$$

Equation (6.29) can be solved numerically to obtain T_s. An approximate but simpler equation for T_s, given initial values of T (in kelvins) and $\mathcal{H}$, is given by (Bolton, 1980)

$$T_s = \frac{1}{\dfrac{1}{T-55} - \dfrac{\ln \mathcal{H}}{2840}} + 55 \tag{6.30}$$

The *saturation pressure*, p_s, can be obtained from (2.22) to be

$$\ln \frac{p_s}{p} = \frac{c_p}{R_d} \ln \frac{T_s}{T}$$

or, taking anti-logs,

$$p_s = p\left(\frac{T_s}{T}\right)^{c_p/R_d} \tag{6.31}$$

The coordinate (T_s, p_s) is known as the *saturation point* of the air mass.

During ascent, the water vapor mixing ratio, w_v, remains constant until saturation occurs. The dew-point temperature, however, decreases slightly during the ascent as pressure decreases. Recall from (6.18) that

$$d(\ln e) = \frac{L_{lv}}{R_v T_D^2} dT_D \tag{6.32}$$

Using Dalton's law of partial pressure (1.13), we can write the hypsometric equation (1.46) as

$$d(\ln e) = -\frac{g}{R_d T} dz \tag{6.33}$$

Combining (6.32) and (6.33), we obtain

$$\frac{dT_D}{dz} = -\frac{T_D^2 g}{\varepsilon L_{lv} T} = \frac{T_D^2 c_p}{\varepsilon L_{lv} T} \Gamma_d \tag{6.34a}$$

For typical atmospheric values, dT_D/dz is approximately one-sixth of the dry adiabatic lapse rate. At saturation level, T becomes equal to T_D and to T_s. The *lifting condensation level*, z_s, corresponds to the level of the saturation pressure, p_s.

Using (6.34a) and the definition of the dry adiabatic lapse rate, $\Gamma_d = g/c_p$, we can write

$$\frac{d(T - T_D)}{dz} = \left(1 + \frac{T_D^2 c_p}{\varepsilon L_{lv} T}\right) \Gamma_d \tag{6.34b}$$

When $T = T_D$, the saturation level has been reached, and a value of z_s can be determined by integrating (6.34b):

$$\int_{T_0 - T_{D0}}^{0} d(T - T_D) = \int_0^{z_s} \left[\left(1 + \frac{T_D^2 c_p}{\varepsilon L_{lv} T}\right) \Gamma_d\right] dz \tag{6.34c}$$

where $T_0 - T_{D0}$ is the dew-point depression at the surface. For a parcel of air lifted from the surface, the value of z_s can be estimated from (6.34c) to be

$$z_s \approx 0.12\left(T_0 - T_{D0}\right) \quad \text{(km)} \tag{6.35}$$

This relation is an approximate expression of the height of the lifting condensation level achieved in an adiabatic ascent where T_0 and T_{D0} represent the initial temperature and dew-point temperature of the air mass that is being lifted. Note that z_s can be determined directly from (1.45) if p_s and T_s are known. Calculation of the lifting condensation level provides a good estimate of the cloud base height for clouds that form by adiabatic ascent.

Once saturation occurs, further lifting of the air mass results in condensation. Because of the latent heat released during condensation, the decrease of temperature with height will be smaller than that in dry adiabatic ascent. In addition, the potential temperature, θ, which was conserved in a reversible dry adiabatic ascent, is no longer conserved once condensation occurs.

A derivation of an approximate form of the *saturated adiabatic lapse rate*, Γ_s, is given here by starting with the adiabatic entropy equation (6.12) in the following approximate form:

$$0 = c_{pd}\, d(\ln T) - R_d\, d(\ln p) + \frac{L_{lv}}{T}\, dw_s \tag{6.36}$$

Using the hypsometric equation (1.46)

$$\frac{dp}{p} = -\frac{g}{R_d T}\, dz$$

and logarithmically differentiating the equation for saturation mixing ratio (4.37),

$$\frac{dw_s}{w_s} = \frac{de_s}{e_s} - \frac{dp}{p}$$

we can rewrite (6.36) as

$$-L_{lv} w_s \left(\frac{de_s}{e_s} - \frac{dp}{p} \right) = c_p\, dT + g\, dz \tag{6.37}$$

Dividing by an incremental dz and solving for $-dT/dz$, we obtain

$$-\frac{dT}{dz} = \frac{L_{lv}}{c_p} w_s \left(\frac{1}{e_s}\frac{de_s}{dz} + \frac{g}{RT} \right) + \frac{g}{c_p} \tag{6.38}$$

Using the chain rule, we can write the term de_s/dz as

$$\frac{de_s}{dz} = \frac{de_s}{dT}\frac{dT}{dz} \tag{6.39}$$

and substitute into (6.38) to obtain

$$-\frac{dT}{dz}\left(1 + \frac{de_s}{dT}\frac{L_{lv}}{c_p}\frac{w_s}{e_s} \right) = \frac{g}{c_p}\left(\frac{L_{lv} w_s}{RT} + 1 \right)$$

Incorporating the Clausius–Clapeyron equation (4.19), solving for $dT/dz = -\Gamma_s$ and noting that $\Gamma_d = -g/c_p$ (2.68), we obtain finally

$$\Gamma_s = \Gamma_d \left(\frac{1 + \dfrac{L_{lv} w_s}{R_d T}}{1 + \dfrac{\varepsilon L_{lv}^2 w_s}{c_{pd} R_d T^2}} \right) \tag{6.40}$$

The denominator of (6.40) is larger than the numerator, and thus $\Gamma_s < \Gamma_d$. Table 6.1 shows values of Γ_s for selected values of T and p. It is seen that the temperature variation of Γ_s exceeds the pressure variation. At low temperatures and high pressures, Γ_s approaches Γ_d.

Values of Γ_s determined from (6.40) are within about 0.5% of the values determined from a more exact form of the entropy equation (6.11). Because of the approximate nature of (6.40), Γ_s is sometimes called the *pseudo-adiabatic lapse rate*.

The amount of water condensed in saturated adiabatic ascent, called the *adiabatic liquid water mixing ratio*, can be determined from the adiabatic enthalpy equation (6.7b):

$$0 = c_p\, dT - L_{lv}\, dw_l - v\, dp$$

Table 6.1 Γ_s for selected values of temperature and pressure.

T (°C)	p (hPa)		
	1000	700	500
–30	9.2	9.0	8.7
–20	8.6	8.2	7.8
–10	7.7	7.1	6.4
0	6.5	5.8	5.1
10	5.3	4.6	4.0
20	4.3	3.7	3.3

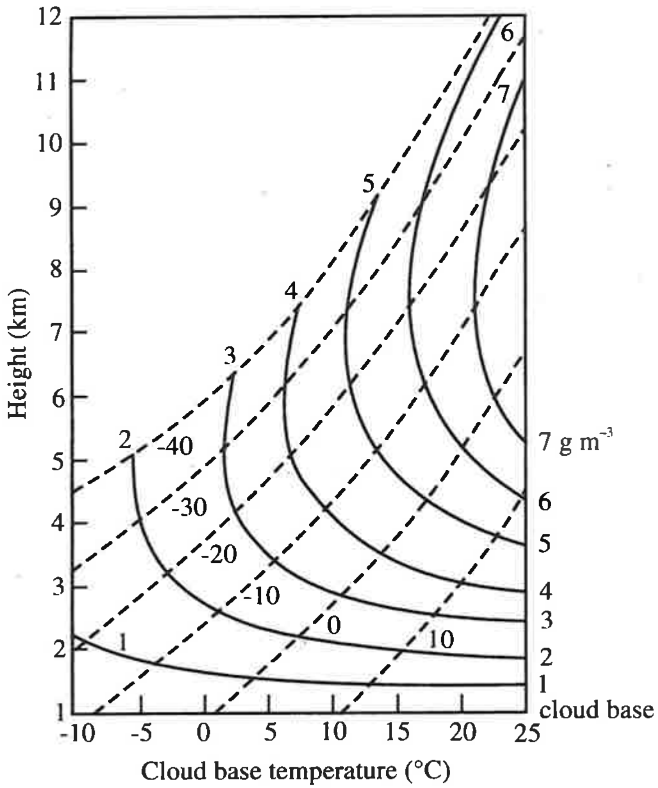


Figure 6.5 Adiabatic liquid water mixing ratio as a function of height above the cloud base and cloud base temperature. (After Goody, 1995.)

Solving for dw_l and incorporating the hydrostatic equation (1.33), we obtain

$$dw_l = \frac{c_p}{L_{lv}}\left(\frac{dT}{dz} + \frac{g}{c_p}\right) dz$$

Substituting $\Gamma_d = g/c_p$ and $\Gamma_s = -dT/dz$ yields

$$dw_l = \frac{c_p}{L_{lv}}\left(\Gamma_d - \Gamma_s\right) dz \tag{6.41}$$

Integrating (6.41) from cloud base to height z gives the adiabatic liquid water mixing ratio at height z. Because of the complicated form of Γ_s, this equation must be integrated numerically. Integration of (6.41) shows that the adiabatic liquid water content increases with height above the cloud base and increasing cloud base temperature (Figure 6.5). Because of the variation of Γ_s with temperature, clouds with warmer bases have larger values of $\Gamma_d - \Gamma_s$ and thus larger values of the adiabatic liquid water content. The adiabatic liquid water content represents an upper bound on the liquid water that can be produced in a cloud by rising motion. Processes such as precipitation and mixing with dry air reduce the cloud liquid water content relative to the adiabatic value.

6.6 The Ice Phase

As isobaric or adiabatic cooling proceeds, the cloud may eventually cool to the point where ice crystals form. Assuming that a water cloud is present initially, then the formation of ice crystals releases latent heat during fusion. Once the cloud glaciates, it is supersaturated with respect to ice, and deposition occurs on the ice crystals, releasing the latent heat of sublimation, until the ambient relative humidity is at ice saturation. Further cooling will result in the increase of ice water content in the cloud and the release of the latent heat of sublimation into the atmosphere.

Assuming that the thermodynamic system consists of moist air plus the condensate, and that the freezing and subsequent deposition occur isobarically and adiabatically, then the enthalpy of the system will not change during this transformation. Since enthalpy is an exact differential, the enthalpy change depends only on the initial and final states (but not on the path). Consider the following path for the warming of the system associated with the phase change:

Step 1. Water freezes at constant T_1:

$$\Delta h_1 = -L_{il} w_l \tag{6.42}$$

Step 2. Vapor deposits on the ice at constant T_1, until the water vapor pressure reaches the saturation value over ice at T_2:

$$\Delta h_2 = -L_{iv}\left[w_s(T_1) - w_{si}(T_2)\right] = -L_{iv}\frac{\varepsilon}{p}\left[e_s(T_1) - e_{si}(T_2)\right]$$

Assuming that $(T_2 - T_1)$ is small enough to treat as a differential, we can approximate $e_{si}(T_2)$ as

$$e_{si}(T_2) = e_{si}(T_1) + \frac{L_{iv} e_{si}(T_1)}{R_v T^2}\left(T_2 - T_1\right)$$

and Δh_2 becomes

$$\Delta h_2 = -w_s L_{iv}\left[1 - \frac{e_{si}(T_1)}{e_s(T_1)}\right] + \frac{L_{iv}^2 w_{si}(T_1)}{R_v T^2}(T_2 - T_1) \tag{6.43}$$

where w_s has been adopted in favor of e_s using $w_s \approx \varepsilon e_s/p$.

Step 3. The system is heated from T_1 to T_2:

$$\Delta h_3 = c_p\left(T_2 - T_1\right) \tag{6.44}$$

Since $\Delta h_1 + \Delta h_2 + \Delta h_3 = 0$, we can incorporate (6.42), (6.43), and (6.44) and solve for $\Delta T = T_2 - T_1$:

$$\Delta T = \frac{L_{il} w_l + L_{iv} w_s\left(1 - \dfrac{e_{si}}{e_s}\right)}{c_p + \dfrac{\varepsilon w_i L_{iv}^2}{R_d T^2}} \tag{6.45}$$

Equation (6.45) gives the increase in temperature due to the freezing of cloud water and the subsequent deposition of water vapor onto the ice crystals. In clouds that cool by adiabatic ascent, the freezing does not occur isobarically, but gradually over a temperature interval.

Once the cloud has glaciated, further adiabatic ascent results in deposition of water vapor onto the ice crystals. Analogously to (6.40), the *ice-saturation adiabatic lapse rate* is determined to be

$$\Gamma_{si} = \Gamma_d\left(\frac{1 + \dfrac{L_{iv} w_{si}}{R_d T}}{1 + \dfrac{\varepsilon L_{iv}^2 w_{si}}{c_{pd} R_d T^2}}\right) \tag{6.46}$$

The melting process is distinctly different from the freezing process. Melting may occur as ice particles fall to temperatures that are above the melting point. In contrast to freezing, which may be distributed through a considerable vertical depth, melting of ice particles can be quite localized, occurring in a very narrow layer around the freezing point. Cooling of the atmosphere from the melting can result in an isothermal layer near 0°C. Because of their large size and density, hailstones do not melt at the freezing level in the same manner as a small ice crystal or a snowflake with a low density, but melt over a deeper layer. If atmospheric relative humidities are low in the atmosphere below the melting level, then the melting water will evaporate, cooling the hailstone and retarding the melting.

6.7 Conserved Moist Thermodynamic Variables

As shown in Section 3.1, conserved variables are commonly used in time-dependent equations. The concept of potential temperature becomes less useful when applied to a cloud, since potential temperature is not conserved during phase changes of water. Derivation of a potential temperature that is conserved in moist adiabatic ascent eliminates the need to include latent heat source terms in the time-dependent thermodynamic equation. Additionally, a potential temperature that is conserved in moist adiabatic ascent can be used to interpret graphically numerous cloud processes and characteristics (see Sections 6.8, 7.3, and 8.5).

Recall that for a reversible, adiabatic process in dry air, the entropy equation is written as (2.26b)

$$0 = c_{pd}\, d\left(\ln T\right) - R_d\, d\left(\ln p\right)$$

It was shown in Section 2.4 that integration of the above equation gives the potential temperature (2.62)

$$\theta = T\left(\frac{p_0}{p}\right)^{R_d/c_{pd}}$$

which is conserved for dry adiabatic motions.

We seek an analogous variable that is conserved for a cloud so that the variation of temperature with pressure can be determined in a saturated adiabatic process. We begin with the adiabatic form of the complete equation for the combined first and second laws for a moist air with cloud that includes both the liquid and ice phases (6.13):

$$0 = \left(c_{pd} + w_t c_l\right) d(\ln T) - R_d\, d(\ln p_d) - w_v\, d\left(\frac{A_{lv}}{T}\right) + d\left(\frac{L_{lv} w_v}{T}\right) + w_i\, d\left(\frac{A_{il}}{T}\right) - d\left(\frac{L_{il} w_i}{T}\right)$$

A conserved potential temperature for clouds will obviously be far more complex than the potential temperature derived for a dry adiabatic process, since (6.13) is considerably more complex than (2.26b). A number of different conserved potential temperatures have been derived for clouds that employ various approximate forms of (6.13).

The simplest possible case is that in which saturation conditions are maintained, ice is not present, and heat capacity of the water vapor and condensed water are neglected relative to that of dry air. Using these approximations, the entropy equation (6.13) becomes:

$$0 = c_{pd}\, d(\ln T) - R_d\, d(\ln p) + d\left(\frac{L_{lv} w_s}{T}\right) \tag{6.47}$$

Recall that we have for a dry adiabatic process from (2.63)

$$c_{pd}\, d(\ln\theta) = c_{pd}\, d(\ln T) - R_d\, d(\ln p)$$

Equating (2.63) with (6.47) yields

$$-d\left(\frac{L_{lv} w_s}{T}\right) = c_{pd}\, d(\ln\theta)$$

This expression is integrated to a height in the atmosphere where all of the water vapor has been condensed out by adiabatic cooling. The corresponding temperature is called the *equivalent potential temperature*, θ_e. Integration of

$$-L_{lv}\int_{w_s}^{0} d\left(\frac{w_s}{T}\right) = c_p \int_{\theta}^{\theta_e} d(\ln\theta)$$

yields

$$\frac{L_{lv} w_s}{T} = c_{pd} \ln\left(\frac{\theta_e}{\theta}\right)$$

or

$$\theta_e = \theta \exp\left(\frac{L_{lv} w_s}{c_{pd} T}\right) \tag{6.48}$$

It is easily determined that $\theta_e > \theta$, which arises from the latent heat released from the condensation of water vapor. Because of the approximations made in (6.47), the

equivalent potential temperature is only approximately conserved in a saturated adiabatic process. Although approximate, (6.48) retains the essential physics of the process, whereby the condensation of water vapor provides energy to the moist air and increases its temperature relative to what the temperature would have been in dry adiabatic ascent.

An alternative but analogous potential temperature, the *liquid water potential temperature*, is derived as follows. Writing (6.47) as

$$0 = c_{pd}\, d(\ln T) - R_d\, d(\ln p) - d\left(\frac{L_{lv} w_l}{T}\right)$$

we can follow a procedure analogous to the derivation of θ_e and show that (Betts, 1973)

$$\theta_l = \theta \exp\left(-\frac{L_{lv} w_l}{c_{pd} T}\right) \tag{6.49}$$

One advantage of θ_l over θ_e is that θ_l reverts to θ, the dry potential temperature, in the absence of liquid water.

In the presence of ice, an *ice–liquid water potential temperature* can be derived from the following approximate form of (6.13):

$$0 = c_{pd}\, d(\ln T) - R_d\, d(\ln p) - d\left(\frac{L_{lv} w_l}{T}\right) - d\left(\frac{L_{iv} w_i}{T}\right)$$

to be (Tripoli and Cotton, 1981)

$$\theta_{il} = \theta \exp\left(-\frac{L_{lv} w_l}{c_{pd} T} - \frac{L_{iv} w_i}{c_{pd} T}\right) \tag{6.50}$$

The derivation of the ice–liquid water potential temperature implies that it is applicable only under conditions of equilibrium, since the affinity terms were not included. Since ice and liquid are both at equilibrium only at the triple point, use of the ice–liquid water potential temperature is inconsistent physically at temperatures away from the triple point. Nevertheless, the ice–liquid water potential temperature is an economical and not too inaccurate way to treat ice processes in a numerical cloud model.

The *entropy potential temperature*, θ_η, includes ice processes and is derived from the complete form of the adiabatic entropy equation (6.13) to be (Hauf and Holler, 1987):

$$\theta_\eta = T\left(\frac{p_0}{p}\right)^{R_d/(c_{pd}+w_t c_l)} \exp\left[\frac{(L_{iv}+A_{iv})\,w_l}{\left[(c_{pd}+w_t c_l)T\right]} - \frac{(L_{il}+A_{il})w_i}{\left[(c_{pd}+w_t c_l)T\right]}\right] \tag{6.51}$$

The entropy potential temperature is thus the most general potential temperature considered here. Unlike θ_l and θ_{il}, θ_η is applicable to nonequilibrium conditions such as subsaturated or supersaturated environments.

A major application of the conserved potential temperatures is their use as prognostic variables in cloud models (Section 8.6). Use of the more complex potential temperatures such as θ_{il} and θ_η is desirable in terms of their accuracy; however, a nontrivial calculation is required to invert (6.50) and (6.51) to obtain the physical temperature, T. When various other uncertainties are introduced into a calculation or model, the more approximate forms of the potential temperature can be justified.

Another moist thermodynamic variable that is often used is the *moist static energy*, $\hbar$. It is conserved in hydrostatic saturated adiabatic processes. We start from the following adiabatic form of the first law of thermodynamics:

$$0 = (c_{pd} + w_t c_l)\,dT + d(L_{iv} w_v) - v\,dp$$

Using the hydrostatic equation (1.33), we may write

$$0 = (c_{pd} + w_t c_l)\,dT + d(L_{iv} w_v) + g\,dz \equiv d\hbar$$

where the term $(1 + w_t)$ accounts for the contribution of the condensed water to the atmospheric density. Upon integration, the moist static energy is shown to be

$$\hbar = (c_{pd} + w_t c_l)\,T + L_{lv} w_v + (1 + w_t)\,gz \tag{6.52}$$

The moist static energy is conserved for adiabatic, saturated or unsaturated transformations for a closed system in which the pressure change is hydrostatic.

It is important to note the conditions under which θ_e and the other conserved thermodynamic variables are not conserved. Examples include cases where external radiative heating or conduction takes place, since these alter the entropy. Other examples include atmospheric conditions in which latent heating occurs externally, such as the evaporation of water into air from the ocean or when precipitation falls out.

In this chapter, we have considered numerous temperatures and potential temperatures, which are defined in the context of their conservative properties regarding certain moist atmospheric processes. Table 6.2 summarizes how various temperature, humidity, and other thermodynamic parameters vary in response to certain types of moist processes.

Table 6.2 Conservative properties of several parameters (C=conservative; N=nonconservative).

Parameter	Isobaric cooling no condensation	Isobaric cooling with condensation	Adiabatic expansion no condensation	Adiabatic expansion with condensation
w_v	C	N	C	N
$\mathcal{H}$	N	C	N	C
T_D	C	N	N	N
θ	N	N	C	N
θ_e	N	N	C	C
η	N	N	C	C

6.8 Aerological Diagrams

The principal function of a thermodynamic diagram is to provide a graphical display of a thermodynamic process. The following examples of thermodynamic diagrams have been used thus far in the text: (T, s) diagram (Section 1.9); (p, V) diagram (Sections 2.4 and 4.2); and (e, T) diagram (Sections 4.2 and 6.4). Here we consider a special class of thermodynamic diagrams called *aerological diagrams*. An aerological diagram is used to represent the vertical structure of the atmosphere and major types of processes to which moist air may be subjected, including isobaric cooling, dry adiabatic processes, and saturated adiabatic processes.

The simplest and most common form of the aerological diagram has pressure as the ordinate and temperature as the abscissa. While the temperature scale is linear, it is usually desirable to have the ordinate approximately representative of height above the surface. Thus the ordinate may be proportional to $-\ln p$ (the *Emagram*) or to p^{R/c_p} (the *Stuve diagram*). The Emagram has the advantage over the Stuve diagram in that area on the diagram is proportional to energy. Before the advent of computers, aerological diagrams were used widely in weather forecasting applications and the energy–area equivalence of the diagram was an important consideration. For the present purposes, we use the aerological diagram to illustrate certain moist atmospheric processes, and the energy–area equivalence is not an important consideration. Because of the simplicity of its construction, we use the Stuve diagram (sometimes referred to as a *pseudo-adiabatic chart*) to illustrate the utility of aerological diagrams in understanding moist thermodynamic processes.

The construction of the pseudo-adiabatic chart is illustrated in Figure 6.6 (see also Appendix E). The temperature scale is linear, while the pressure scale is proportional to p^{R/c_p}. From (2.62), it is easily seen that the dry adiabats or lines of constant potential temperature are straight lines. Pseudo-adiabats (θ_e = constant), are shown by the curved dashed lines. Lines of constant saturated water vapor mixing ratio (w_s = constant) are given by the thin solid lines in Figure 6.6. The ordinate p^{R/c_p} can be interpreted in terms of altitude, z, using (1.45). The use of the pseudo-adiabatic chart is illustrated with the following examples.

Figure 6.7 illustrates vertical profiles of temperature and dew-point temperature plotted on an aerological diagram. Such observations are obtained using balloons, aircraft or remote sensing. From the definition of dew-point temperature (6.15), it is easily seen that by reading off the saturation mixing ratio at the dew-point temperature at a given level on the diagram, one obtains the actual water vapor mixing ratio. Conversely, if the mixing ratio is given, the dew-point temperature may be read off the diagram.

The adiabatic ascent of a parcel from the surface is represented schematically in Figure 6.8. Consider a parcel with $p = p_0$, $T = T_0$, and $w_v = w_0$. The potential temperature of this parcel corresponds to the value of the dry adiabat that passes through T_0,

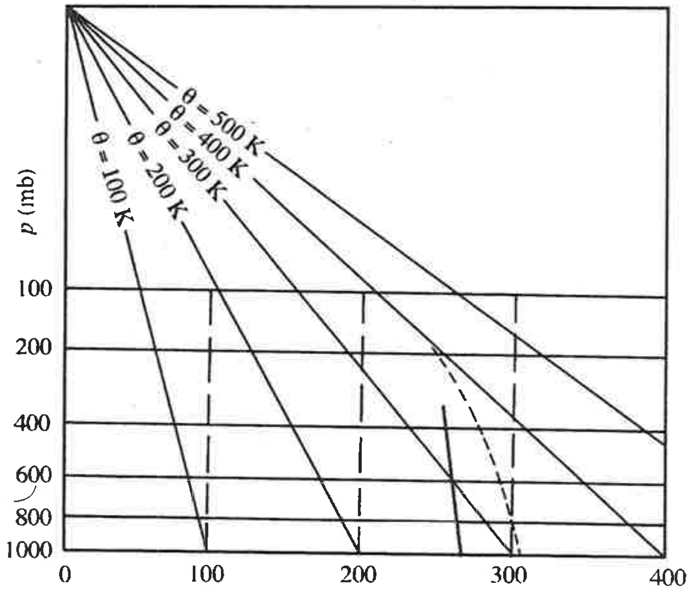


Figure 6.6 Construction of the pseudo-adiabatic chart.

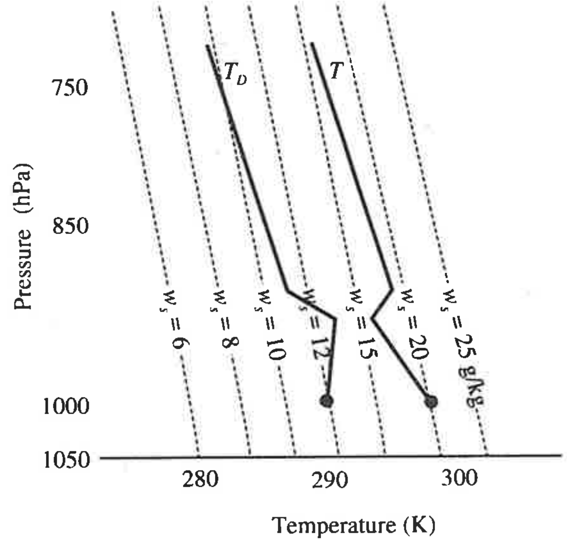


Figure 6.7 Determination of w, w_s, and T_D given the vertical profiles of temperature and dew-point temperature.

p_0. In adiabatic ascent, the parcel will be lifted dry adiabatically along an isopleth of constant θ, that passes through p_0, T_0. In this ascent, the temperature and saturation mixing ratio decrease while the actual mixing ratio remains the same. The level where the saturation mixing ratio equals the actual mixing ratio (the intersection of the constant θ line with the constant w_0 line) corresponds to T_s, p_s, z_s; the saturation temperature and pressure and the lifting condensation level. The thermodynamic properties of air that continues to ascend above the saturation point is found by following the pseudo-adiabat (line of constant θ_e) that passes through T_s, p_s. The mixing ratio of the parcel in pseudo-adiabatic ascent corresponds to the saturation mixing ratio at that level (the intersection of the pseudo-adiabat that passes through T_s, p_s with the constant mixing ratio line). The adiabatic liquid water content at a given level above the saturation point is approximated by subtracting the saturation mixing ratio from the original mixing ratio, w_0.

The equivalent potential temperature, θ_e, corresponding to T_0, p_0, is determined by following the pseudo-adiabat through T_s, p_s to very low pressure, until the pseudo-adiabat is essentially parallel to the dry adiabat. By following the dry adiabat down to a pressure of p_0 and reading off the corresponding temperature, the equivalent temperature, T_e, is obtained; by continuing to follow this dry adiabat down to $p = 1000$ mb, the equivalent potential temperature, θ_e, is obtained. The *equivalent temperature* is related to the equivalent potential temperature analogously to (2.62) as

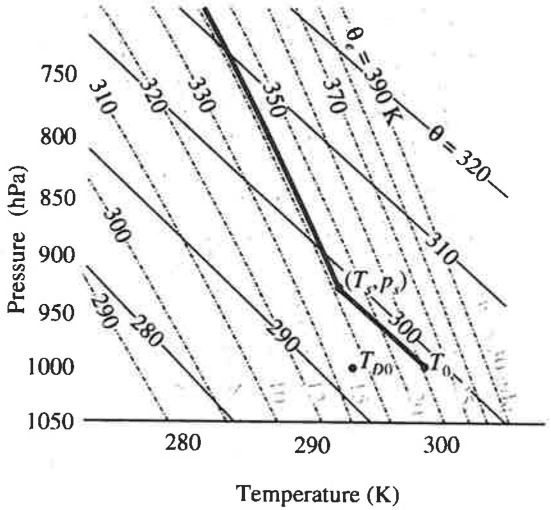


Figure 6.8 Adiabatic ascent of a parcel from p_0. The parcel initially ascends dry adiabatically along the constant potential temperature line that passes through (T_0, 1000 hPa). As the parcel ascends, the saturation mixing ratio decreases while the actual mixing ratio remains the same. At the point at which the actual mixing ratio of the parcel is equal to the saturation mixing ratio, the parcel becomes saturated. Further lifting of the parcel occurs along the saturated adiabat that passes through the point, (T_s, p_s).

$$\theta_e = T_e\left(\frac{p_0}{p}\right)^{R_d/c_p}$$

The wet-bulb temperature, T_W, can be approximated by following the pseudo-adiabat that passes through p_s, T_s down to the level of p_0 and reading the corresponding temperature. By continuing to follow this pseudo-adiabat down to $p = 1000$ mb, the wet-bulb potential temperature, θ_W, is determined. Note that while the pseudo-adiabatic wet-bulb temperature is almost numerically equivalent to the adiabatic isobaric wet-bulb temperature defined in Section 6.3, they are slightly different. In the case of the pseudo-adiabatic wet-bulb temperature, water is evaporated into the air through pseudo-adiabatic descent, while water is evaporated isobarically in the atmosphere in the determination of the adiabatic isobaric wet-bulb temperature.

While aerological diagrams are useful for illustrating schematically the results of thermodynamic transformations of moist air, their use as a computational tool has been superseded by computers.

Notes

General reference sources for this chapter include *Atmospheric Thermodynamics* (1981, Chapters IV and VII) by Iribarne and Godson, *Atmospheric Convection* (1994, Chapter 4) by Emanuel, *Clouds and Storms* (1980, Chapter 3) by Ludlam, and *The Ceaseless Wind* (1986, Chapter 4) by Dutton.

A more detailed discussion of aerological diagrams is given in *Atmospheric Thermodynamics* (1981, Chapter VI) by Iribarne and Godson.

Problems

1. For a pressure of 1000 mb, determine the following. You may use the e_s table in Appendix D. Given:
a) $w_s = 5$ g kg^{-1}, find T.
b) $T = 25°$C, find w_s.
c) $T = 30°$C and $w = 15$ g kg^{-1}, find $\mathcal{H}$.
d) $T = 20°$C and $T_D = 15$, find $\mathcal{H}$.
e) $T = 15°$C and $\mathcal{H} = 0.8$, find T_D.
f) $w = 20$ g kg^{-1}, find T_D.
g) $T_F = -10°$C, find T_D.

2. Consider a 1 kg parcel of moist air at $p = 1000$ mb, $T = 30°$C and $\mathcal{H} = 0.95$. The parcel passes over a cold ocean so that the parcel cools to 25°C. Assume that only heat (no moisture) is transferred between the ocean and the parcel.
a) What is the initial vapor pressure and mixing ratio of the parcel?
b) What is the dew-point temperature?
c) How much water condenses?

3. During the formation of a radiation fog, 4000 J kg^{-1} is lost after saturation started, at 10°C. The pressure is 1000 mb. Estimate the following:
a) final temperature;
b) vapor pressure;
c) liquid mixing ratio.

4. Home humidifiers, or "swamp coolers," operate by evaporating water into the air in the house, and thereby raise its relative humidity. Consider a house having a volume of 200 m^3 in which the air temperature is initially 21°C and the relative humidity is 10%. Compute the amount of water that must be evaporated to raise the relative humidity to 60%. Assume a constant pressure process at 1010 hPa in which the heat required for evaporation is supplied by the air itself.