

4

Applications of the Equations of Motion

Objectives

In the previous chapter we derived expressions for the equations of motion as well as the continuity of mass. The present chapter will concentrate on illustrating various simple applications of these equations to observable phenomena in the mid-latitude atmosphere. Many of the applications of the equations of motion, in particular, will be greatly simplified by first recasting the expressions in one of three new Cartesian coordinate systems. The first two use pressure (p) or potential temperature (θ) as the vertical coordinate and are known as **isobaric** or **isentropic coordinates**, respectively. Upon transforming the equations to these new coordinate systems we will illustrate the utility of each with some examples. Next, by combining the geostrophic and hydrostatic balances, we will find that the vertical shear of the geostrophic wind is directly linked to the magnitude of the horizontal temperature gradient in the **thermal wind relationship**.

Finally, we shall adopt a third new Cartesian coordinate system, known as **natural coordinates**, in which we employ the direction of the flow at each point in the fluid as the basis for defining the horizontal directions. We will use this natural coordinate perspective to examine a number of additional balanced flows including inertial, cyclostrophic, and gradient balances. Applications of each of these balances will also be considered. We begin by transforming the equations of motion and the continuity equation into isobaric coordinates.

4.1 Pressure as a Vertical Coordinate

Recall that in height coordinates, the vector form of the frictionless horizontal momentum equation is

$$\frac{d\vec{V}}{dt} = -\frac{1}{\rho}\nabla p - f\hat{k} \times \vec{V}.$$

In order to recast this expression in isobaric coordinates we must convert the pressure gradient force term into an equivalent expression in isobaric coordinates. This is done most easily by considering the differential (dp) on a constant pressure surface:

$$(dp)_p = \left(\frac{\partial p}{\partial x}\right)_{y,z} dx_p + \left(\frac{\partial p}{\partial y}\right)_{x,z} dy_p + \left(\frac{\partial p}{\partial z}\right)_{x,y} dz_p \quad (4.1a)$$

where the subscripts indicate differentiation carried out holding the subscripted variable constant. Now, since there is no change in pressure on an isobaric (i.e. constant pressure) surface, then $(dp)_p = 0$ so that

$$0 = \left(\frac{\partial p}{\partial x}\right)_{y,z} dx_p + \left(\frac{\partial p}{\partial y}\right)_{x,z} dy_p + \left(\frac{\partial p}{\partial z}\right)_{x,y} dz_p. \quad (4.1b)$$

Next, we expand dz_p as a function of x and y to yield

$$0 = \left(\frac{\partial p}{\partial x}\right)_{y,z} dx_p + \left(\frac{\partial p}{\partial y}\right)_{x,z} dy_p + \left(\frac{\partial p}{\partial z}\right)_{x,y} \left[\left(\frac{\partial z}{\partial x}\right)_{y,p} dx_p + \left(\frac{\partial z}{\partial y}\right)_{y,p} dy_p \right]$$

which can be rearranged into

$$\begin{aligned} 0 = & \left[\left(\frac{\partial p}{\partial x}\right)_{y,z} + \left(\frac{\partial p}{\partial z}\right)_{x,y} \left(\frac{\partial z}{\partial x}\right)_{y,p} \right] dx_p \\ & + \left[\left(\frac{\partial p}{\partial y}\right)_{x,z} + \left(\frac{\partial p}{\partial z}\right)_{x,y} \left(\frac{\partial z}{\partial y}\right)_{y,p} \right] dy_p. \end{aligned} \quad (4.1c)$$

Since this statement is true for all dx and dy , the terms in square brackets in (4.1c) must both be zero. Hence,

$$\left(\frac{\partial p}{\partial x}\right)_{y,z} = - \left(\frac{\partial p}{\partial z}\right)_{x,y} \left(\frac{\partial z}{\partial x}\right)_{y,p} \quad \text{and} \quad \left(\frac{\partial p}{\partial y}\right)_{x,z} = - \left(\frac{\partial p}{\partial z}\right)_{x,y} \left(\frac{\partial z}{\partial y}\right)_{y,p}. \quad (4.1d)$$

With the help of the hydrostatic equation, these expressions become

$$- \left(\frac{\partial p}{\partial x}\right)_{y,z} = -\rho g \left(\frac{\partial z}{\partial x}\right)_{y,p} \quad \text{and} \quad - \left(\frac{\partial p}{\partial y}\right)_{x,z} = -\rho g \left(\frac{\partial z}{\partial y}\right)_{y,p}. \quad (4.1e)$$

Dividing both sides of (4.1e) by ρ yields

$$\begin{aligned} - \frac{1}{\rho} \left(\frac{\partial p}{\partial x}\right)_z &= -g \left(\frac{\partial z}{\partial x}\right)_p = - \left(\frac{\partial \phi}{\partial x}\right)_p \\ - \frac{1}{\rho} \left(\frac{\partial p}{\partial y}\right)_z &= -g \left(\frac{\partial z}{\partial y}\right)_p = - \left(\frac{\partial \phi}{\partial y}\right)_p \end{aligned}$$

where, for convenience, we have dropped the subscripts x and y on the LHS derivatives. The LHS expressions represent the height coordinate versions of the

x - and y -direction pressure gradient force terms. Thus, the RHS expressions represent the x - and y -direction pressure gradient force terms in isobaric coordinates. Therefore, the isobaric coordinate expression for the pressure gradient force in vector form is

$$P G F_p = -\nabla_p \phi \quad (4.2)$$

where

$$\nabla_p = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j}.$$

The isobaric coordinate form of the pressure gradient force involves no reference to density and is therefore much more amenable to operational use. The removal of density from the expression for the pressure gradient force is a major advantage of isobaric coordinates and provides the motivation for their use.

With the result in (4.2), the vector form of the horizontal equation of motion can be rewritten as

$$\frac{d\vec{V}}{dt} = -\nabla_p \phi - f \hat{k} \times \vec{V} \quad (4.3)$$

where, importantly,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \hat{i} + v \frac{\partial}{\partial y} \hat{j} + \omega \frac{\partial}{\partial p} \hat{k}.$$

The component velocity in the last term, ω , is equal to

$$\omega = \frac{dp}{dt} \quad (4.4)$$

and is a measure of vertical velocity in units of hPa s^{-1} (or, more commonly in operations, $\mu\text{bar s}^{-1}$, where $1 \mu\text{b} = 10^{-3} \text{ hPa}$).

By neglecting the horizontal acceleration vector, a new expression for the geostrophic balance arises from (4.3): namely,

$$f \hat{k} \times \vec{V} = -\nabla_p \phi. \quad (4.4a)$$

Taking $-\hat{k} \times$ (4.4a) and dividing by f on both sides yields an expression for the geostrophic wind in isobaric coordinates,

$$\vec{V}_g = \frac{\hat{k}}{f} \times \nabla_p \phi. \quad (4.4b)$$

Without reference to ρ , (4.4b) provides a much simpler expression for calculating the geostrophic wind from observations. The simplicity is illustrated in Figure 4.1 which shows an example of the 500 hPa geopotential height contours and actual wind vectors from the middle latitudes. The geostrophic wind is parallel to the geopotential height contours with a magnitude dependent on the magnitude of $\nabla_p \phi$. For the most

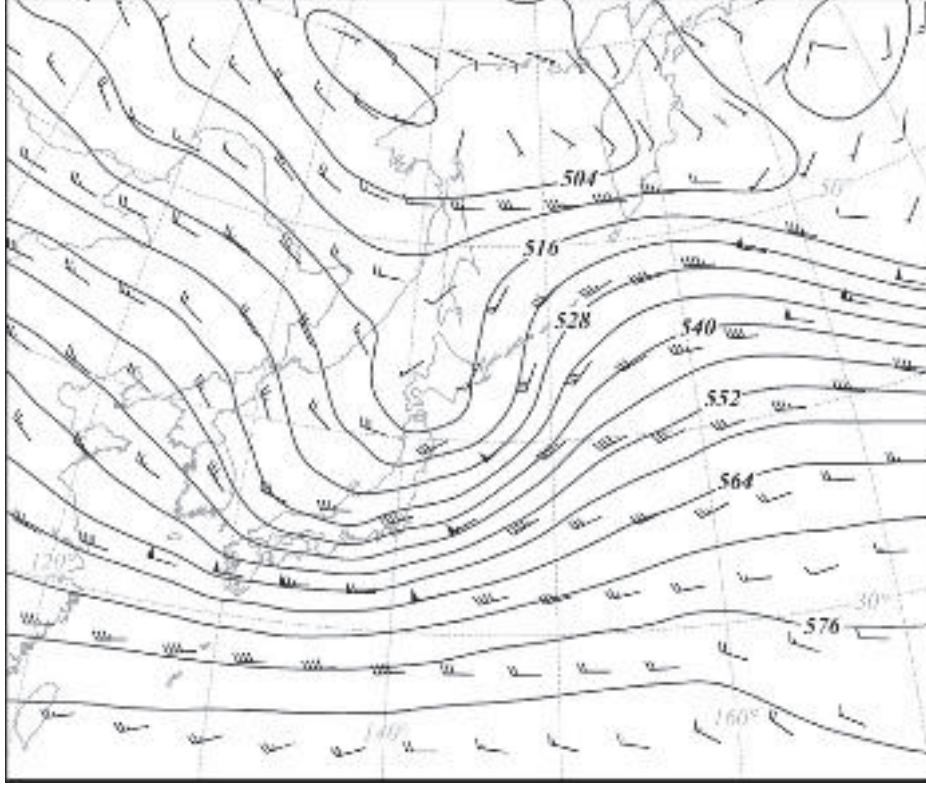


Figure 4.1 The 500 hPa geopotential height and winds at 0000 UTC 23 February 2004. Geopotential height is labeled in dam (deca meters) and contoured every 6 dam. Winds are represented as flags pointing into the direction from which the wind is coming. Speed of the winds is indicated by the barbs on the flags as: half barb, less than 5 m s^{-1} ; full barb, 5 m s^{-1} ; pennant, 25 m s^{-1}

part, the actual wind is close to the geostrophic wind with notable exceptions being in regions of wind speed maxima and strong curvature.

Another consequence of the simplicity of (4.4b) is that, so long as we consider f to be constant, then

$$\begin{aligned}
 \nabla \cdot \vec{V}_g &= \nabla \cdot \left(\frac{\hat{k}}{f} \times \nabla_p \phi \right) \\
 &= \nabla \cdot \left(-\frac{1}{f} \frac{\partial \phi}{\partial y} \hat{i} + \frac{1}{f} \frac{\partial \phi}{\partial x} \hat{j} \right) \\
 &= \frac{1}{f} \left[\frac{\partial}{\partial x} \left(-\frac{\partial \phi}{\partial y} \right) + \frac{\partial}{\partial y} \left(\frac{\partial \phi}{\partial x} \right) \right] \\
 &= \frac{1}{f} \left(-\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial x \partial y} \right) = 0
 \end{aligned} \tag{4.5}$$

so that the geostrophic wind is non-divergent. This is an extremely important property of the geostrophic wind and its importance for understanding mid-latitude weather systems will be amplified when we examine the continuity equation in isobaric coordinates.

Rather than transform the continuity equation from height to pressure coordinates, we will derive the isobaric form of the continuity equation by considering a control volume ($\delta V = \delta x \delta y \delta z$) as before. Using the hydrostatic equation ($\delta p = -\rho g \delta z$) we can rewrite δV as

$$\delta V = \frac{-\delta x \delta y \delta p}{\rho g}.$$

Let us adopt the Lagrangian perspective that the mass of the control volume (given by $\delta M = \rho \delta V = -\delta x \delta y \delta p / g$) does not change following the parcel. Then the rate of change of mass (per unit mass) is given by

$$\frac{1}{\delta M} \frac{d(\delta M)}{dt} = 0 = \frac{-g}{\delta x \delta y \delta p} \frac{d}{dt} \left(\frac{-\delta x \delta y \delta p}{g} \right). \quad (4.6a)$$

Applying the chain rule to the RHS of (4.6a) yields

$$\frac{1}{\delta x \delta y \delta p} \left[\frac{d(\delta x)}{dt} \delta y \delta p + \frac{d(\delta y)}{dt} \delta x \delta p + \frac{d(\delta p)}{dt} \delta x \delta y \right] = 0. \quad (4.6b)$$

Since, as we saw before,

$$\frac{d(\delta x)}{dt} = \delta u, \quad \frac{d(\delta y)}{dt} = \delta v, \quad \text{and} \quad \frac{d(\delta p)}{dt} = \delta \omega,$$

(4.6b) can be simplified to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0 \quad (4.7)$$

as δx , δy , and δp approach zero. This is the isobaric form of the continuity equation. This form of the continuity equation is much simpler than the height coordinate version ((3.46) and (3.47)) since, similar to the isobaric expression for the pressure gradient force, density does not appear in it.

A simple rearrangement of (4.7) produces

$$\nabla \cdot \vec{V}_h = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial \omega}{\partial p} \quad (4.8a)$$

which relates the fact that the horizontal divergence on an isobaric surface is directly related to the vertical motion (ω), a variable of exceptional importance in creating the sensible weather. If we know the vertical (p -direction) distribution of horizontal divergence in an atmospheric column, then we can determine the vertical motion distribution in that column as well. Consider the hypothetical situation depicted in Figure 4.2 in which horizontal convergence of air occurs near the surface ($\nabla \cdot \vec{V}_h < 0$)

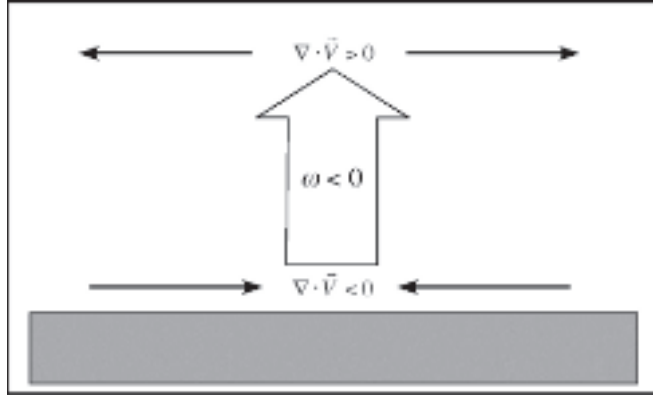


Figure 4.2 Illustration of the vertical distribution of divergence associated with upward vertical motion. The divergence values are measured on isobaric surfaces and $\omega < 0$ corresponds to ascent

and horizontal divergence of air occurs at the top of the column ($\nabla \cdot \vec{V}_h > 0$). In accord with the continuity of mass, such a circumstance must be accompanied by upward vertical motion in the intervening column of air. Integrating (4.8a) with respect to pressure yields

$$\int_{p_s}^{p_t} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dp = - \int_{p_s}^{p_t} \partial \omega \quad (4.8b)$$

or

$$(\nabla \cdot \vec{V}_h)_{p_t} - (\nabla \cdot \vec{V}_h)_{p_s} = -[\omega_{p_t} - \omega_{p_s}]. \quad (4.8c)$$

There can be no vertical motion precisely at ground level, so $\omega_{p_s} = 0$. Since $(\nabla \cdot \vec{V}_h)_{p_t} - (\nabla \cdot \vec{V}_h)_{p_s} > 0$, we find that $\omega_{p_t} < 0$ (i.e. there is upward vertical motion at the top of the hypothetical column) as we suspected.

Another useful physical insight arises from (4.8a) by considering the horizontal wind as the sum of its geostrophic and ageostrophic components. By substituting $\vec{V}_h = \vec{V}_g + \vec{V}_{ag}$ into (4.8a) we get

$$\nabla \cdot \vec{V}_h = \nabla \cdot (\vec{V}_g + \vec{V}_{ag}) = \nabla \cdot \vec{V}_g + \nabla \cdot \vec{V}_{ag} = -\frac{\partial \omega}{\partial p}. \quad (4.9a)$$

Recall from (4.5) that so long as f is constant, the geostrophic wind is non-divergent so that (4.9a) becomes

$$\nabla \cdot \vec{V}_{ag} = -\frac{\partial \omega}{\partial p} \quad (4.9b)$$

which states that the divergence of the ageostrophic wind determines the distribution of vertical motion in the atmosphere. Thus, it is the ageostrophic wind that is entirely responsible for the distribution of cyclones, anticyclones, clouds, and precipitation

in the atmosphere. The ramifications of this statement are profound. Despite the fact that the mid-latitude atmosphere is predominantly in geostrophic balance, all of the important weather with which we are confronted develops as a direct result of the often relatively small ageostrophic portion of the wind.

Finally, we can quite easily express the thermodynamic energy equation in isobaric coordinates by writing the first law of thermodynamics in pressure coordinates as

$$c_p \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + \omega \frac{\partial T}{\partial p} \right) - \alpha \omega = \dot{Q}. \quad (4.10a)$$

Rearranging the LHS and dividing by c_p yields

$$\left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) - \left(\frac{\alpha}{c_p} - \frac{\partial T}{\partial p} \right) \omega = \frac{\dot{Q}}{C_p} \quad (4.10b)$$

which can be rewritten as

$$\left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) - \sigma_p \omega = \frac{\dot{Q}}{C_p} \quad (4.10c)$$

where

$$\sigma_p = \left(\frac{\alpha}{c_p} - \frac{\partial T}{\partial p} \right)$$

is a measure of the static stability in isobaric coordinates. If an atmospheric flow is assumed to be (1) adiabatic ($\dot{Q} = 0$), (2) steady state ($\partial T / \partial t = 0$), and (3) stably stratified ($\sigma_p > 0$), then (4.10c) can be written in a physically illuminating manner as

$$\frac{(-\vec{V}_h \cdot \nabla T)}{-\sigma_p} = \omega. \quad (4.11)$$

This expression states that the horizontal temperature advection is related to the vertical motion such that warm (cold) air advection is associated with upward (downward) vertical motions. The cyclone depicted in Figure 4.3 illustrates that these relationships, though based upon some troubling assumptions (most notably the steady-state assumption), do tend to be observed in the real mid-latitude atmosphere. For this reason meteorologists are often very interested in the sign of the horizontal temperature advection.

4.2 Potential Temperature as a Vertical Coordinate

Though adopting pressure as a vertical coordinate simplifies a number of the basic equations by removing reference to density, air parcels are no more constrained to remain on an isobaric surface than they are to remain on a geometric height surface. In many applications it is desirable to choose potential temperature (θ) as the vertical coordinate since (1) for statically stable stratifications, θ is a monotonic