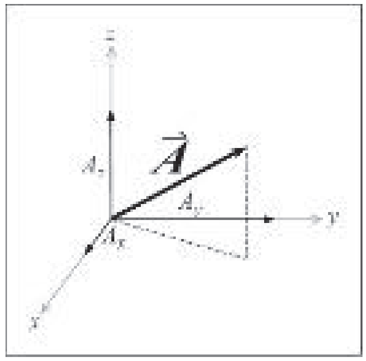

Figure 1.1 The 3-D representation of a vector, $\vec{A}$. The components of $\vec{A}$ are shown along the coordinate axes

1.2.1 Elements of vector calculus

Many physical quantities with which we are concerned in our experience of the universe are described entirely in terms of *magnitude*. Examples of these types of quantities, known as **scalars**, are area, volume, money, and snowfall total. There are other physical quantities such as velocity, the force of gravity, and slopes to topography which are characterized by both magnitude and direction. Such quantities are known as **vectors** and, as you might guess, any description of the fluid atmosphere necessarily contains reference to both scalars and vectors. Thus, it is important that we familiarize ourselves with the mathematical descriptions of these quantities, a formalism known as vector analysis.[2]

Employing a Cartesian coordinate system in which the three directions (x, y, and z) are mutually orthogonal (i.e. perpendicular to one another), an arbitrary vector, $\vec{A}$, has components in the x, y, and z directions labeled A_x, A_y, and A_z, respectively. These components themselves are scalars since they describe the magnitude of vectors whose directions are given by the coordinate axes (as shown in Figure 1.1). If we denote the direction vectors in the x, y, and z directions as $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, respectively (where the $\hat{}$ symbol indicates the fact that they are vectors with magnitude 1 in the respective directions – so-called **unit vectors**), then

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k} \tag{1.1a}$$

is the component form of the vector, $\vec{A}$. In a similar manner, the component form of an arbitrary vector $\vec{B}$ is given by

$$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}. \tag{1.1b}$$

---

[2] Vector analysis is generally considered to have been invented by the Irish mathematician Sir William Rowan Hamilton in 1843. Despite its enormous value in the physical sciences, vector analysis was met with skepticism in the nineteenth century. In fact, Lord Kelvin wrote, in the 1890s, that vectors were 'an unmixed evil to those who have touched them in any way . . vectors . . have never been of the slightest use to any creature'. Remember, no matter how great a thinker one may be, one cannot always be right!

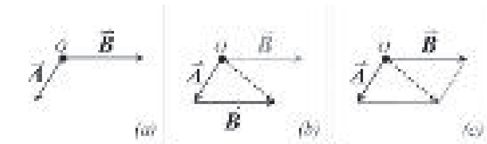

Figure 1.2 (a) Vectors $\vec{A}$ and $\vec{B}$ acting upon a point O. (b) Illustration of the tail-to-head method for adding vectors $\vec{A}$ and $\vec{B}$. (c) Illustration of the parallelogram method for adding vectors $\vec{A}$ and $\vec{B}$

The vectors $\vec{A}$ and $\vec{B}$ are equal if $A_x = B_x$, $A_y = B_y$, and $A_z = B_z$. Furthermore, the magnitude of a vector $\vec{A}$ is given by

$$\left|\vec{A}\right| = \left(A_x^2 + A_y^2 + A_z^2\right)^{1/2} \tag{1.2}$$

which is simply the 3-D Pythagorean theorem and can be visually verified with the aid of Figure 1.1.

Vectors can be added to and subtracted from one another both by graphical methods as well as by components. Graphical addition is illustrated with the aid of Figure 1.2. Imagine that the force vectors $\vec{A}$ and $\vec{B}$ are acting at point O as shown in Figure 1.2(a). The total force acting at O is equal to the sum of $\vec{A}$ and $\vec{B}$. Graphical construction of the vector sum $\vec{A} + \vec{B}$ can be accomplished either by using the tail-to-head method or the parallelogram method. The tail-to-head method involves drawing $\vec{B}$ at the head of $\vec{A}$ and then connecting the tail of $\vec{A}$ to the head of the redrawn $\vec{B}$ (Figure 1.2b). Alternatively, upon constructing a parallelogram with sides $\vec{A}$ and $\vec{B}$, the diagonal of the parallelogram between $\vec{A}$ and $\vec{B}$ represents the vector sum, $\vec{A} + \vec{B}$ (Figure 1.2c).

If we know the component forms of both $\vec{A}$ and $\vec{B}$, then their sum is given by

$$\vec{A} + \vec{B} = (A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath} + (A_z + B_z)\hat{k}. \tag{1.3a}$$

Thus, the sum of $\vec{A}$ and $\vec{B}$ is found by simply adding like components together. It is clear from considering the component form of vector addition that addition of vectors is commutative ($\vec{A} + \vec{B} = \vec{B} + \vec{A}$) and associative ($(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$).

Subtraction is simply the opposite of addition so $\vec{B}$ can be subtracted from $\vec{A}$ by simply adding $-\vec{B}$ to $\vec{A}$. Graphical subtraction of $\vec{B}$ from $\vec{A}$ is illustrated in Figure 1.3. Notice that $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$ results in a vector directed from the head of $\vec{B}$ to the head of $\vec{A}$ (the lighter dashed arrow in Figure 1.3). Component subtraction involves

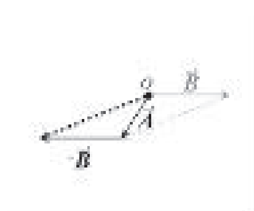

Figure 1.3 Graphical subtraction of vector $\vec{B}$ from vector $\vec{A}$

Figure 1.4 (a) Vectors $\vec{A}$ and $\vec{B}$ with an angle α between them. (b) Illustration of the relationship between vectors $\vec{A}$ and $\vec{B}$ (gray arrows) and their cross-product, $\vec{A}\times\vec{B}$ (bold arrow). Note that $\vec{A}\times\vec{B}$ is perpendicular to both $\vec{A}$ and $\vec{B}$

subtracting like components and is given by

$$\vec{A}-\vec{B}=(A_x-B_x)\hat{\imath}+(A_y-B_y)\hat{\jmath}+(A_z-B_z)\hat{k}. \tag{1.3b}$$

Vector quantities may also be multiplied in a variety of ways. The simplest vector multiplication involves the product of a vector, $\vec{A}$, and a scalar, F. The resulting expression for $F\vec{A}$ is given by

$$F\vec{A}=FA_x\hat{\imath}+FA_y\hat{\jmath}+FA_z\hat{k}, \tag{1.4}$$

a vector with direction identical to the original vector, $\vec{A}$, but with a magnitude F times larger than the original magnitude.

It is also possible to multiply two *vectors* together. In fact, there are two different vector multiplication operations. One such method renders a scalar as the product of the vector multiplication and is thus known as the **scalar** (or **dot**) product. The dot product of the vectors $\vec{A}$ and $\vec{B}$ shown in Figure 1.4(a) is given by

$$\vec{A}\cdot\vec{B}=|A|\,|B|\cos\alpha \tag{1.5}$$

where α is the angle between $\vec{A}$ and $\vec{B}$. Clearly this product is a scalar. Using this formula, we can determine a less mystical form of the dot product of $\vec{A}$ and $\vec{B}$. Given that $\vec{A}=A_x\hat{\imath}+A_y\hat{\jmath}+A_z\hat{k}$ and $\vec{B}=B_x\hat{\imath}+B_y\hat{\jmath}+B_z\hat{k}$, the dot product is given by

$$\vec{A}\cdot\vec{B}=(A_x\hat{\imath}+A_y\hat{\jmath}+A_z\hat{k})\cdot(B_x\hat{\imath}+B_y\hat{\jmath}+B_z\hat{k}) \tag{1.6}$$

which expands to the following nine terms:

$$\begin{aligned}\vec{A}\cdot\vec{B}=\;&A_xB_x(\hat{\imath}\cdot\hat{\imath})+A_xB_y(\hat{\imath}\cdot\hat{\jmath})+A_xB_z(\hat{\imath}\cdot\hat{k})\\&+A_yB_x(\hat{\jmath}\cdot\hat{\imath})+A_yB_y(\hat{\jmath}\cdot\hat{\jmath})+A_yB_z(\hat{\jmath}\cdot\hat{k})\\&+A_zB_x(\hat{k}\cdot\hat{\imath})+A_zB_y(\hat{k}\cdot\hat{\jmath})+A_zB_z(\hat{k}\cdot\hat{k}).\end{aligned}$$

Now, according to (1.5), $\hat{\imath}\cdot\hat{\imath}=\hat{\jmath}\cdot\hat{\jmath}=\hat{k}\cdot\hat{k}=1$ since the angle between like unit vectors is 0°. However, the dot products of all other combinations of the unit vectors are zero since the unit vectors are mutually orthogonal. Thus, only three terms survive out of the nine-term expansion of $\vec{A}\cdot\vec{B}$ to yield

$$\vec{A}\cdot\vec{B}=A_xB_x+A_yB_y+A_zB_z. \tag{1.7}$$

Given this result, it is easy to show that the dot product is commutative ($\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$) and distributive ($\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$).

Two vectors can also be multiplied together to produce another vector. This vector multiplication operation is known as the **vector** (or **cross-**)product and is signified

$$\vec{A} \times \vec{B}.$$

The magnitude of the resultant vector is given by

$$|A|\,|B| \sin\alpha \tag{1.8}$$

where α is the angle between the vectors. Note that since the resultant of the cross-product is a vector, there is also a direction to be discerned. The resultant vector is in a plane that is perpendicular to the plane that contains $\vec{A}$ and $\vec{B}$ (Figure 1.4b). The direction in that plane can be determined by using the **right hand rule**. Upon curling the fingers of one's right hand in the direction from $\vec{A}$ to $\vec{B}$, the thumb points in the direction of the resultant vector, $\vec{A} \times \vec{B}$, as shown in Figure 1.4(b). Because the resultant direction depends upon the order of multiplication, the cross-product has different properties than the dot product. It is not commutative ($\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$; instead $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$) and it is not associative ($\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$) but it is distributive ($\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$).

Given the vectors $\vec{A}$ and $\vec{B}$ in their component forms, the cross-product can be calculated by first setting up a 3×3 determinant using the unit vectors as the first row, the components of $\vec{A}$ as the second row, and the components of $\vec{B}$ as the third row:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}. \tag{1.9a}$$

Evaluating this determinant involves evaluating three 2×2 determinants, each one corresponding to a unit vector $\hat{i}$, $\hat{j}$, *or* $\hat{k}$. For the $\hat{i}$ component of the resultant vector, only the components of $\vec{A}$ and $\vec{B}$ in the $\hat{j}$ and $\hat{k}$ columns are considered. Multiplying the components along the diagonal (upper left to lower right) first, and then subtracting from that result the product of the terms along the anti-diagonal (lower left to upper right) yields the $\hat{i}$ component of the vector $\vec{A} \times \vec{B}$, which equals $(A_y B_z - A_z B_y)\hat{i}$. The same operation done for the $\hat{k}$ component yields $(A_x B_y - A_y B_x)\hat{k}$. For the $\hat{j}$ component, the first and third columns are used to form the 2×2 determinant and since the columns are non-consecutive, the result must be multiplied by –1 to yield $-(A_x B_z - A_z B_x)\hat{j}$. Adding these three components together yields

$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{i} + (A_z B_x - A_x B_z)\hat{j} + (A_x B_y - A_y B_x)\hat{k}. \tag{1.9b}$$

Vectors, just like scalar functions, can be differentiated as long as the rules of vector addition and multiplication are obeyed. One simple example is Newton's second law (which we will see again soon) that states that an object's momentum will not change unless a force is applied to the object. In mathematical terms,

$$\vec{F} = \frac{d}{dt}(m\vec{V}) \tag{1.10}$$

where m is the object's mass and $\vec{V}$ is its velocity. Using the chain rule of differentiation on the right hand side of (1.10) renders

$$\vec{F} = m\frac{d\vec{V}}{dt} + \vec{V}\frac{dm}{dt} \text{ or } \vec{F} = m\vec{A} + \vec{V}\frac{dm}{dt} \tag{1.11}$$

where $\vec{A}$ is the object's acceleration. Exploitation of the second term of this expansion is what made Einstein famous!

Let us consider a more general example. Consider a velocity vector defined as $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$. In such a case, the acceleration will be given by

$$\frac{d\vec{V}}{dt} = \frac{du}{dt}\hat{i} + u\frac{d\hat{i}}{dt} + \frac{dv}{dt}\hat{j} + v\frac{d\hat{j}}{dt} + \frac{dw}{dt}\hat{k} + w\frac{d\hat{k}}{dt}. \tag{1.12}$$

The terms involving derivatives of the unit vectors may seem like mathematical baggage but they will be extremely important in our subsequent studies. Physically, such terms will be non-zero only when the coordinate axes used to reference motion are not fixed in space. Our reference frame on a rotating Earth is clearly not fixed and so we will eventually have to make some accommodation for the acceleration of our rotating reference frame. Thus, all six terms in the above expansion will be relevant in our examination of the mid-latitude atmosphere.

The last stop on the review of vector calculus is perhaps the most important one and will examine a tool that is extremely useful in fluid dynamics. We will often need to describe both the magnitude and direction of the derivative of a scalar field. In order to do so we employ a mathematical operator known as the **del operator**, defined as

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}. \tag{1.13}$$

If we apply this partial differential del operator to a scalar function or field, the result is a vector that is known as the **gradient** of that scalar. Consider the 2-D plan view of an isolated hill in an otherwise flat landscape. If the elevation at each point in the landscape is represented on a 2-D projection, a set of elevation contours results as shown in Figure 1.5. Such contours are lines of equal height above sea level, Z. Given such information, we can determine the gradient of elevation, ∇Z, as

$$\nabla Z = \frac{\partial Z}{\partial x}\hat{i} + \frac{\partial Z}{\partial y}\hat{j}.$$

Note that the gradient vector, ∇Z, points up the hill *from low values of elevation to high values.* At the top of the hill, the derivatives of Z in both the x and y

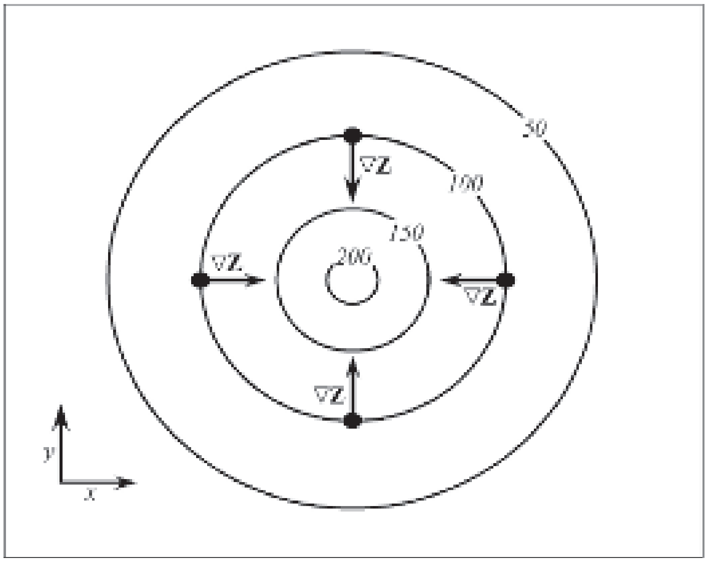

Figure 1.5 The 2-D plan view of an isolated hill in a flat landscape. Solid lines are contours of elevation (Z) at 50m intervals. Note that the gradient of Z points from low to high values of the scalar Z

directions are zero so there is no gradient vector there. Thus the gradient, ∇Z, not only measures magnitude of the elevation difference but assigns that magnitude a direction as well. Any scalar quantity, Φ, is transformed into a vector quantity, $\nabla\Phi$, by the del operator. In subsequent chapters in this book we will concern ourselves with the gradients of a number of scalar variables, among them temperature and pressure.

The del operator may also be applied to vector quantities. The dot product of ∇ with the vector $\vec{A}$ is written as

$$\nabla \cdot \vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})$$

$$\nabla \cdot \vec{A} = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) \tag{1.14}$$

which is a scalar quantity known as the **divergence of** $\vec{A}$. Positive divergence physically describes the tendency for a vector field to be directed away from a point whereas negative divergence (also known as **convergence**) describes the tendency for a vector field to be directed toward a point. Regions of convergence and divergence in the atmospheric fluid are extremely important in determining its behavior.

The cross-product of ∇ with the vector $\vec{A}$ is given by

$$\nabla \times \vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}). \tag{1.15a}$$

The resulting vector can be calculated using the determinant form we have seen previously,

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \tag{1.15b}$$

where the second row of the 3×3 determinant is filled by the components of ∇ and the third row is filled by the components of $\vec{A}$. This vector is known as the **curl of** $\vec{A}$. The curl of the velocity vector, $\vec{V}$, will be used to define a quantity called **vorticity** which is a measure of the rotation of a fluid.

Quite often in a study of the dynamics of the atmosphere, we will encounter second-order partial differential equations. Some of these equations will contain a mathematical operator (which will operate on scalar quantities) known as the **Laplacian** operator. The Laplacian is the **divergence of the gradient** and so takes the form

$$\textit{Laplacian} = \nabla \cdot (\nabla F) = \nabla^2 F = \left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} \right). \tag{1.16}$$

It is also possible to combine the vector $\vec{A}$ with the del operator to form a new operator that takes the form

$$\vec{A} \cdot \nabla = A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y} + A_z \frac{\partial}{\partial z}$$

and is known as the scalar invariant operator. This operator, which can be used with both vector and scalar quantities, is important because it is used to describe a process known as **advection**, a ubiquitous topic in the study of fluids.

1.2.2 The Taylor series expansion

It is sometimes convenient to estimate the value of a continuous function, $f(x)$, about the point $x = 0$ with a power series of the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n. \tag{1.17}$$

The fact that this can actually be done might appear to be an assumption so we must identify conditions for which this assumption is true. These conditions are that (1) the polynomial expression (1.17) passes through the point $(0,\ f(0))$ and (2) its first n derivatives match the first n derivatives of $f(x)$ at $x = 0$. Implicit in this second condition is the fact that $f(x)$ is differentiable at $x = 0$. In order for these conditions to be met, the coefficients a_0, a_1,..., a_n must be chosen properly. Substituting $x = 0$ into (1.17) we find that $f(0) = a_0$. Taking the first derivative of

(1.17) with respect to x and substituting $x = 0$ into the resulting expression we get $f'(0) = a_1$. Taking the second derivative of (1.17) with respect to x and substituting $x = 0$ into the result leaves $f''(0) = 2a_2$, or $f''(0)/2 = a_2$. If we continue to take higher order derivatives of (1.17) and evaluate each of them at $x = 0$ we find that, in order that the n derivatives of (1.17) match the n derivatives of $f(x)$, the coefficients, a_n, of the polynomial expression (1.17) must take the general form

$$a_n = \frac{f^n(0)}{n!}.$$

Thus, the value of the function $f(x)$ at $x = 0$ can be expressed as

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^n(0)}{n!}x^n. \tag{1.18}$$

Now, if we want to determine the value of $f(x)$ near the point $x = x_0$, the above expression can be generalized into what is known as the Taylor series expansion of $f(x)$ about $x = x_0$, given by

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^n(x_0)}{n!}(x - x_0)^n. \tag{1.19}$$

Since the dependent variables that describe the behavior of the atmosphere are all continuous variables, use of the Taylor series to approximate the values of those variables will prove to be a nifty little trick that we will exploit in our subsequent analyses. Most often we consider Taylor series expansions in which the quantity $(x - x_0)$ is very small in order that all terms of order 2 and higher in (1.19), the so-called **higher order terms**, can be effectively neglected. In such cases, we will approximate the given functions as

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$

1.2.3 Centered difference approximations to derivatives

Though the atmosphere is a continuous fluid and its observed state at any time *could theoretically* be represented by a continuous function, the reality is that actual observations of the atmosphere are only available at discrete points in space and time. Given that much of the subsequent development in this book will arise from consideration of the spatial and temporal variation of observable quantities, we must consider a method of approximating derivative quantities from discrete data. One such method is known as **centered differencing**[3] and it follows directly from the prior discussion of the Taylor series expansion.

[3] Centered differencing is a subset of a broader category of such approximations known as **finite differencing**.

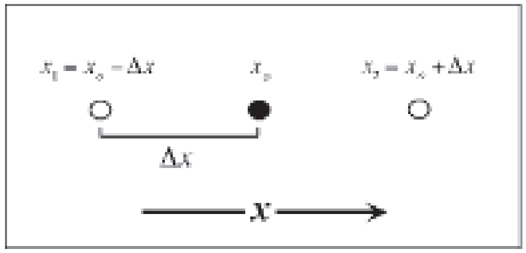

Figure 1.6 Points x_1 and x_2 defined with respect to a central point x_0

Consider the two points x_1 and x_2 in the near vicinity of a central point, x_0, as illustrated in Figure 1.6. We can apply (1.19) at both points to yield

$$f(x_1) = f(x_0 - \Delta x) = f(x_0) + f'(x_0)(-\Delta x) + \frac{f''(x_0)}{2!}(-\Delta x)^2 + \cdots + \frac{f^n(x_0)}{n!}(-\Delta x)^n \tag{1.20a}$$

and

$$f(x_2) = f(x_0 + \Delta x) = f(x_0) + f'(x_0)(\Delta x) + \frac{f''(x_0)}{2!}(\Delta x)^2 + \cdots + \frac{f^n(x_0)}{n!}(\Delta x)^n. \tag{1.20b}$$

Subtracting (1.20a) from (1.20b) produces

$$f(x_0 + \Delta x) - f(x_0 - \Delta x) = 2f'(x_0)(\Delta x) + 2f'''(x_0)\frac{(\Delta x)^3}{6} + \cdots. \tag{1.21}$$

Isolating the expression for $f'(x_0)$ on one side then leaves

$$f'(x_0) = \frac{f(x_0 + \Delta x) - f(x_0 - \Delta x)}{2\Delta x} - f'''(x_0)\frac{(\Delta x)^2}{6} - \cdots$$

which, upon neglecting terms of second order and higher in Δx, can be approximated as

$$f'(x_0) \approx \frac{f(x_0 + \Delta x) - f(x_0 - \Delta x)}{2\Delta x}. \tag{1.22}$$

The foregoing expression represents the centered difference approximation to $f'(x)$ at x_0 accurate to second order (i.e. the neglected terms are at least quadratic in Δx).

Adding (1.20a) to (1.20b) gives a similarly approximated expression for the second derivative as

$$f''(x_0) \approx \frac{f(x_0 + \Delta x) - 2f(x_0) + f(x_0 - \Delta x)}{\Delta x^2}. \tag{1.23}$$

Such expressions will prove quite useful in evaluating a number of relationships we will encounter later.

1.2.4 Temporal changes of a continuous variable

The fluid atmosphere is an ever evolving medium and so the fundamental variables discussed in Section 1.1 are ceaselessly subject to temporal changes. But what does it really mean to say 'The temperature has changed in the last hour'? In the broadest sense this statement could have two meanings. It could mean that the temperature of an individual air parcel, moving past the thermometer on my back porch, is changing as it migrates through space. In this case, we would be considering the change in temperature experienced *while moving with a parcel of air*. However, the statement could also mean that the temperature of the air parcels currently in contact with my thermometer is lower than that of air parcels that used to reside there but have since been replaced by the importation of these colder ones. In this case we would be considering the changes in temperature as measured *at a fixed geographic point*. These two notions of temporal change are clearly not the same, but one might wonder if and how they are physically and mathematically related. We will consider a not so uncommon example to illustrate this relationship.

Imagine a winter day in Madison, Wisconsin characterized by biting northwesterly winds which are importing cold arctic air southward out of central Canada. From the fixed geographical point of my back porch, the temperature (or potential temperature) drops with the passage of time. If, however, I could ride along with the flow of the air, I would likely find that the temperature does not change over the passage of time. In other words, a parcel with $T = 270^\circ\text{K}$ passing my porch at 8 a.m. still has $T = 270^\circ\text{K}$ at 2 p.m. even though it has traveled nearly to Chicago, Illinois by that time. Therefore, *the steady drop in temperature I observe at my porch is a result of the continuous importation of colder air parcels from Canada*. Phenomenologically, therefore, we can write an expression for this relationship we've developed:

$$\begin{matrix}\textit{Change with Time} \\ \textit{Following an Air} \\ \textit{Parcel}\end{matrix} = \begin{matrix}\textit{Change with Time} \\ \textit{at a Fixed} \\ \textit{Location}\end{matrix} - \begin{matrix}\textit{Rate of Importation} \\ \textit{of Temperature by} \\ \textit{Movement of Air.}\end{matrix} \tag{1.24}$$

This relationship can be made mathematically rigorous. Doing so will assist us later in the development of the equations of motion that govern the mid-latitude atmosphere. The change following the air parcel is called the **Lagrangian** rate of change while the change at a fixed point is called the **Eulerian** rate of change. We can quantify the relationship between these two different views of temporal change by considering an arbitrary scalar (or vector) quantity that we will call Q. If Q is a function of space and time, then

$$Q = Q(x, y, z, t)$$

and, from the differential calculus, the *total differential of* Q is

$$dQ = \left(\frac{\partial Q}{\partial x}\right)_{y,z,t} dx + \left(\frac{\partial Q}{\partial y}\right)_{x,z,t} dy + \left(\frac{\partial Q}{\partial z}\right)_{x,y,t} dz + \left(\frac{\partial Q}{\partial t}\right)_{x,y,z} dt \tag{1.25}$$

where the subscripts refer to the independent variables that are held constant whilst taking the indicated partial derivatives. Upon dividing both sides of (1.25) by dt, the total differential of t which represents a time increment, the resulting expression is

$$\frac{dQ}{dt} = \left(\frac{\partial Q}{\partial t}\right)\frac{dt}{dt} + \left(\frac{\partial Q}{\partial x}\right)\frac{dx}{dt} + \left(\frac{\partial Q}{\partial y}\right)\frac{dy}{dt} + \left(\frac{\partial Q}{\partial z}\right)\frac{dz}{dt} \tag{1.26}$$

where the subscripts on the partial derivatives have been dropped for convenience. The rates of change of x, y, or z with respect to time are simply the component velocities in the x, y, or z directions. We will refer to these velocities as u, v, and w and define them as $u = dx/dt$, $v = dy/dt$, and $w = dz/dt$, respectively. Substituting these expressions into (1.26) yields

$$\frac{dQ}{dt} = \left(\frac{\partial Q}{\partial t}\right) + u\left(\frac{\partial Q}{\partial x}\right) + v\left(\frac{\partial Q}{\partial y}\right) + w\left(\frac{\partial Q}{\partial z}\right) \tag{1.27}$$

which can be rewritten in vector notation as

$$\frac{dQ}{dt} = \left(\frac{\partial Q}{\partial t}\right) + \vec{V} \cdot \nabla Q \tag{1.28}$$

where $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$ is the 3-D vector wind. The three terms in (1.27) involving the component winds and derivatives of Q physically represent the horizontal and vertical transport of Q by the flow. Thus, we see that dQ/dt corresponds to the Lagrangian rate of change noted in (1.24). The Eulerian rate of change is represented by $\partial Q/\partial t$. The rate of importation by the flow (recall it was subtracted from the Eulerian change on the RHS of (1.24)) is represented by $-\vec{V} \cdot \nabla Q$ (*minus* the dot product of the velocity vector and the gradient of Q). In subsequent discussions in this book, $-\vec{V} \cdot \nabla Q$ will be referred to as **advection of** Q. Next we show that the mathematical expression $-\vec{V} \cdot \nabla Q$ actually describes the rate of importation of Q by the flow.

Consider the isotherms (lines of constant temperature) and wind vector shown in Figure 1.7. The gradient of temperature (∇T) is a vector that always points from lowest temperatures to highest temperatures as indicated. The wind vector, clearly drawn in Figure 1.7 so as to transport warmer air toward point A, is directed opposite to ∇T. Recall that the dot product is given by $\vec{V} \cdot \nabla T = |\vec{V}||\nabla T| \cos\alpha$ where α is the angle between the vectors $\vec{V}$ and ∇T. Given that the angle between $\vec{V}$ and ∇T is 180° in Figure 1.7, the dot product $\vec{V} \cdot \nabla T$ returns a negative value. Therefore, the sign of $\vec{V} \cdot \nabla T$ does not accurately reflect the reality of the physical situation depicted in Figure 1.7 – that is, that importation of *warmer air* is occurring at point A.

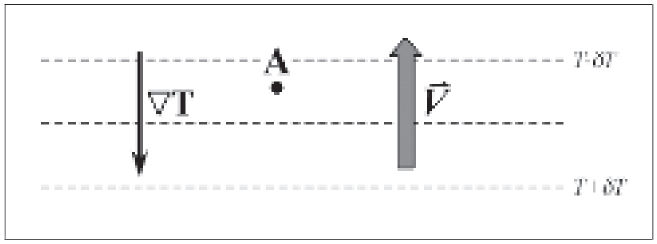

Figure 1.7 Isotherms (dashed lines) and wind vector $\vec{V}$ (filled arrow) surrounding point A. The thin black arrow is the horizontal temperature gradient vector

Thus, we define temperature advection, a measure of the rate (and sign) of importation of temperature to point A, as $-\vec{V} \cdot \nabla T$. The physical situation depicted in Figure 1.7, therefore, is said to be characterized by positive temperature (or warm air) advection.

To round out this discussion, we now return to the example that motivated the mathematical development: measuring the temperature change on my back porch. Rearranging (1.28) and substituting T (temperature) for Q we get

$$\left(\frac{\partial T}{\partial t}\right) = \frac{dT}{dt} - \vec{V} \cdot \nabla T$$

which shows that the Eulerian (fixed location) change is equal to the sum of the Lagrangian (parcel following) change and advection. In the prior example we imagined a temperature drop at my back porch. We also surmised that the temperature of individual air parcels did not undergo any change as the day wore on. Thus, the advective change at the porch must be negative – there must be negative temperature advection, or cold air advection (i.e. $-\vec{V} \cdot \nabla T < 0$), occurring in Madison on this day. Clearly, the situation of northwesterly winds importing cold air southward out of Canada fits the bill.

1.3 Estimating with Scale Analysis

In many fluid dynamical problems, it is convenient and insightful to estimate which physical terms are likely to contribute most to a particular process under study. For instance, in assessing the threat to coastal property in Hawaii in the face of a major tsunami, it is not likely that the ambient wind speed will figure into the problem in any significant way. In the development of the equations of motion in subsequent chapters, a variety of physical processes will be confronted, each of which has some bearing on the behavior of the fluid atmosphere. At many junctures, however, we will attempt to simplify those equations by estimating the magnitude of the mathematical terms that comprise them. A formal process known as scale analysis is employed in