2

Fundamental and Apparent Forces

Objectives

The fluid atmosphere is a physical object and its motion is therefore governed by the laws of physics. From among these laws, Newton's second law states that the rate of change of momentum of an object (i.e. its acceleration) equals the sum of all the forces acting on that object:

$$\frac{d(Momentum)}{dt} = \sum \textit{Forces Acting on the Object.}$$

This powerful statement is valid only for motions measured in a non-accelerating coordinate system – one that is fixed in space. Such a coordinate system is known as an **inertial frame of reference**. The most convenient x, y, and z coordinates by which we measure motions on Earth refer to a grid based upon latitude and longitude (for the x and y coordinate directions) and elevation above sea level (for the z coordinate direction). Since the Earth rotates on its axis and revolves around the Sun, this Earth-based x, y, and z coordinate system undergoes constant acceleration. This fact is easily proven using a globe. After finding your location on the globe, consider the fact that what you view at that location as the immutable direction east is, in fact, constantly changing direction (to an observer fixed in space) as the Earth rotates on its axis. Thus our Earth-based coordinates are non-inertial (i.e. accelerating). This being the case, Newton's second law can only be applied to the motion of objects on Earth if we correct for the acceleration of our coordinate system.

The collection of forces required to adequately represent Newton's second law on the rotating Earth can therefore be split into two broad categories. The first of these includes forces that would affect objects even in the absence of rotation, the so-called **fundamental forces**. The most important of these fundamental forces are (1) the pressure gradient force, (2) the gravitational force, and (3) the frictional force, all of which we will investigate in this chapter. The other group of forces that we must

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consider in a full treatment of Newton's second law arises from the need to correct for the acceleration of our terrestrial coordinate system. We will refer to such forces as **apparent forces**. Two important apparent forces to be investigated in this chapter are (1) the centrifugal force and (2) the Coriolis force. We begin this examination by considering the fundamental forces.

2.1 The Fundamental Forces

Understanding the fundamental forces is essential to gaining insight into the behavior of the fluid atmosphere. Most people have a solid intuitive feel for the gravitational and friction forces since both are so widely recognized as manifest in our daily experience. As it turns out, the effects of the often less familiar pressure gradient force are equally ubiquitous and readily detectable. We begin our examination of the fundamental forces by considering the nature of this pressure gradient force.

2.1.1 The pressure gradient force

In order to examine the pressure gradient force (PGF) we will consider the pressure exerted by the atmosphere on sides A and B of the infinitesimal fluid element illustrated in Figure 2.1. The pressure exerted on sides A and B arises from the fact

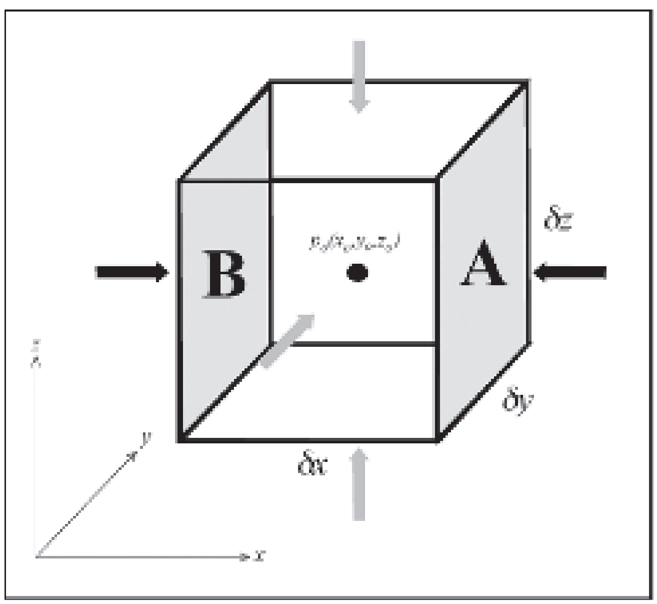

Figure 2.1 The pressure forces acting on the sides of an infinitesimal fluid element. The sides A and B are referenced in the text and the forces acting on those sides are indicated by the black arrows. The forces on other sides are indicated by the gray arrows

that random molecular motions compel molecules to strike the sides. Each time a molecule strikes the side of the fluid element, a certain amount of momentum is transferred to that side. The total momentum transfer is the sum of all the individual momentum transfers. The total momentum transferred each second defines the force exerted by the atmosphere on the side of the fluid element. Dividing this total force by the area of the side of the fluid element defines the pressure that is exerted on that side. The volume of the fluid element is given by $V = \delta x\, \delta y\, \delta z$ and its mass is given by $M = \rho \delta x\, \delta y\, \delta z$ where ρ is the density of the fluid. Let us define the pressure at the center of the fluid element to be $p(x_0, y_0, z_0) = p_0$. Assuming the pressure is continuous, we can use a Taylor series expansion to determine the pressure on sides A and B:

$$p_A = p_0 + \frac{\partial p}{\partial x}\left(\frac{\delta x}{2}\right) + \textit{Higher Order Terms} \tag{2.1a}$$

$$p_B = p_0 - \frac{\partial p}{\partial x}\left(\frac{\delta x}{2}\right) + \textit{Higher Order Terms.} \tag{2.1b}$$

Now, the x-direction pressure force acting on side A has magnitude $p_A \times (Area\ of\ A)$ and is directed toward the center of the infinitesimal fluid element. Thus, this force can be expressed as

$$F_{A_x} = -\left(p_0 + \frac{\partial p}{\partial x}\frac{\delta x}{2}\right)\delta y\, \delta z. \tag{2.2a}$$

By similar reasoning, the x-direction pressure force acting on side B is given by

$$F_{B_x} = \left(p_0 - \frac{\partial p}{\partial x}\frac{\delta x}{2}\right)\delta y\, \delta z \tag{2.2b}$$

so that the net x-direction pressure force acting on the fluid element is

$$F_x = F_{A_x} + F_{B_x} = -\frac{\partial p}{\partial x}\delta x\, \delta y\, \delta z. \tag{2.3}$$

Thus, the net force *per unit mass* acting in the x direction on the fluid element is

$$\frac{F_x}{M} = -\frac{1}{\rho}\frac{\partial p}{\partial x}. \tag{2.4}$$

Similar expressions can be derived in exactly the same way for the y- and z-direction components of the pressure gradient force per unit mass. Therefore, the total pressure gradient force per unit mass can be expressed as

$$\frac{\vec{F}}{M} = -\frac{1}{\rho}\nabla p. \tag{2.5}$$

2.1.2 The gravitational force

Newton's law of universal gravitation says that any two elements of mass in the universe attract each other with a force proportional to their masses and inversely

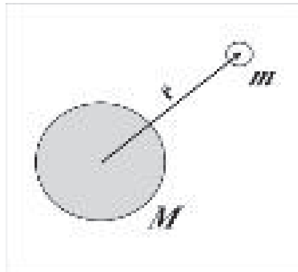

Figure 2.2 Two masses, *M* and *m*, used to illustrate Newton's law of universal gravitation. The vector $\vec{r}$ is the position vector directed from the center of mass of *M* to the center of mass of *m*

proportional to the distance between their centers of mass. This is represented symbolically, with the aid of the illustration in Figure 2.2, as

$$\vec{F}_g = -\frac{GMm}{r^2}\left(\frac{\vec{r}}{r}\right) \tag{2.6}$$

where $G = 6.673 \times 10^{-11}\ N\,m^2\,kg^{-2}$ is the universal gravitational constant, and M pulls m toward its center. For a fluid parcel of the atmosphere, M is the mass of the Earth and m is the mass of the fluid parcel. Thus, we can express the gravitational force per unit mass as

$$\frac{\vec{F}_g}{m} = -\frac{GM}{r^2}\left(\frac{\vec{r}}{r}\right). \tag{2.7}$$

Many applications in atmospheric dynamics use height above sea level (Z) as the vertical coordinate. This suggests that a parcel of air at a high elevation in the atmosphere might experience a smaller gravitational force than one located at sea level (i.e. nearer the center of gravity of the Earth). Though this conjecture is strictly true, the difference is very small from the surface to any level in the troposphere (lowest 10–12 km of the atmosphere) and we use a constant value of the gravitational force, g_0^*, where

$$g_0^* = -\frac{GM}{a^2}\left(\frac{\vec{r}}{r}\right) \tag{2.8}$$

with a being the radius of the Earth, as a consequence. It is left to the reader to demonstrate that this is an entirely reasonable simplification.

2.1.3 The frictional force

Most of us have some conceptual understanding of friction and its effect on the behavior of solids. A textbook, for instance, that is pushed across a table feels the effect of the friction between itself and the tabletop and begins to decelerate immediately. In fact, the only reason the textbook does not continue to slide along the table for ever is that a force, the friction force, is applied opposite to its motion. The frictional force in this simple example is quantified in terms of a **coefficient of friction** which is a measure of the resistance to motion that results from pushing the book over the table. This simplistic view of friction has to be modified when one considers the frictional force acting on a fluid parcel. Fluids, being collections of discrete atoms or molecules, are subject to internal friction among these particles which cause the fluid to resist the tendency to flow. We will try to gain some insight into the nature of this resistance and how to express the physics in mathematical terms.

Another analogy here may help set the stage for our more formal exploration of friction in fluids. Nearly all of us have, at one time or another, experienced traffic on a multi-lane highway. Generally cars in such traffic may pass other cars in a passing lane (on the left in North America) while passing on the right (in the cruising lanes) is discouraged. Occasionally, a driver who has just used the passing lane will decide to move to the adjacent cruising lane, in which the average speed is lower. When this happens, the passer's car imports high momentum into the cruising lane, often upsetting the smooth flow of traffic. A similar disruption occurs when a driver enters the passing lane at an insufficient speed. In the worst case (i.e. when a number of passers decide to change lanes simultaneously), the rapid flux of momentum from the passing lane to the cruising lane can cause a slowdown of the entire flow of traffic. If one considers the individual cars in this example as molecules in a fluid flow, one can see that momentum transfer between layers of a fluid (accomplished by molecules or clumps of molecules) may lie at the conceptual heart of fluid friction.

Consider, for instance, the situation depicted in Figure 2.3 in which a plate, moving at speed u_0, is placed atop a column of fluid with depth, l. The top layer of fluid moves at the velocity of the plate while the fluid at the bottom of the column has zero motion. Thus, a shearing stress exists in the fluid and a force must be exerted on the plate in order that it be kept moving at speed u_0 along the top surface of the fluid. The requisite force is proportional to u_0 since a greater force will be required for a greater speed. Additionally, since molecules of fluid that reside at the bottom of the column

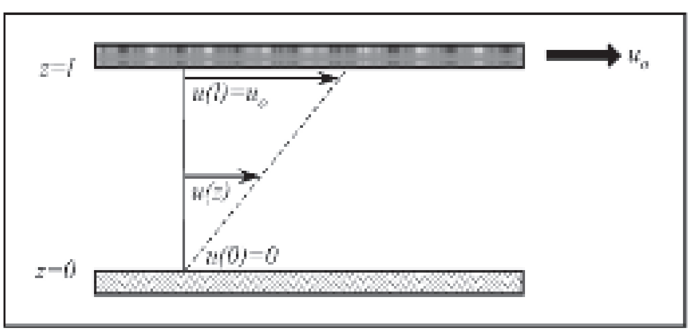

Figure 2.3 Flow beneath a moving plate illustrating 1-D, steady-state, viscous shear flow. The top plate, at height $z = l$, is moving across the top of the fluid with speed u_0 while the bottom plate is fixed. The vertical shear of the flow speed is indicated with arrows between the plates

can influence the movement of the plate through momentum transport in the fluid column, the requisite force is also inversely proportional to the depth of the fluid. The force is also proportional to the area of the plate since a larger plate makes contact with more fluid than a smaller one. The actual force required to keep the plate moving can therefore be written as $F = \mu A u_0/l$, where μ is the dynamic viscosity coefficient measured empirically and expressed in kg m^{-1} s^{-1}. If we represent the vertical shear within the fluid as $\delta u/\delta z = u_0/l$, then the force can be expressed as

$$F = \mu A \frac{\delta u}{\delta z}. \tag{2.9a}$$

Here F represents the x-direction force required to overcome the viscous effect of the vertical shear of the x-direction velocity component. Hence, as $\delta z \to 0$, the shearing stress, or viscous force per unit area, is given by

$$\tau_{zx} = \mu \frac{\partial u}{\partial z}, \tag{2.9b}$$

where the subscript '*zx*' indicates that this is the component of the shearing stress (in the x direction) that arises from the vertical shear (z) of the x-direction (x) velocity component. From the molecular viewpoint, a molecule moving to smaller z (i.e. toward the bottom of the fluid column) transports high momentum that it acquired from the motion of the plate to the surrounding fluid. Thus, there is a net downward transport of x-direction momentum and this momentum transport per unit time per unit area is the shearing stress, τ_{zx}.

The prior example considered the *steady* movement of a plate across the top of a fluid column. In nature, viscous forces result from *non-steady* shear flows. In recognition of this fact, let us consider the volume element depicted in Figure 2.4 which represents the case of non-steady, 2-D shear flow in a fluid of constant density. Analogous to our treatment of the pressure gradient force, we expand the

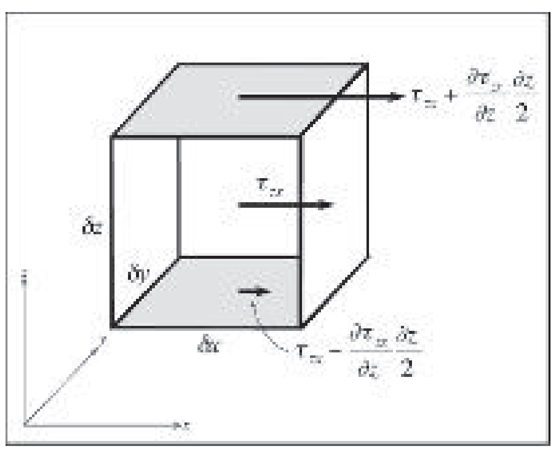

Figure 2.4 Illustration of the *x* component of the vertical shearing stress on a fluid element

shearing stress in a Taylor series in order to determine its value at the top and bottom (z-direction) facing sides of the volume element. The stress acting across the upper boundary on the fluid *below* it can be approximated as

$$\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{\delta z}{2} \tag{2.10a}$$

while the stress acting across the bottom boundary on the fluid *below* it can be approximated as

$$\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{\delta z}{2}. \tag{2.10b}$$

According to Newton's third law, this stress must be equal and opposite to the stress acting across the bottom boundary on the fluid *above* it. Since we are interested in the net stress acting on the volume element in Figure 2.4, we want to sum the forces that act on fluid *within* the volume element. Thus, we find that the net viscous force on the volume element acting in the x direction is given by

$$\left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{\delta z}{2}\right)\delta x \delta y - \left(\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{\delta z}{2}\right)\delta x \delta y = \frac{\partial \tau_{zx}}{\partial z}\delta x \delta y \delta z. \tag{2.11a}$$

Dividing this expression by the mass of volume element, $\rho \delta x \delta y \delta z$, we have the viscous force per unit mass arising from the vertical shear of the x-direction motion:

$$\frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z} = \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu \frac{\partial u}{\partial z}\right). \tag{2.11b}$$

If μ is constant, (2.11b) can be simplified to

$$\frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu \frac{\partial u}{\partial z}\right) = \upsilon \frac{\partial^2 u}{\partial z^2} \tag{2.12}$$

where $\upsilon = \mu/\rho$ is known as the **kinematic viscosity coefficient** and has an empirically determined value of 1.46×10^{-5} m^2 s^{-1}.

Analogous derivations can be performed to determine the viscous stresses acting in the other directions. The resulting frictional force components per unit mass in the x, y, and z directions are

$$\begin{aligned}
F_{rx} &= \upsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \\
F_{ry} &= \upsilon\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \\
F_{rz} &= \upsilon\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right).
\end{aligned} \tag{2.13}$$

For the lowest 100 km of the atmosphere, υ is so small that molecular viscosity is entirely negligible except within a few millimeter of the Earth's surface where the vertical shear is very large (on the order of 10^3 s^{-1}!). Above about 10 mm we need an entirely separate treatment of fluid friction in which it is useful to conceptualize eddies as discrete 'blobs' of fluid which move around like molecules and transfer momentum toward or away from the surface of the Earth in a manner analogous to molecules in molecular viscosity. A mixing length, defined as the average length through which an eddy can travel before mixing out its momentum, can be defined by analogy to the mean free path for molecular diffusion. With this adjustment, the dissipative effects of small-scale turbulence can be represented by defining an eddy viscosity coefficient so that

$$\frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z} \approx K\frac{\partial^2 u}{\partial z^2} \tag{2.14}$$

where K is the eddy viscosity coefficient.

2.2 Apparent Forces

In expressing his first law, Sir Isaac Newton states: 'Every body persists in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by forces impressed on it.' In other words, a mass in uniform motion *relative to a coordinate system fixed in space* will remain in uniform motion in the absence of any forces. Any motion relative to a coordinate system fixed in space is known as **inertial motion** and the reference frame in which that motion is measured is known as an **inertial reference frame**. Most of us live at a single location long enough to become accustomed to thinking of north, south, east, and west as fixed directions. In reality, however, the direction I call 'north' at Madison, Wisconsin is not the same, as viewed from the perspective of a space traveler orbiting Earth, as the 'north' known to a resident of Jakarta, Indonesia. If one considers the intersection of latitude and longitude lines on a globe as the intersections of a Cartesian x and y grid describing the Earth, then it is clear that since the Earth rotates, this coordinate system is accelerating and thus provides us Earthlings with a *non-inertial* reference frame. It might appear that given our non-inertial reference frame we are not able to apply Newton's laws of motion to motion relative to the Earth. Of course, this is not true, but we do have to make some correction for the non-inertial nature of the reference frame by which we measure all such motion. We will make the necessary corrections by introducing the centrifugal and Coriolis forces, the so-called 'apparent forces'. But first, it is instructive to consider physically why the coordinate system matters at all. We can do this by considering application of Newton's laws to experiments conducted inside a closed elevator car.

In the first case, let us imagine that the car is stationary *or* moving with a constant velocity, $\vec{V}$. Under such conditions imagine that a weight is dropped within the moving car. Upon making the appropriate measurements and calculations, you would determine that the weight had fallen toward the floor of the car with a measurable, constant acceleration of 9.81 m s^{-2}. This acceleration would be observed relative to the walls and floor of the elevator car in a Cartesian coordinate system defined by the dimensions of the elevator car. In such a case, an observer in the elevator car would note complete agreement between the results of the experiment and Newton's laws of motion since the *constant velocity* elevator car provides an inertial reference frame for this experiment.

In the second case, we remotely observe the elevator car falling freely through the elevator shaft. If a similar weight is dropped within the car the weight appears to remain suspended in mid-air, at a constant elevation above the floor of the car. Measured relative to the coordinate frame of the car, the weight has zero acceleration even though to us remote observers it is clearly accelerating toward the ground at a rate of 9.81 m s^{-2}. Viewed from inside the car, Newton's laws seem to fail here, but this is because the coordinate system itself is accelerating and is therefore non-inertial.

The latitude/longitude coordinate system on a rotating Earth is also accelerating and so we have to take that acceleration into account in order to apply Newton's laws accurately to objects moving relative to that Earth-based coordinate system.

2.2.1 The centrifugal force

Each of us is located a certain distance from the axis of rotation of the Earth. Depending upon the exact distance, we are rotating around that axis at a very high, but constant speed (at Madison, Wisconsin that speed is 330 m s^{-1}!). Each of us is, therefore, not unlike the ball on the end of the string depicted in Figure 2.5. The speed of the ball is constant, equal to the rotation rate, ω, times the radius of rotation, r ($r = |\vec{r}|$). The direction of the ball changes continuously, however, and so, as viewed from the perspective of the ball, there is a uniform acceleration directed toward the axis of rotation equal to

$$\frac{d\vec{V}}{dt} = -\omega^2\vec{r}. \tag{2.15}$$

This acceleration is called the **centripetal acceleration** and is caused by the force of the string pulling on the ball. Suppose you are on the ball and rotating with it. From

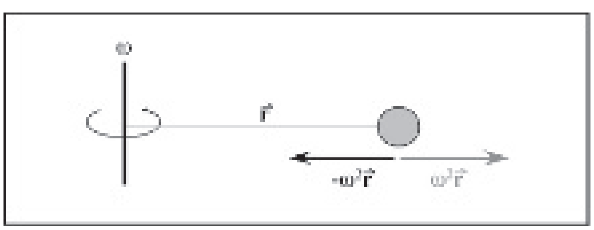

Figure 2.5 The rotating ball on a string experiences an inward-directed centripetal acceleration, indicated by the dark arrow. To the observer on the ball, a compensating centrifugal force, indicated by the gray arrow, must be included to describe accurately motions on the ball itself

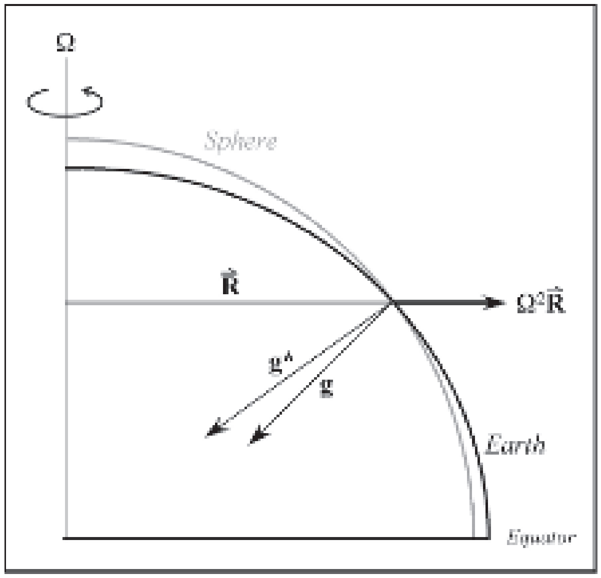

Figure 2.6 Relationship between the centrifugal force, gravitation (g^*), and effective gravity (g). The effect of the centrifugal force is to deform the Earth's shape into an oblate spheroid on which the local vertical direction is perpendicular to effective gravity as shown

your perspective the ball is stationary but, in reality, a centripetal acceleration is still being exerted upon it. In order for a person on the ball to apply Newton's laws under this condition, an apparent force that exactly balances the true centripetal force must be included in the physics; this apparent force is known as the **centrifugal force**.

In order to balance the centripetal acceleration, the centrifugal acceleration is directed *outward* along the radius of rotation and is given by

$$CEN = \omega^2 \vec{r}. \tag{2.16}$$

As depicted in Figure 2.6, on a rotating Earth, the centrifugal force affects the vertical force balance. When the centrifugal force and gravitational forces (g^*) are added, the result is called **effective gravity** (g) and is given by

$$g = g^* + \Omega^2 \vec{R} \tag{2.17}$$

where Ω is the rotation rate of the Earth and $\vec{R}$ is the position vector from the axis of rotation to the object in question. Note that effective gravity, thus defined, is directed perpendicular to the local tangent of the surface of the Earth – not necessarily toward the center of the Earth. In fact, since $\Omega^2 \vec{R}$ is directed away from the axis of rotation, g is *not* directed toward the center of the Earth except at the poles and the equator! Were the Earth a perfect sphere, this fact would result in the existence of a horizontal, equatorward-directed component of gravity. The relatively malleable crust of the Earth has long since responded to this circumstance and adopted its oblate spheroidal shape with an equatorial radius some 21 km larger that its polar radius. Given such a slightly distorted shape, the local vertical direction everywhere on Earth is defined parallel to g. The centrifugal force component of effective gravity is an example of the effect of rotation on objects *at rest* with respect to the Earth-based rotating frame of reference. In order to apply Newton's laws accurately to the motion of objects relative to that rotating frame an additional apparent force, the Coriolis force, must be considered.

2.2.2 The Coriolis force

Consider a dynamics field experiment in which one student takes a position on a merry-go-round and another student takes a position some distance above the ground in an adjacent tree. The merry-go-round is set spinning and a ball is pushed from the center of the merry-go-round toward the spinning student. From the vantage point of the tree, the motion of the ball appears as a straight line, as it should since a uniform force was administered to it. But from the perspective of the rotating frame, the ball appears to accelerate in a curved path, away from the observer in a direction opposite to the direction of rotation. Upon consulting each other's notes, the students conclude that an apparent force, arising from the rotation of the merry-go-round, deflects the ball from its path. This apparent force is the Coriolis force. How can the Coriolis force be quantified on the rotating Earth?

Suppose a hockey puck is given an impulse in the eastward direction on a frozen, frictionless Earth. Under these circumstances, the puck is rotating faster than the solid Earth beneath it so that, for its latitude, the centrifugal force acting on the puck will be increased to

$$CEN = \left(\Omega + \frac{u}{R}\right)^2 \vec{R} = \Omega^2 \vec{R} + 2\Omega u \frac{\vec{R}}{R} + \frac{u^2 \vec{R}}{R^2} \tag{2.18}$$

where u/R represents the incremental change in rotation rate resulting from the eastward impulse. The first term on the RHS of (2.18) is the already familiar centrifugal force, included in effective gravity. The second and third terms, however, are deflecting forces acting outward along $\vec{R}$ (perpendicular to the axis of rotation). For normal synoptic-scale motions on Earth, $u \ll \Omega R$ (remember, $\Omega R = 330\ \mathrm{m\,s^{-1}}$ at Madison), allowing neglect of the third term to hardly compromise the result. The remaining term, $2\Omega u \vec{R}/R$ (the excess centrifugal force), represents the Coriolis force resulting from relative motion *parallel to a latitude circle.* This Coriolis force has two components as suggested by Figure 2.7. The vertical and horizontal components are given by

$$\frac{dw}{dt} = 2\Omega u \cos\phi \text{ and } \frac{dv}{dt} = -2\Omega u \sin\phi, \tag{2.19}$$

respectively, where ϕ is the latitude. Using a shorthand in which f, the so-called Coriolis parameter, is given by $f = 2\Omega \sin\phi$, we can rewrite the horizontal component of the Coriolis force resulting from relative zonal motion as $dv/dt = -fu$. We see

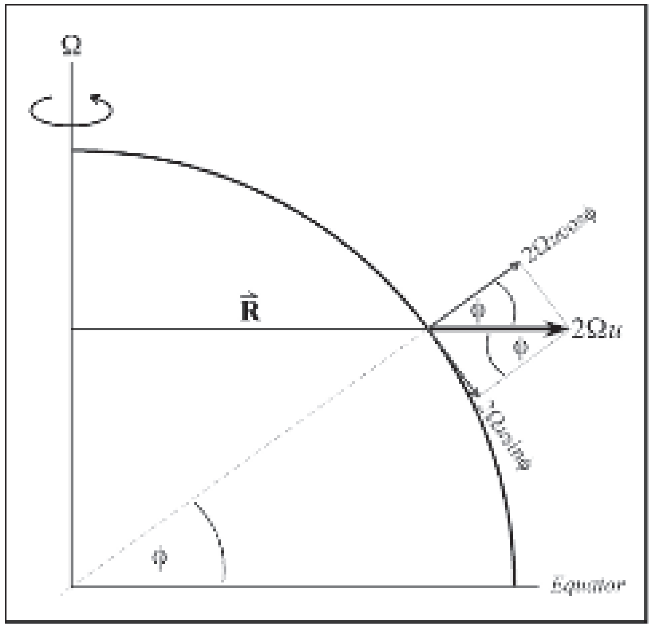

Figure 2.7 For east to west relative motions on Earth, the Coriolis force arises as excess centrifugal force

that given an eastward (westward) directed impulse, the Coriolis force will deflect the object to the south (north), or to the right of its original path, in the northern hemisphere (where ϕ is positive by convention).

What happens if we consider the hockey puck moving equatorward relative to the Earth? In the absence of applied forces, it must conserve angular momentum ($\Omega \vec{R}^2$). Upon being pushed equatorward in the northern hemisphere, the radius of rotation of the puck begins to increase. Consequently, an anti-rotational relative motion develops in order to conserve angular momentum. We can quantify this simple physics by considering a balance between the initial angular momentum of the puck and its angular momentum after displacement equatorward toward larger $\vec{R}$. (Note that displacement toward larger $\vec{R}$ also occurs if the puck is compelled to move in the relative *vertical* direction.) If we let δu signify the induced westward motion at the new radius of rotation, $\vec{R} + \delta R$, then conservation of angular momentum is given by

$$\Omega \vec{R}^2 = \left(\Omega + \frac{\delta u}{R + \delta R}\right)(\vec{R} + \delta R)^2. \tag{2.20a}$$

Expansion of (2.20a) yields

$$\Omega \vec{R}^2 = \left(\Omega + \frac{\delta u}{R + \delta R}\right)(\vec{R}^2 + 2\vec{R}\delta R + \delta R^2). \tag{2.20b}$$

Since δR (and δu) are so small, we will neglect the products of such differential terms so that (2.20b) becomes

$$\Omega \vec{R}^2 = \left(\Omega + \frac{\delta u}{R + \delta R}\right)(\vec{R}^2 + 2\vec{R}\,\delta R) \tag{2.20c}$$

or

$$\Omega \vec{R}^2 = \Omega \vec{R}^2 + 2\Omega \vec{R}\,\delta R + \frac{\vec{R}^2 \delta u}{R + \delta R} \tag{2.20d}$$

which reduces to

$$2\Omega \vec{R}\delta R = -\frac{\vec{R}^2 \delta u}{R + \delta R} \text{ or } 2\Omega \vec{R}\,\delta R = -\vec{R}\delta u. \tag{2.20e}$$

In the end, we find that

$$\delta u = -2\Omega\,\delta R. \tag{2.21}$$

The incremental zonal velocity δu can be induced by both meridional (i.e. north/south) motion or by vertical motion as illustrated in Figure 2.8. The incremental radius of rotation, δR, has components in the vertical and meridional directions. By the similar triangles in Figure 2.8, we see that $\sin\phi = \delta R / -\delta y$ and $\cos\phi = \delta R/\delta z$. Thus, for meridional motions,

$$\delta u = -2\Omega(-\delta y \sin\phi) = 2\Omega \sin\phi(\delta y). \tag{2.22a}$$

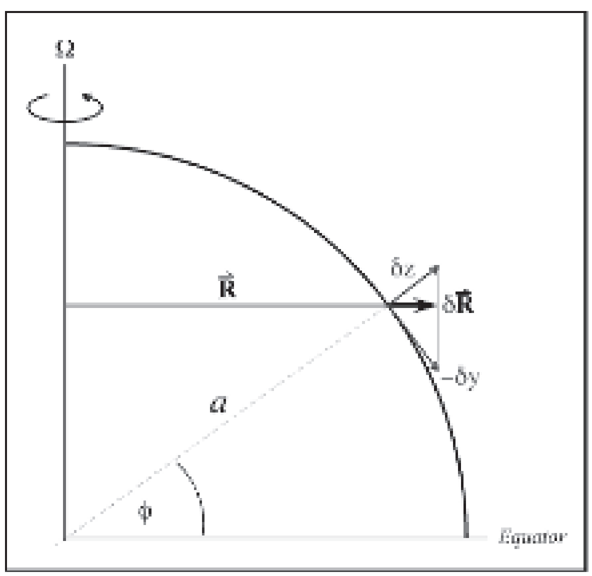

Figure 2.8 Illustration of the effect of vertical and meridional motions on the radius of rotation, $\vec{R}$. Upward and equatorward displacements produce an incremental increase in $\vec{R}$, indicated by $\delta \vec{R}$

As can be seen in Figure 2.8, however, $\delta y = a\delta\phi$, so (2.22a) can be rewritten as

$$\delta u = 2\Omega \sin\phi a\delta\phi. \tag{2.22b}$$

If we divide both sides of (2.22b) by the incremental δt and take the limit as $\delta t \to 0$, we get

$$\frac{du}{dt} = 2\Omega \sin\phi \left(a\frac{d\phi}{dt} \right). \tag{2.23a}$$

Since $ad\phi/dt = v$ and $f = 2\Omega \sin\phi$, (2.23a) can be rewritten as

$$\frac{du}{dt} = fv. \tag{2.23b}$$

It is clear from (2.23b) that equatorward motion in the northern hemisphere ($v < 0$) will induce a westward-directed zonal motion in accord with our physical intuition in the face of angular momentum conservation. Such a circumstance implies that the Coriolis force, in this instance, again compels an object to the right of its intended path.

Considering Figure 2.8, and (2.21), we see that for vertical motions

$$\delta u = -2\Omega \cos\phi\delta z. \tag{2.24a}$$

Once again, dividing both sides by δt and taking the limit as $\delta t \to 0$ results in

$$\frac{du}{dt} = -2\Omega \cos\phi \left(\frac{dz}{dt} \right) \quad \text{or} \quad \frac{du}{dt} = -2\Omega \cos\phi w. \tag{2.24b}$$

Thus, the full expression for the Coriolis force arising from meridional motions is given by

$$\frac{du}{dt} = fv - 2\Omega \cos\phi w \tag{2.25}$$

while the full 3-D Coriolis force is given by

$$\begin{aligned} \frac{du}{dt} &= fv - 2\Omega \cos\phi w \\ \frac{dv}{dt} &= -fu \\ \frac{dw}{dt} &= 2\Omega \cos\phi u. \end{aligned} \tag{2.26}$$

The Coriolis parameter, $f = 2\Omega \sin\phi$, is worth some special consideration before we leave this subject. The Coriolis parameter's dependence on latitude squares with our intuitive sense that the effect of rotation does indeed vary with latitude. We notice that the Coriolis parameter is identically zero at the equator and is a maximum at the poles. Since the Coriolis force is an apparent force arising from the acceleration of our Earth-based coordinate system, assigning a value for Ω, the rotation rate, is rather more involved than you might think.

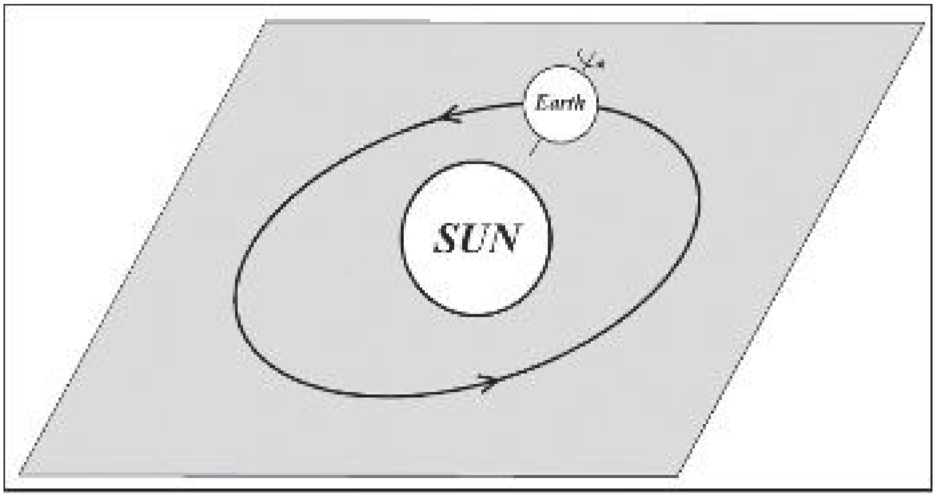

Figure 2.9 Illustration of the rotation of Earth on its axis with respect to its revolution around the Sun. The thick black line represents the Earth's revolution while the curved thin arrow represents the rotation. Gray shading is the plane of the ecliptic

The **solar day** represents the amount of time between successive local noontimes (i.e. moments at which the Sun in highest in the sky at a given location) and is 24 h long. As shown in Figure 2.9, the Earth revolves around the Sun in a counterclockwise fashion as viewed from above the plane of the ecliptic. Even if the Earth were not rotating on its axis, the revolution would provide one rotation each year – from east to west! In addition, during the year the Earth rotates (from west to east) through 365.25 solar days. Thus, as viewed from the perspective of the distant, fixed stars, the Earth must actually rotate 366.25 times (from west to east) on its axis in one year's time. Each rotation with respect to the fixed stars is therefore completed in

$$\frac{(365.25\, solar\ days) \times (24 \cdot 3600\, s\ solar\ day^{-1})}{366.25\, rotations} = 86\,156.09\, s\ \ rotation^{-1},$$

the length of the **sidereal day**. In order to apply Newton's laws accurately, we have to correct for the acceleration of our Earth-based coordinate system as viewed from the perspective of the fixed stars. Thus, Ω is determined using the length of the sidereal day as

$$\Omega = \frac{2\pi}{86\,156.09\, s} = 7.292 \times 10^{-5}\, s^{-1}.$$

Finally, it is important to note that since the Coriolis force always acts perpendicular to the motion vector, it can do no work on the moving particle since work is the scalar product of a force and a vector distance. Thus, the Coriolis force can only change the direction of motion but cannot initiate motion in an object at rest. We have now considered all the forces necessary to formulate the equations of motion on the rotating Earth from which we will investigate the fluid dynamics of the mid-latitude atmosphere. We will see in the next chapter that these equations are simply an expression of the conservation of momentum in the fluid atmosphere.

Selected References

Holton, *An Introduction to Dynamic Meteorology*, provides a thorough discussion and derivation of the fundamental and apparent forces.
Hess, *Introduction to Theoretical Meteorology*, offers similar material conveyed lucidly.

Problems

2.1. So long as it is shallow, water is a fluid with constant density. Use this fact to help solve the following problem.

(a) Develop a relationship for the horizontal pressure gradient force in terms of depth (h) of water in a shallow container.

A cylindrical tank of water is set on a turntable. The radius of the tank is r_0 and the depth of the water is z_0.

(b) The turntable is turned on (with rotation rate ω) and the system is allowed to equilibrate. Derive an expression for the height of the water surface, h, as a function of radius.

(c) Express $h(r)$ in terms of z_0 (Hint: consider the volume of fluid in the container.)

(d) If $r_0 = 1$ m, what rotation rate is required to raise the water level on the outer edge of the tank to $h = 2z_0$?

2.2. A baseball player at 30°N latitude throws a ball northward a horizontal distance of 75 m in 2 s. In what direction, and by how much, is the ball deflected laterally as a result of the rotation of the Earth?

2.3. Given the picture in Figure 2.1A, prove that $\alpha = \beta$.

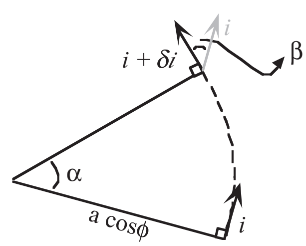

Figure 2.1A

2.4. While taking an eastbound train to work, a passenger of fixed mass finds that she weighs 542 N. On the way home she weighs herself again while the train is at full speed and finds she weighs 543 N. If she works 50 km from home, how long is her commute if she lives at 40°S? (You may assume that the average speed of the train is its full speed.)