3

Mass, Momentum, and Energy: The Fundamental Quantities of the Physical World

Objectives

Study of the physical world tends to be focused on the quantities known as **mass**, **momentum**, and **energy**. The behavior of the atmosphere is no exception to this rule. In this chapter we will investigate the manner in which these quantities and their various interactions serve to describe the building blocks of a dynamical understanding of the atmosphere at middle latitudes. We must first consider the distribution of mass in the atmosphere and the force balance that underlies this distribution. A number of insights concerning the vertical structure of the atmosphere proceed directly from this understanding.

Beginning with Newton's second law, we will construct expressions for the conservation of momentum in the three Cartesian directions. These expressions are commonly known as the equations of motion and will serve as *the* fundamental set of physical relationships for all subsequent inquiry in this book. Scale analysis of the horizontal equations of motion will reveal that a simple diagnostic relationship between the mass and momentum fields, geostrophy, characterizes the mid-latitude atmosphere on Earth. Finally, employing these equations of motion we will develop expressions for the conservation of mass and the conservation of energy. We begin by considering the distribution of mass in the atmosphere.

3.1 Mass in the Atmosphere

For our purposes, we shall define mass as *the measure of the substance of an object* and make that measurement in kilograms (kg). Though it was not clear to ancient

Mid-Latitude Atmospheric Dynamics Jonathan E. Martin

thinkers like Aristotle,[1] the atmosphere has mass. In fact the Earth's atmosphere has a mass of 5.265×10^{18} kg! The pressure exerted by this object decreases with increasing distance away from the surface as the depth of the fluid decreases. As a consequence, there is a vertical pressure gradient force given by

$$PGF_{vertical} = -\frac{1}{\rho}\frac{\partial p}{\partial z}\hat{k} \tag{3.1}$$

which compels atmospheric fluid from higher pressure (near the surface) to lower pressure (above the surface) and so is directed upward. The fact that the atmosphere does not race away into space under this forcing is a consequence of the fact that there is also the force of effective gravity acting on the fluid parcel, pulling it downwards. This force is given by

$$Gravity = -g\hat{k}. \tag{3.2}$$

The sum of the vertical pressure gradient force and gravity is zero for an atmosphere at rest. In mathematical terms

$$0 = \left(-g - \frac{1}{\rho}\frac{\partial p}{\partial z}\right)\hat{k}$$

or, after rearranging the terms and dropping the $\hat{k}$ designation for notational simplicity,

$$\frac{\partial p}{\partial z} = -\rho g. \tag{3.3}$$

This expression is known as the hydrostatic equation and represents a fundamental balance characteristic of the Earth's atmosphere: namely, that the vertical pressure gradient force is perfectly balanced by gravity. Though strictly true only for an atmosphere at rest (hence the *static* portion of the name), this **hydrostatic balance** is obeyed to great accuracy under nearly all conditions in the Earth's atmosphere.

In order to construct a vertical equation of motion we must take account of all the forces with components in the local vertical direction. The vertical pressure gradient force and gravity (combined in the hydrostatic balance) comprise the largest fraction of these forces. Surely friction, slight though it may be, will also affect motions in the vertical direction. Also, we have already shown that there is a vertical Coriolis acceleration induced by zonal motions. Thus, we can write a first approximation to the vertical equation of motion as

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + \vec{F}_z + (2\Omega\cos\phi)u. \tag{3.4}$$

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[1] The theories of the ancient Greek natural philosopher Aristotle (384–322 BC) held sway in many disciplines for nearly 2000 years! He reputedly conducted an experiment to determine the weight of air. Undoubtedly using a crude scale, he 'filled' a leather bag with air, weighed it, and then compared that measurement to the weight of an 'empty' leather bag. Noting no difference between the two, he concluded that air had no weight.

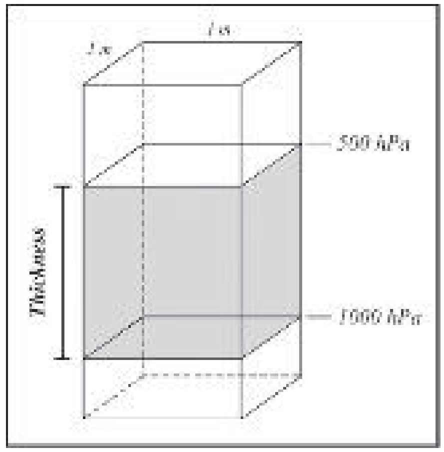

Figure 3.1 The amount of mass between any two isobaric surfaces is the same regardless of the thickness of the layer

3.1.1 The hypsometric equation

Consider the unit area column of atmosphere contained between pressure levels 1000 and 500 hPa shown in Figure 3.1. Since pressure is defined as force per unit area, we have isolated in that column an atmospheric mass sufficient to exert 500 hPa of pressure. Such a slab of the atmosphere has a unique mass whether it extends from 1000 to 500 hPa or from 812 to 312 hPa. In fact, the mass of this column can be precisely calculated as

$$Mass = (500\ hPa) \times \left(\frac{100\ N\,m^{-2}}{hPa}\right) \times (1\ m^2) \times \left(\frac{1}{9.81\ m\,s^{-2}}\right) = 5102.04\ kg.$$

Though the mass of a 500 hPa, unit area slab of the atmosphere is unique, its depth might be different from one day to the next. We will refer to this geometric depth as the *thickness* between two isobaric surfaces. Clearly, if the thickness varies, then so does the volume of the unit area slab. The variation of the volume of the slab dictates that the density of the air contained within the slab varies as well: less (more) dense air corresponding to a greater (smaller) thickness. By the ideal gas law, less (more) dense air will correspond to a higher (lower) column average virtual temperature, $\overline{T_v}$.[2] Thus, column average virtual temperature should have a bearing on the thickness between two isobaric levels.

Combining the hydrostatic equation with the ideal gas law provides convincing evidence to support this supposition. Recall that the ideal gas law can be written as $p = \rho R_d T_v$ where p is the pressure, ρ is the density, R_d is the gas constant for dry

[2] See Appendix A for a discussion and derivation of virtual temperature, T_v.

air,[3] and T_v is the virtual temperature. Using this expression, the hydrostatic equation can be rewritten as

$$\frac{\partial p}{\partial z} = -\frac{pg}{R_d T_v} \tag{3.5a}$$

which can be rearranged into

$$-\frac{R_d T_v}{g} \partial \ln p = \partial z. \tag{3.5b}$$

If we integrate this expression between pressure levels p_1 and p_2 ($p_1 > p_2$) at which the heights are z_1 and z_2 ($z_2 > z_1$) we get

$$-\int_{p_1}^{p_2} \frac{R_d T_v}{g} \partial \ln p = \int_{z_1}^{z_2} \partial z. \tag{3.5c}$$

Inverting the order of integration on the LHS of (3.5c) yields

$$\int_{p_2}^{p_1} \frac{R_d T_v}{g} \partial \ln p = \int_{z_1}^{z_2} \partial z$$

which can be integrated to give

$$\frac{R_d \overline{T}_v}{g} \ln\left(\frac{p_1}{p_2}\right) = z_2 - z_1 = \Delta z \tag{3.6}$$

where $\overline{T}_v$ is the pressure-weighted, column average virtual temperature, given by

$$\overline{T}_v = \frac{\int_{p_2}^{p_1} T_v \partial \ln p}{\int_{p_2}^{p_1} \partial \ln p}.$$

The foregoing expression is known as the **hypsometric equation** and it quantifies and verifies our suspicion regarding the influence of column average temperature on the thickness of an isobaric column.

We can express the hypsometric equation (and, therefore, the hydrostatic equation also) in terms of a quantity called **geopotential**, Φ. The geopotential is defined as the work required to raise a unit mass a distance dz above sea level. It quantifies the work (per unit mass) that is done against gravity in doing so. Mathematically, therefore, geopotential is given as $d\Phi = gdz$. Employing this expression, we can rewrite the

[3] R_d has a value of $287\,\mathrm{J\,kg^{-1}K^{-1}}$ and is equal to the universal gas constant ($R^* = 8.3143 \times 10^3\,\mathrm{J\,K^{-1}\,kmol^{-1}}$) divided by the molecular weight of the atmospheric mixture ($28.97\,\mathrm{kg\,kmol^{-1}}$). 'Dry' air refers to the mixture without the variable water vapor included.

hydrostatic equation as

$$\partial p = -\rho\partial\Phi \quad \text{or} \quad \frac{\partial\Phi}{\partial p} = -\alpha = -\frac{R_d T_v}{p}.$$

Correspondingly, the hypsometric equation can also be written as

$$R_d \overline{T}_v \ln\left(\frac{p_1}{p_2}\right) = \Phi_2 - \Phi_1 = \Delta\Phi.$$

We will often refer to geopotential height (Z) in subsequent discussions. The geopotential height is simply given by

$$Z = \frac{\Phi}{g_0} \tag{3.7}$$

where g_0 is the global average gravity at sea level (9.81 m s^{-2}). Thus, geometric height (z) and Z are just about equal in the troposphere.

There are several important applications of the hydrostatic and hypsometric equations that have a bearing on the analysis and understanding of mid-latitude weather systems. One of the most common analysis products used to characterize and understand the weather is a sea level pressure map. This is a map on which isobars of sea-level pressure are contoured in an attempt to identify and characterize the major circulation systems in a given location at a given time. In geographical regions characterized by high terrain, such as the Rocky Mountains of North America or the high steppe of Mongolia, the elevation is so far above sea level that use of the station pressure (i.e. the pressure actually measured with a barometer at the station) does not effectively contribute to this goal. In such regions the hypsometric equation can be used to calculate a **reduced sea-level pressure** (i.e. an estimate of what the sea-level pressure would be were the surface elevation 0 m). Consider the following example.

The station pressure at St Louis, Missouri (STL), a city close to sea level, on a certain day is measured to be 995 hPa. Meanwhile, the station pressure at Denver, Colorado (DEN), whose elevation is 1609 m above sea level, is measured at 825 hPa. There is not a *horizontal* pressure difference of 180 hPa between STL and DEN. Most of the observed pressure difference is a consequence of the *vertical* variation of pressure. By reducing the station pressure to sea level at DEN, we attempt to discover how much of the observed pressure difference actually is a horizontal pressure difference.

We begin with the hypsometric equation,

$$\frac{R_d \overline{T}_v}{g} \ln\left(\frac{p_1}{p_2}\right) = z_2 - z_1 = \Delta z$$

with $z_2 = z_{DEN}$ and $z_1 = 0$ (the geometric height at sea level). Correspondingly, $p_2 = p_{STA\,at\,DEN}$ (observed station pressure) and $p_1 = p_{SLP\,at\,DEN}$ (the desired value we will calculate as sea level pressure at DEN). Finally, $\overline{T}_v$ represents the average column temperature between sea level at DEN and the station elevation. This is clearly a fictitious quantity but we can estimate it by assuming the standard atmosphere lapse rate ($6.5\,\mathrm{K\,km^{-1}}$) throughout the fictitious column. Rearranging the hypsometric equation using the given definitions we have

$$\frac{g\,z_{DEN}}{R_d\,\overline{T}_v} = \ln\left(\frac{p_{SLP\ at\ DEN}}{p_{STA\ at\ DEN}}\right). \tag{3.8a}$$

Taking anti-logs of both sides yields

$$\left(\frac{p_{SLP\ at\ DEN}}{p_{STA\ at\ DEN}}\right) = e^{\frac{g\,z_{DEN}}{R_d\,\overline{T}_v}}$$

so that

$$p_{SLP\ at\ DEN} = p_{STA\ at\ DEN}\,e^{\frac{g\,z_{DEN}}{R_d\,\overline{T}_v}}. \tag{3.8b}$$

The above expression is known as the **altimeter equation** and is the standard expression for reducing station pressure to sea level. Supposing that the surface T_v at Denver is 20°C, we find that the reduced sea-level pressure at Denver would be 998.6 hPa. This value can be usefully compared to the sea-level pressure at St Louis on a synoptic weather chart.

The hypsometric equation can also be used to gain insights into the large-scale structure of mid-latitude weather systems. If, for instance, we consider the thickness between 1000 and 500 hPa at a given station, then (3.6) becomes

$$\Delta z = \frac{R_d\,\overline{T}_v}{g}\ln\left(\frac{1000}{500}\right) = \frac{R_d\,\overline{T}_v}{g}\ln(2) = 20.3\,\overline{T}_v. \tag{3.9}$$

Thus, a change of 60 m in the 1000–500 hPa thickness corresponds to a 2.96°C mean temperature change. This fact implies that pressure drops off more rapidly with height in a cold column of air than in a warm column. The ramifications of this fact are illustrated in Figure 3.2. in which a cold core cyclone is depicted in a vertical cross-section. Since the air column in the middle of the cyclone is colder relative to its surroundings at all levels, the thickness in that column is smaller than anywhere else. Consequently, the horizontal pressure gradient force, directed inward toward the center of the cyclone, increases in magnitude with increasing height. Thus, cold core cyclones, like those that populate the mid-latitudes on Earth, intensify with height. This characteristic of mid-latitude cyclones will prove to be a major influence on the dynamics of the cyclone life cycle.

Now that we have acquired a perspective on the distribution of mass in the atmosphere, we turn to an investigation of the basic conservation laws that govern its behavior. The atmosphere, like all physical systems, obeys the laws of conservation of energy and mass, as well as the slightly more restrictive conservation of momentum. We begin by considering the conservation of momentum.

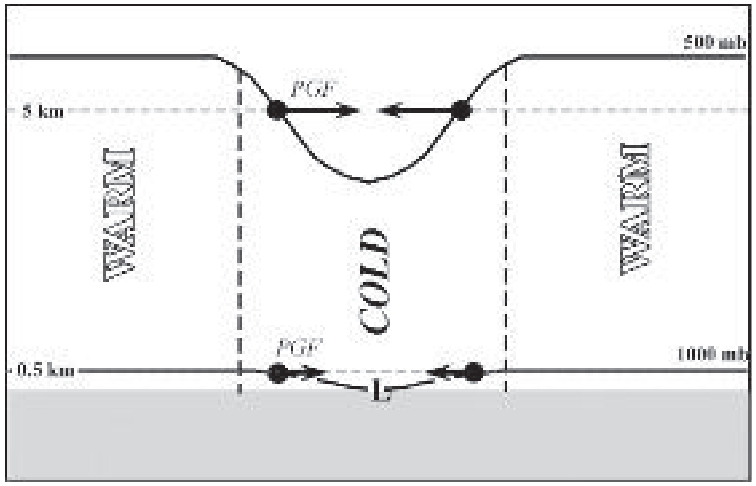

Figure 3.2 Vertical cross-section through a cold core cyclone. 'Warm' and 'Cold' refer to the column average temperatures in the three columns. Solid lines are isobars, thin dashed lines are the 0.5 km and 5 km elevation lines. The thick arrows represent the PGF, which is much larger at the top black dots. 'L' is the location of the lowest sea-level pressure

3.2 Conservation of Momentum: The Equations of Motion

Newton's second law is a statement of the conservation of momentum:

$$\frac{d}{dt}(m\vec{V}) = \sum \textit{Forces Acting on a Parcel},$$

but it is strictly true, as we have already considered, only in an inertial frame of reference. Since we will find it most convenient to use the x, y, and z coordinates fixed to Earth for our descriptions of motions, and these coordinates are accelerating, we have to relate the Lagrangian derivative of a vector in an inertial frame to the corresponding Lagrangian derivative in a rotating frame. Let $\vec{A}$ be an arbitrary vector whose Cartesian components in an inertial frame are

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

and whose components in a coordinate frame rotating with an angular velocity $\vec{\Omega}$ are

$$\vec{A} = A'_x\hat{i}' + A'_y\hat{j}' + A'_z\hat{k}'.$$

Now, let $d_a\vec{A}/dt$ be the total derivative of $\vec{A}$ in the inertial (absolute) frame, expressed as

$$\frac{d_a\vec{A}}{dt} = \frac{dA_x}{dt}\hat{i} + \frac{dA_y}{dt}\hat{j} + \frac{dA_z}{dt}\hat{k}.$$

Notice that in the inertial frame the coordinate directions $\hat{i}$, $\hat{j}$, and $\hat{k}$ are unchanging. Taking the same derivative in the rotating frame, however, yields

$$\frac{d_a\vec{A}}{dt} = \frac{dA'_x}{dt}\hat{i}' + \frac{dA'_y}{dt}\hat{j}' + \frac{dA'_z}{dt}\hat{k}' + A'_x\frac{d\hat{i}'}{dt} + A'_y\frac{d\hat{j}'}{dt} + A'_z\frac{d\hat{k}'}{dt}$$

which can be rewritten as

$$\frac{d_a\vec{A}}{dt} = \frac{d\vec{A}}{dt} + A'_x\frac{d\hat{i}'}{dt} + A'_y\frac{d\hat{j}'}{dt} + A'_z\frac{d\hat{k}'}{dt} \tag{3.10}$$

given that

$$\frac{d\vec{A}}{dt} = \frac{dA'_x}{dt}\hat{i}' + \frac{dA'_y}{dt}\hat{j}' + \frac{dA'_z}{dt}\hat{k}'$$

where $d\vec{A}/dt$ represents the rate of change of $\vec{A}$ following the *relative* motion in the rotating frame.

The derivatives $d\hat{i}'/dt$, $d\hat{j}'/dt$, and $d\hat{k}'/dt$ on the RHS of (3.10) represent the rates of change of the unit vectors $\hat{i}'$, $\hat{j}'$, and $\hat{k}'$ that arise because the coordinate system is accelerating. It is important to note that each of these derivative terms describes only the change in *direction* of the unit vectors since, by definition, the vector magnitudes are always equal to one. Thus, full expressions for these derivatives are achieved upon describing the change in direction experienced by each of the unit vectors as a result of rotation of the Earth.

Figure 3.3(a) illustrates a view of the change of $\hat{i}'$ as viewed from the North Pole. The rotation vector, $\vec{\Omega}$, points upward out of the page. By similar triangles, we find that $\delta\hat{i}' = \hat{i}'\delta\theta$. Now, upon dividing both sides of this equality by the amount of time (δt) it takes to rotate through $\delta\theta$ degrees, and taking the limit of the resulting

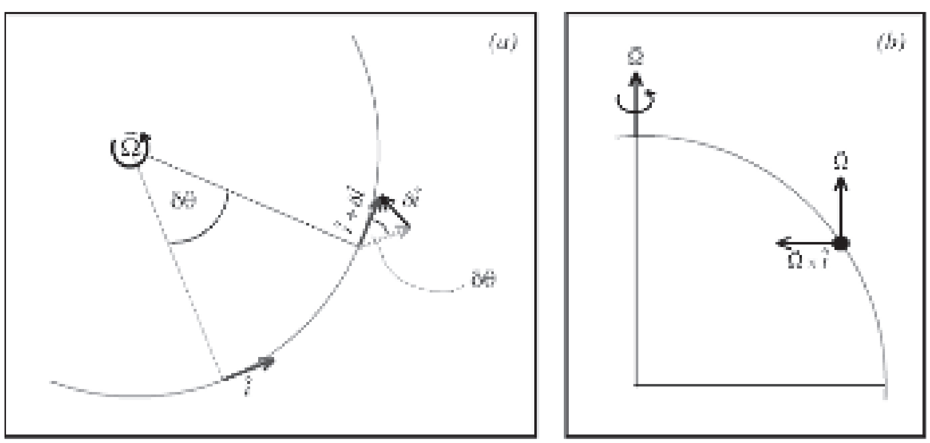

Figure 3.3 **(a) View from the North Pole of the change in the $\hat{i}$ unit vector ($\delta\hat{i}$) and (b) cross-sectional view of the same vector, $\delta\hat{i}$. $\vec{\Omega}$ is the rotation vector**

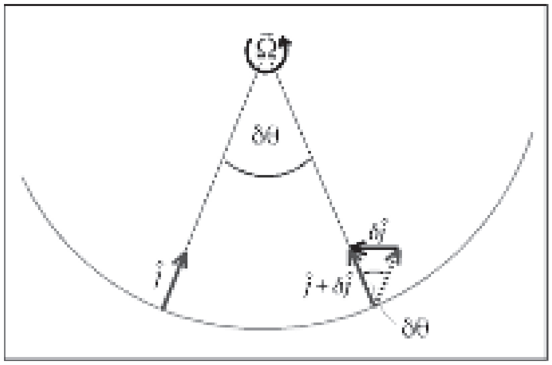

Figure 3.4 View from the North Pole of the change in the $\hat{j}$ unit vector ($\delta\hat{j}$). $\vec{\Omega}$ is the rotation vector

expression as $\delta t \to 0$, we get

$$\lim_{\delta t \to 0} \frac{\delta \hat{i}'}{\delta t} = \left| \frac{d\hat{i}'}{dt} \right| = \left| \hat{i}' \frac{d\theta}{dt} \right| = \left| \hat{i}' \vec{\Omega} \right| \tag{3.11}$$

so the magnitude of the vector $d\hat{i}'/dt$ is equal to $|\vec{\Omega}|$. It is clear from Figure 3.3(b), however, that the vector $d\hat{i}'/dt$ is directed inward toward the axis of rotation. Knowing that $d\hat{i}'/dt$ is a vector that is both perpendicular to $\hat{i}'$ and has magnitude $|\vec{\Omega}|$, we find that its full expression is given by

$$\frac{d\hat{i}'}{dt} = \vec{\Omega} \times \hat{i}'. \tag{3.12}$$

Similar relationships exist for $d\hat{j}'/dt$ and $d\hat{k}'/dt$ as can be seen in Figures 3.4 and 3.5. Consequently, we can rewrite the last three terms on the RHS of (3.10) as

$$A'_x \frac{d\hat{i}'}{dt} = A'_x(\vec{\Omega} \times \hat{i}') = \vec{\Omega} \times (A'_x \hat{i}'),$$

$$A'_y \frac{d\hat{j}'}{dt} = A'_y(\vec{\Omega} \times \hat{j}') = \vec{\Omega} \times (A'_y \hat{j}'), \text{ and}$$

$$A'_z \frac{d\hat{k}'}{dt} = A'_z(\vec{\Omega} \times \hat{k}') = \vec{\Omega} \times (A'_z \hat{k}'),$$

so that

$$A'_x \frac{d\hat{i}'}{dt} + A'_y \frac{d\hat{j}'}{dt} + A'_z \frac{d\hat{k}'}{dt} = \vec{\Omega} \times (A'_x \hat{i}' + A'_y \hat{j}' + A'_z \hat{k}') = \vec{\Omega} \times \vec{A}. \tag{3.13}$$

As a result, (3.10) can be rewritten as

$$\frac{d_a \vec{A}}{dt} = \frac{d\vec{A}}{dt} + \vec{\Omega} \times \vec{A} \tag{3.14}$$

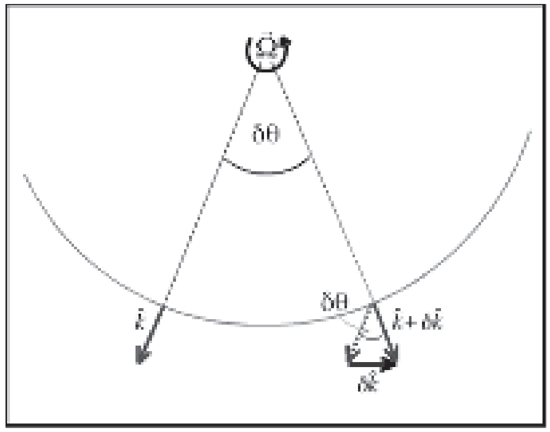

Figure 3.5 View from the North Pole of the change in the $\hat{k}$ unit vector ($\delta\hat{k}$). $\vec{\Omega}$ is the rotation vector

for any vector $\vec{A}$. This expression describes the relationship between the total derivative of a vector in inertial coordinates and its associated derivative in a coordinate system rotating with angular velocity $\vec{\Omega}$.

Employing (3.14), let us now find a relationship between the absolute velocity of an air parcel ($\vec{U}_a$) and the velocity of the same air parcel relative to Earth ($\vec{U}$). We can do this by applying (3.14) to the position vector $\vec{r}$ (where $\vec{r}$ is a vector perpendicular to the axis of rotation with magnitude equal to the distance from the surface of the Earth to the axis of rotation), for a parcel of air on Earth:

$$\frac{d_a\vec{r}}{dt} = \frac{d\vec{r}}{dt} + \vec{\Omega} \times \vec{r}. \tag{3.15a}$$

By definition, $d_a\vec{r}/dt = \vec{U}_a$ and $d\vec{r}/dt = \vec{U}$ so the desired relationship is simply

$$\vec{U}_a = \vec{U} + \vec{\Omega} \times \vec{r} \tag{3.15b}$$

which states that the absolute velocity of an object on the rotating Earth is equal to the sum of its velocity relative to the Earth ($\vec{U}$) and the velocity of the rotating Earth itself ($\vec{\Omega} \times \vec{r}$).

Now if we reapply the previous result to the vector $\vec{U}_a$ we get

$$\frac{d_a\vec{U}_a}{dt} = \frac{d\vec{U}_a}{dt} + \vec{\Omega} \times \vec{U}_a. \tag{3.16a}$$

Substituting (3.15b) for $\vec{U}_a$ above yields

$$\begin{aligned}\frac{d_a\vec{U}_a}{dt} &= \frac{d}{dt}(\vec{U} + \vec{\Omega} \times \vec{r}) + \vec{\Omega} \times (\vec{U} + \vec{\Omega} \times \vec{r}) \\ &= \frac{d\vec{U}}{dt} + \vec{\Omega} \times \frac{d\vec{r}}{dt} + \vec{\Omega} \times \vec{U} + \vec{\Omega} \times \vec{\Omega} \times \vec{r}.\end{aligned} \tag{3.16b}$$

Since $d\vec{r}/dt = \vec{U}$ and $\vec{\Omega} \times \vec{\Omega} \times \vec{r} = -\Omega^2\vec{r}$, this can be simplified to

$$\frac{d_a\vec{U}_a}{dt} = \frac{d\vec{U}}{dt} + 2\vec{\Omega} \times \vec{U} - \Omega^2\vec{r}. \tag{3.17}$$

Equation (3.17) states that the Lagrangian acceleration in an inertial system is equal to the sum of (1) the Lagrangian change of relative $\vec{U}$, plus (2) the Coriolis acceleration from relative motion in the relative frame, plus (3) centripetal acceleration resulting from the rotation of the coordinates. Recalling Newton's second law and the fact that we will consider the pressure gradient force, the frictional force, and gravitational force as the only real forces acting on the atmospheric fluid, we find that

$$\frac{d_a\vec{U}_a}{dt} = \frac{d\vec{U}}{dt} + 2\vec{\Omega} \times \vec{U} - \Omega^2\vec{r} = -\frac{1}{\rho}\nabla p + \vec{g}^* + \vec{F}$$

or, upon rearranging terms,

$$\frac{d\vec{U}}{dt} = -2\vec{\Omega} \times \vec{U} - \frac{1}{\rho}\nabla p + \vec{g} + \vec{F} \tag{3.18}$$

where the centripetal force has been combined with the gravitational force ($\vec{g}^*$) in the gravity term ($\vec{g}$). This expression states that the acceleration following the relative motion in a rotating reference frame is equal to the sum of (1) the Coriolis force, (2) the pressure gradient force, (3) effective gravity, and (4) the friction force. This is a major result but it remains in vectorial form only – a form not particularly amenable to analysis. Since the Earth is nearly a sphere, it will turn out to be quite convenient to recast this vector expression into spherical coordinates.

3.2.1 The equations of motion in spherical coordinates

Spherical coordinates treat the three dimensions in terms of longitude, latitude, and geometric height above sea level (λ, ϕ, z) using unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ in the description of motions. The relative velocity vector becomes $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$ where the components are defined as

$$u \equiv a\cos\phi\frac{d\lambda}{dt}, \quad v \equiv a\frac{d\phi}{dt}, \quad \text{and} \quad w \equiv \frac{dz}{dt}$$

where a is the radius of the Earth.[4] Distances in the zonal and meridional directions are given by $dx = a\cos\phi d\lambda$ and $dy = ad\phi$, respectively. It is important to note that this coordinate system is not a Cartesian system because the unit vectors are not constant; they are, in fact, functions of position on Earth. A simple way of conceptualizing this fact is to consider that all longitude lines converge at the pole. Therefore, the direction 'north' is not pointed in the same absolute direction at every

[4] Formally, a should be replaced with $(r + a)$ where r is the distance above sea level and a is the radius of the Earth. However, for all tropospheric, and nearly all atmospheric, applications, $r \ll a$ so we simply use a.

longitude on Earth. This position dependence must be taken into account when the acceleration vector is expanded into its components

$$\frac{d\vec{V}}{dt} = \frac{du}{dt}\hat{i} + \frac{dv}{dt}\hat{j} + \frac{dw}{dt}\hat{k} + u\frac{d\hat{i}}{dt} + v\frac{d\hat{j}}{dt} + w\frac{d\hat{k}}{dt}. \tag{3.19}$$

We now must determine expressions for the last three terms on the RHS of (3.19).

Beginning with $d\hat{i}/dt$, we simply expand it like any other total derivative to get

$$\frac{d\hat{i}}{dt} = \frac{\partial \hat{i}}{\partial t} + u\frac{\partial \hat{i}}{\partial x} + v\frac{\partial \hat{i}}{\partial y} + w\frac{\partial \hat{i}}{\partial z}. \tag{3.20}$$

We know that $\partial\hat{i}/\partial t = 0$ as there is no local change in the coordinate direction (i.e. at any given location, east *always* points in the same direction). The $\hat{i}$ direction experiences no change as one moves north or south along a given longitude line, nor as one moves up or down in elevation so that $\partial\hat{i}/\partial y$ and $\partial\hat{i}/\partial z$ are both zero. However, as we saw already in Figure 3.3(a), the $\hat{i}$ direction does change as one moves along a latitude circle so that (3.20) can be simplified to

$$\frac{d\hat{i}}{dt} = u\frac{\delta \hat{i}}{\delta x}. \tag{3.21}$$

The problem becomes one of determining the magnitude and direction of $\partial\hat{i}/\partial x$. We can make this determination by considering a horizontal cross-section viewed from the North Pole as shown in Figure 3.6. It is evident that $\delta x = a\cos\phi\delta\lambda$ and that $|\delta\hat{i}| = |\hat{i}|\delta\lambda = \delta\lambda$ since $\hat{i}$ has unit magnitude. Therefore,

$$\left|\frac{\delta \hat{i}}{\delta x}\right| = \frac{\delta\lambda}{a\cos\phi\delta\lambda} = \frac{1}{a\cos\phi} \tag{3.22a}$$

with $\delta\hat{i}$ directed toward the axis of rotation. Thus, we must split $\delta\hat{i}$ into components in order to determine the direction (in terms of $\lambda,\ \phi,\ z$) of $\delta\hat{i}/\delta x$. With the help

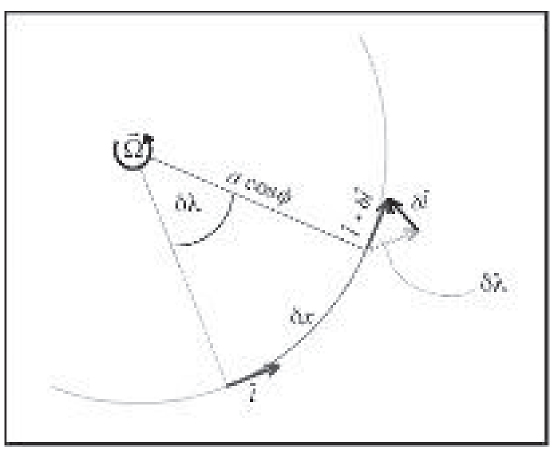

Figure 3.6 Illustration of the derivative $\frac{\delta\hat{i}}{\delta x}$

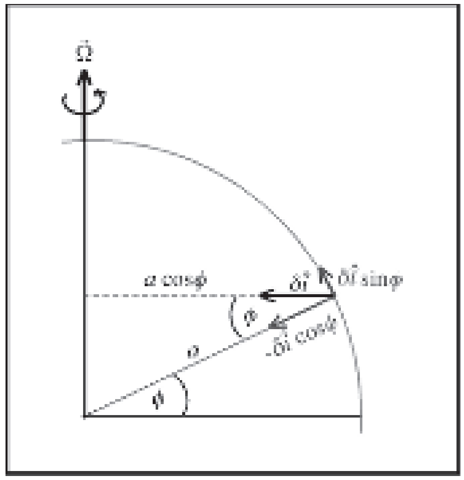

Figure 3.7 The northward and vertical components of $\delta\hat{i}$

of Figure 3.7, we see that $\delta\hat{i}$ has components in the $\hat{j}$ and $-\hat{k}$ directions. The $\hat{j}$ component is a function of $\sin\phi$ while the $-\hat{k}$ component is a function of $\cos\phi$. We find, therefore, that

$$\frac{\delta\hat{i}}{\delta x} = \frac{(\sin\phi\hat{j} - \cos\phi\hat{k})}{a\cos\phi} \tag{3.22b}$$

so that, taking the limit as $\delta x \to 0$,

$$\frac{d\hat{i}}{dt} = \frac{u(\sin\phi\hat{j} - \cos\phi\hat{k})}{a\cos\phi}. \tag{3.22c}$$

Next we consider the component form of $d\hat{j}/dt$. Once again, this term must be expanded like any other Lagrangian derivative into

$$\frac{d\hat{j}}{dt} = \frac{\partial\hat{j}}{\partial t} + u\frac{\partial\hat{j}}{\partial x} + v\frac{\partial\hat{j}}{\partial y} + w\frac{\partial\hat{j}}{\partial z}. \tag{3.23}$$

As was the case with $\hat{i}$, there is no local time derivative of $\hat{j}$ nor is there any change in $\hat{j}$ resulting from a change in elevation. There are, however, changes in $\hat{j}$ that arise from changing position in the x or y direction. Figure 3.8(a) illustrates the geometry involved in determining $\partial\hat{j}/\partial x$. The hypotenuse β of the lightly shaded triangle is given by $\beta = a/\tan\phi$ since $\sin\phi = (a\cos\phi)/\beta$. Knowing this dimension, the darker shaded triangle, shown independently in Figure 3.8(b), can be used to find $\partial\hat{j}/\partial x$. It is clear from Figure 3.8(b) that $\delta x = (a/\tan\phi)\delta\alpha$ and that $\delta\hat{j} = \hat{j}\delta\alpha$ with $\delta\hat{j}$ directed in the $-x$ direction. Thus,

$$\left|\frac{\delta\hat{j}}{\delta x}\right| = \frac{\tan\phi}{a}$$

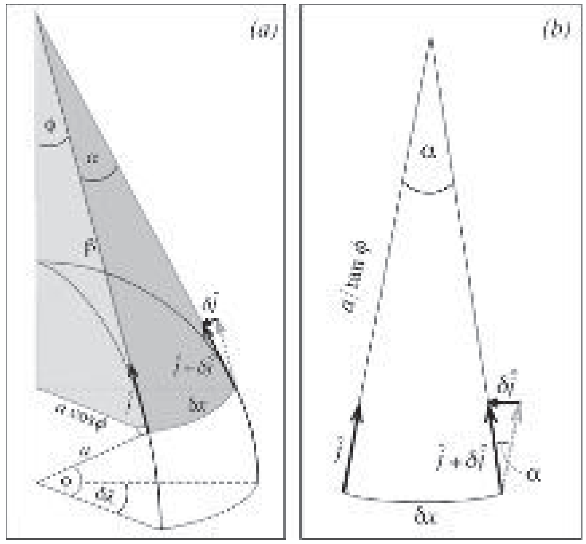

Figure 3.8 Illustration of the x variation of the unit vector $\hat{j}$. (a) A 3-D view of the plane on which $\hat{j}$ sits (darker shading). Dark-shaded triangle in (a) is illustrated in (b)

or, taking the limit as $\delta x \to 0$ and incorporating the direction,

$$\frac{\partial \hat{j}}{\partial x} = -\frac{\tan\phi}{a}\hat{\imath}. \tag{3.24}$$

Figure 3.9 illustrates the dependence of $\hat{j}$ on the y direction. We find that $\delta y = a\delta\phi$ and that $|\delta \hat{j}| = |\hat{j}\delta\phi| = \delta\phi$. Thus, $|\delta \hat{j}/\delta y| = 1/a$ with $\delta \hat{j}$ directed in the $-\hat{k}$

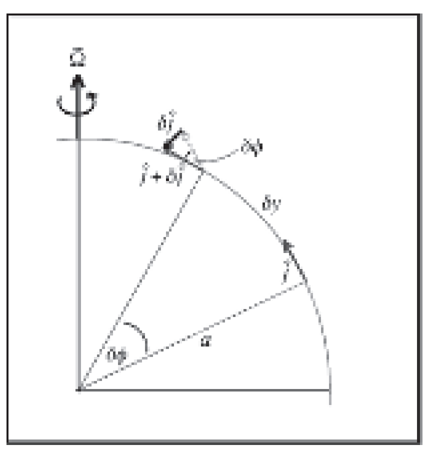

Figure 3.9 The y-direction dependence of $\hat{j}$

direction. Again, taking the limit of this expression as $\delta y \to 0$ yields

$$\frac{\partial \hat{j}}{\partial y} = -\frac{1}{a}\hat{k} \tag{3.25}$$

which, combined with (3.23) and (3.24), results in an expression for $d\hat{j}/dt$:

$$\frac{d\hat{j}}{dt} = \frac{-u\tan\phi}{a}\hat{\imath} - \frac{v}{a}\hat{k}. \tag{3.26}$$

Finally, we turn to $d\hat{k}/dt$ and, recognizing that $\hat{k}$ has no local time derivative nor any vertical derivative, obtain that

$$\frac{d\hat{k}}{dt} = u\frac{\partial \hat{k}}{\partial x} + v\frac{\partial \hat{k}}{\partial y}. \tag{3.27}$$

Figure 3.10 illustrates the x-direction dependence of $\hat{k}$. Since the triangle of interest represents a cross-section originating at the center of the Earth, we find that $\delta x = a\delta\lambda$ and that $|\delta\hat{k}| = |\hat{k}\delta\lambda| = \delta\lambda$ directed in the positive x direction. Consequently, $|\delta\hat{k}/\delta x| = 1/a$ which leads to the differential expression

$$\frac{\partial \hat{k}}{\partial x} = \frac{1}{a}\hat{\imath}. \tag{3.28}$$

Using a cross-section like that shown in Figure 3.9, but concentrating on the change in $\hat{k}$ over the distance δy, yields the expression $\partial\hat{k}/\partial y = (1/a)\,\hat{j}$. Thus, a complete expression for $d\hat{k}/dt$ is given by

$$\frac{d\hat{k}}{dt} = \frac{u}{a}\hat{\imath} + \frac{v}{a}\hat{j}. \tag{3.29}$$

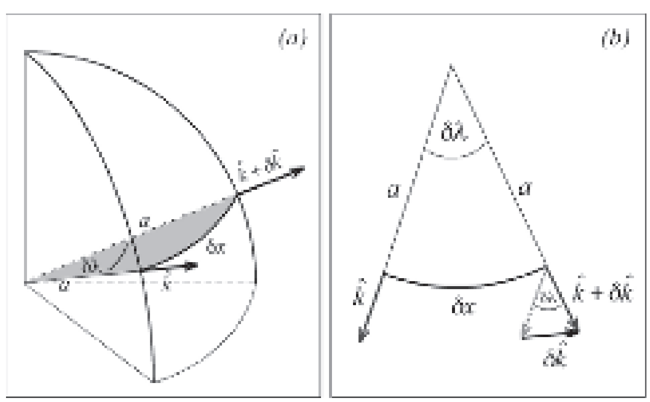

Figure 3.10 The x-direction dependence of the unit vector $\hat{k}$

Combining (3.22c), (3.26), and (3.29) we can rewrite (3.19) in its fully expanded component form as

$$\frac{d\vec{V}}{dt} = \left(\frac{du}{dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a}\right)\hat{i} + \left(\frac{dv}{dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a}\right)\hat{j} + \left(\frac{dw}{dt} - \frac{u^2+v^2}{a}\right)\hat{k}. \tag{3.30}$$

This expression describes only the spherical coordinate components of the Lagrangian derivative of the relative motion. Recall that our vector expression for the equations of motion (3.18) included reference to the pressure gradient, Coriolis, gravity, and friction forces. In order to obtain a complete component expansion of the equations of motion in spherical coordinates we must expand the force terms as well.

The Coriolis force term is given by $-2\vec{\Omega}\times\vec{U}$. Figure 3.11 demonstrates that the rotation vector, $\vec{\Omega}$, is perpendicular to the x direction and so has components only in the positive $\hat{j}$ and positive $\hat{k}$ directions. Considering the trigonometry in Figure 3.11, it is clear that the $\hat{k}$ component of $\vec{\Omega}$ has magnitude $\Omega\sin\phi$ while the $\hat{j}$ component has magnitude $\Omega\cos\phi$. Thus, the component expansion of the Coriolis force term can be determined by assessing the following determinant:

$$-2\vec{\Omega}\times\vec{U} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -2\Omega\cos\phi & -2\Omega\sin\phi \\ u & v & w \end{vmatrix} = -(2\Omega\cos\phi w - 2\Omega\sin\phi v)\hat{i} - 2\Omega\sin\phi u\hat{j} + 2\Omega\cos\phi u\hat{k}. \tag{3.31}$$

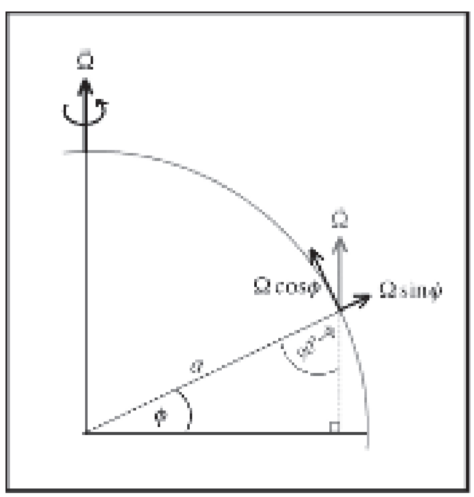

Figure 3.11 Partition of the rotation vector, $\vec{\Omega}$, into its vertical and meridional components

The component form of the pressure gradient force is given by

$$-\frac{1}{\rho}\nabla p = -\frac{1}{\rho}\frac{\partial p}{\partial x}\hat{i} - \frac{1}{\rho}\frac{\partial p}{\partial y}\hat{j} - \frac{1}{\rho}\frac{\partial p}{\partial z}\hat{k}. \tag{3.32}$$

Gravity, which acts downward in the local vertical direction, is represented by

$$\vec{g} = -g\hat{k} \tag{3.33}$$

while friction can be represented as

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}. \tag{3.34}$$

Combining (3.30), (3.31), (3.32), (3.33), and (3.34) and separating the component expression we get the three component equations of motion for flow on the rotating Earth:

$$\frac{du}{dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega\sin\phi v - 2\Omega\cos\phi w + F_x \tag{3.35a}$$

$$\frac{dv}{dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega\sin\phi u + F_y \tag{3.35b}$$

$$\frac{dw}{dt} - \frac{u^2 + v^2}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + 2\Omega\cos\phi u + F_z. \tag{3.35c}$$

The various terms in (3.35) involving $1/a$ arise from the non-flatness of the Earth and are consequently known as **curvature terms**. Each of the curvature terms is quadratic in the dependent variables $(u,\ v,\ w)$ and is thus non-linear and presents difficulty in analysis. It will soon be demonstrated, however, that these curvature terms are entirely negligible in any discussion of the dynamics of mid-latitude weather systems. However, even in the absence of these particular non-linear terms, the remaining elements of (3.35) also contain non-linear elements since, for instance, in the expansion of du/dt we get

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \underline{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}}.$$

The underlined terms are also clearly quadratic in $(u,\ v,\ w)$. These terms are known as the advective acceleration terms and they are comparable to the local acceleration term (in this case, $\partial u/\partial t$). The presence of such non-linear advection processes is one reason why dynamic meteorology is so fascinating (and difficult)!

The equations of motion (3.35) are a complicated set of expressions and it is logical to inquire whether or not they can be simplified. The answer is yes and we will use the method of scale analysis, introduced in Chapter 1, to accomplish this simplification. In order to do so, we must first assign observationally based characteristic values for the set of variables involved in the equations of motion. Considering just the horizontal velocity, which appears in (3.35) as both u and v,

Table 3.1 Characteristic scales of the various terms in the horizontal equations of motion

	1	2	3	4	5	6	7
x equation	$\frac{du}{dt}$	$-2\Omega \sin\phi v$	$2\Omega \cos\phi w$	$\frac{uw}{a}$	$-\frac{uv\tan\phi}{a}$	$-\frac{1}{\rho}\frac{\partial p}{\partial x}$	F_x
y equation	$\frac{dv}{dt}$	$2\Omega \sin\phi u$		$\frac{uv}{a}$	$\frac{u^2\tan\phi}{a}$	$-\frac{1}{\rho}\frac{\partial p}{\partial y}$	F_y
Scales	$\frac{U^2}{L}$	$f_0 U$	$f_0 W$	$\frac{UW}{a}$	$\frac{U^2}{a}$	$\frac{\delta p}{\rho L}$	$\frac{\nu U}{H^2}$
Magnitude ($\mathrm{m\,s^{-2}}$)	10^{-4}	10^{-3}	10^{-6}	10^{-8}	10^{-5}	10^{-3}	10^{-12}

we know from observations that characteristically the horizontal velocity at middle latitudes is not as small as $1\ \mathrm{m\,s^{-1}}$ nor is it as large as $100\ \mathrm{m\,s^{-1}}$. Therefore, a characteristic *scale* for the horizontal velocity is something close to $10\ \mathrm{m\,s^{-1}}$. Performing a similar analysis for the other variables in (3.35) results in the following reasonable set of characteristic values for the relevant variables:

$U \sim 10\ \mathrm{m\,s^{-1}}$ characteristic horizontal velocity

$W \sim 1\ \mathrm{cm\,s^{-1}}$ characteristic vertical velocity

$L \sim 10^6\ \mathrm{m}$ characteristic length scale of synoptic-scale features

$H \sim 10^4\ \mathrm{m}$ characteristic depth (i.e. depth of the troposphere)

$\frac{\delta p}{\rho} \sim 10^3\ \mathrm{m^2\,s^{-2}}$ characteristic horizontal pressure fluctuation

$\frac{L}{U} \sim 10^5\ \mathrm{s}$ characteristic time scale.

Of the above values, the one that seems most foreign is the characteristic horizontal pressure fluctuation. If the characteristic length scale of synoptic-scale features is 10^6 m, what this variable says is that the ratio of the pressure difference between adjacent synoptic-scale features is characteristically of order 1000 Pa (10 mb).[5] The density of the air is order $1\ \mathrm{kg\,m^{-3}}$, so the characteristic ratio across the size of a typical synoptic-scale disturbance is $\sim 1000\ \mathrm{m^2\,s^{-2}}$. Given such characteristic values, we are able to estimate the scale of all terms appearing in (3.35). Since our entire analysis is designed to uncover a simplification of (3.35) that is valid for mid-latitude synoptic-scale disturbances, we will assume a latitude (ϕ_0) of 45° implying that a characteristic Coriolis parameter is given by $f_0 = 2\Omega \sin\phi_0 = 2\Omega \cos\phi_0 \cong 10^{-4}\ \mathrm{s^{-1}}$. Table 3.1 lists the approximate magnitude of each term in (3.35) based upon the characteristic scales just described. Note that the friction term is represented by (2.7) and so involves ν, the kinematic viscosity coefficient, in its formulation. Recall that this parameter has a value of $\sim 1.5 \times 10^{-5}\ \mathrm{m^2\,s^{-1}}$ at sea level.

It is clear from Table 3.1 that with scaling appropriate for mid-latitude synoptic-scale motions, only two terms in the horizontal equations of motion are of order

[5] This is consistent with synoptic experience in which the pressure difference between adjacent sea-level high- and low-pressure centers is not as small as 1 hPa nor as large as 100 hPa!

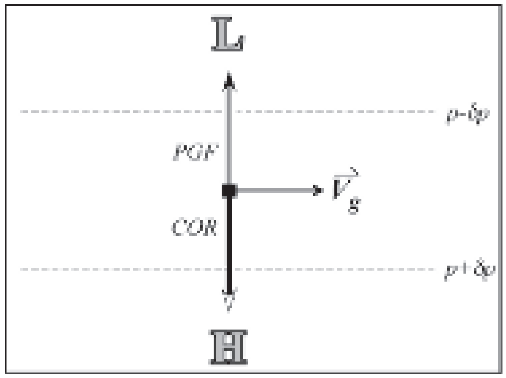

Figure 3.12 Illustration of the force balance resulting in the geostrophic wind, V_g. Arrow *PGF* represents the pressure gradient force and arrow *COR* represents the Coriolis force. The thin dashed lines are isobars and H and L represent regions of high and low pressure, respectively

10^{-3} or larger: the pressure gradient force and Coriolis force terms. This result implies that, as a first approximation to the full equations of motion (3.35), we can consider the PGF and Coriolis force terms to be in approximate balance with one another. This balance is known as the **geostrophic balance** and it represents the fundamental diagnostic balance for mid-latitude synoptic-scale flow. What kind of flow does this geostrophic balance describe? We can get some insight into this question by considering the balance of forces involved. Consider the set of sea-level isobars depicted in Figure 3.12. As we noted in Chapter 2, the PGF vector is always directed from high to low pressure, perpendicular to the isobars as depicted in Figure 3.12. In order that there be a force balance between the pressure gradient and Coriolis forces, the Coriolis force vector must be equal and opposite to the PGF vector as depicted. Since Figure 3.12 represents a hypothetical situation in the northern hemisphere, we know that the Coriolis force must be directed perpendicular to the motion of the air parcel and *to the right*. Consequently, as shown in Figure 3.12, the resulting geostrophic wind flows parallel to the isobars. Were the isobars more closely spaced in the horizontal, the magnitude of the PGF vector would be larger and a correspondingly larger Coriolis force would be required to achieve geostrophic balance. Therefore, the resulting geostrophic wind, though still oriented parallel to the isobars, would be of larger magnitude as well. Thus, to a fairly high degree of accuracy, the wind field (a vector quantity of great importance) can be uniquely specified by a 2-D representation of the scalar quantity, pressure. The mid-latitude atmosphere on Earth need not have been so accommodating to our desire for simplicity, but it is! Let us now examine the mathematical expression for the geostrophic wind.

Considering (3.35a) and (3.35b) we can write component expressions for the geostrophic balance as

$$-fv_g = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad \text{or} \quad v_g = \frac{1}{\rho f}\frac{\partial p}{\partial x} \tag{3.36a}$$

and

$$fu_g = -\frac{1}{\rho}\frac{\partial p}{\partial y} \quad \text{or} \quad u_g = -\frac{1}{\rho f}\frac{\partial p}{\partial y}. \tag{3.36b}$$

We see from (3.36) that the zonal (meridional) component of the geostrophic wind depends on the corresponding meridional (zonal) gradient of pressure in accord with our previous physical examination. In vector form, (3.36) becomes

$$\vec{V}_g = -\frac{1}{\rho f}\frac{\partial p}{\partial y}\hat{i} + \frac{1}{\rho f}\frac{\partial p}{\partial x}\hat{j} = \frac{1}{\rho f}\hat{k} \times \nabla p \tag{3.37}$$

which clearly demonstrates that the geostrophic wind ($\vec{V}_g$) must always be parallel to the isobars (i.e. perpendicular to ∇p) with a magnitude dependent on the inverse of density, the inverse of the Coriolis parameter, as well as the magnitude of the pressure gradient. Some other conclusions regarding the nature of the geostrophic flow can also be determined from (3.37). For a given magnitude of pressure gradient, the resulting geostrophic wind will be larger at lower latitude where the Coriolis parameter is smaller. However, the geostrophic balance cannot be considered at the equator (or very near it either) as at such low latitudes, the inverse of the Coriolis parameter becomes very large and the resulting $\vec{V}_g$ no longer bears a resemblance to the actual wind, $\vec{V}$. For mid-latitude flow, however, the geostrophic wind is usually within 10–15% of the observed wind. This observation does not imply that the mid-latitude atmosphere has a predilection for this simple balance, it instead testifies to the enormity of the two forces, PGF and COR, at middle latitudes.

Given that geostrophy is a balance between the PGF and Coriolis forces, we might inquire under what conditions is geostrophic balance met? Note that in (3.36) there is no reference to du/dt or dv/dt. As a consequence, the geostrophic wind is only strictly valid in regions of *zero* wind acceleration. Since the wind is a vector quantity, with magnitude and direction, if either of those properties is changed over time, the wind has been accelerated. Thus, two broad categories of flow in the atmosphere will violate the geostrophic balance: those characterized by (1) wind speed changes along the flow, and/or (2) wind direction changes along the flow. Figure 3.13 is a randomly selected northern hemisphere analysis of isobars and isotachs (lines of constant wind speed) at 9 km elevation. It is immediately clear that regions of along-flow speed variation and/or along-flow curvature are so numerous as to be the rule rather than the exception. The along-flow speed changes are most prominent in the vicinity of the local wind speed maxima known as **jet streaks**. Along-flow direction changes are most obvious in the vicinity of troughs and ridges in the pressure field. These locations, as we will show presently, are commonly associated with sensible weather in the form of circulation systems, clouds, and precipitation. The degree of departure from geostrophic balance that characterizes these regions can be assessed by considering the difference between the actual wind at a location and the calculated geostrophic wind at the same point. This difference is known as the

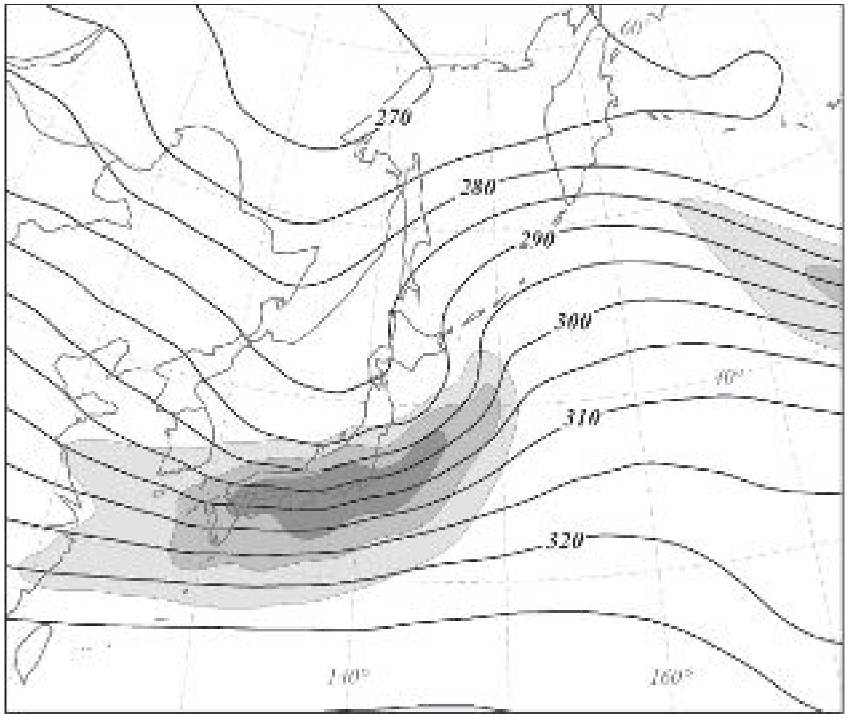

Figure 3.13 Isobars and isotachs at 9 km elevation from the National Center for Environmental Prediction's Global Forecast System initialization at 0000 UTC 23 February 2004. The isobars are labeled and contoured every 5 hPa and the isotachs are shaded every 10 m s^{-1} starting at 30 m s^{-1}

ageostrophic wind, $\vec{V}_{ag}$, and is defined mathematically as

$$\vec{V}_{ag} = \vec{V} - \vec{V}_g. \tag{3.38}$$

We can introduce some prognostic power to our simplified versions of (3.35) by retaining the next largest order terms from Table 3.1: namely, du/dt and dv/dt. The resulting expressions are

$$\frac{du}{dt} = fv - \frac{1}{\rho}\frac{\partial p}{\partial x} \tag{3.39a}$$

$$\frac{dv}{dt} = -fu - \frac{1}{\rho}\frac{\partial p}{\partial y}. \tag{3.39b}$$

If we now substitute (3.36) into (3.39) we get

$$\frac{du}{dt} = fv - fv_g = f(v - v_g) = fv_{ag} \tag{3.40a}$$

$$\frac{dv}{dt} = -fu + fu_g = -f(u - u_g) = -fu_{ag} \tag{3.40b}$$

Table 3.2 Characteristic scales for the terms in the vertical equation of motion

	1	2	3	4	5	6
	$\frac{dw}{dt}$	$-2\Omega\cos\phi u$	$\frac{-(u^2+v^2)}{a}$	$-\frac{1}{\rho}\frac{\partial p}{\partial z}$	$-g$	F_z
Characteristic scales	$\frac{UW}{L}$	$f_0 U$	$\frac{U^2}{a}$	$\frac{p}{\rho H}$	g	$\frac{\nu W}{H^2}$
Magnitudes (m s^{-2})	10^{-7}	10^{-3}	10^{-5}	10	10	10^{-15}

which can be written in vector form as

$$\frac{d\vec{V}}{dt} = -f\hat{k} \times \vec{V}_{ag}. \tag{3.41}$$

This expression clearly shows that the ageostrophic flow is associated with regions of Lagrangian acceleration of the wind. In the next section we will demonstrate why this ageostrophic wind is of such vital importance to understanding the dynamics of the mid-latitude atmosphere.

Given that geostrophic balance is such a strong constraint in the middle latitudes, there are many settings in which the ageostrophic wind is a very small portion of the actual wind. Therefore, it would be convenient if there were some easy way to characterize a flow to determine if it is likely to be nearly in geostrophic balance. Physically, a given flow will be nearly in geostrophic balance if the Lagrangian acceleration term (du/dt or dv/dt) is small compared to the Coriolis force term, as suggested by our scaling and Table 3.1. Recalling that the acceleration term is represented as U^2/L and the Coriolis force is scaled as $f_0 U$, then the ratio of these two accelerations is given by

$$\frac{\textit{Lagrangian Accel.}}{\textit{Coriolis Accel.}} = \frac{U^2/L}{f_0 U} = \frac{U}{f_0 L}. \tag{3.42}$$

Notice that this ratio is non-dimensional (i.e. it is just a number without units) and that if it is less than 0.1 for a given flow it testifies to the fact that the Coriolis acceleration is at least 10 times larger than the Lagrangian acceleration. In such a case, it is quite reasonable to approximate the flow as nearly geostrophic. The ratio defined in (3.42) is known as the **Rossby number** (R_0), after the famous atmospheric/oceanic scientist Carl Gustav Rossby.[6] We will hereafter often refer to flows that are nearly in geostrophic balance as low-R_0 flows. High-R_0 flows will, conversely, be characterized as rather far from geostrophic balance.

Thus far we have discussed the results of a scaling of the horizontal equations of motion. A similar exercise must now be performed on (3.35c), the vertical equation of motion. Table 3.2 shows the characteristic scales of the various terms in (3.35c)

[6] Carl Gustav Rossby (1898–1957) was a Swedish–American scientist who founded the first meteorology department in the United States at the Massachusetts Institute of Technology (MIT) in 1928. Rossby uncovered many of the basic principles of modern dynamical meteorology during the decades of the 1930s and 1940s.

along with their usual magnitudes for mid-latitude weather systems. Even more robustly than was the case for the horizontal equations, the vertical equation of motion is dominated by two terms: the vertical PGF and gravity. We have already seen that these two vertical forces are combined in the hydrostatic balance. Thus, a formal scaling of the equations of motion for mid-latitude synoptic-scale motions renders the following fundamental statement regarding the nature of the mid-latitude atmosphere on Earth:

To a first order, the mid-latitude atmosphere on Earth is in hydrostatic and geostrophic balance.

3.2.2 Conservation of mass

Imagine trying to fill a small basin with water from a hose. If there is a leak in the basin then one needs to know both the inflow rate from the hose as well as the outflow rate through the leak in order to accurately gauge the filling rate. If the inflow rate is suddenly increased while the outflow rate remains the same it is simple to conclude that the mass of water in the basin will increase. If we designate the mass of water in the basin as M_w, then a simple expression of the mass continuity equation becomes

$$\frac{\partial M_w}{\partial t} = \textit{Inflow Rate} - \textit{Outflow Rate}.$$

We can think of a slightly more abstract representation of this idea, illustrated in Figure 3.14, by considering an infinitesimal cube, fixed in space, through which air flows. The x-direction **mass flux** (i.e. the product of the x-direction velocity and the density of the fluid) at the center of the cube is given by ρu. Upon expanding this

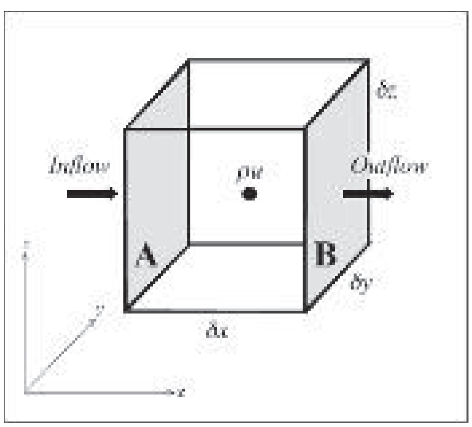

Figure 3.14 Schematic of x-direction flow through a cube fixed in space. The rate of mass flux is given by the product ρu. Accumulation of mass at the center point occurs when the inflow rate exceeds the outflow rate

function in a Taylor series about the center point we find that the rate of mass inflow through side A of the cube is given by

$$\left[\rho u - \frac{\partial}{\partial x}(\rho u)\frac{\delta x}{2}\right]\delta y \delta z \tag{3.43a}$$

while the rate of mass outflow through side B of the cube is given by

$$\left[\rho u + \frac{\partial}{\partial x}(\rho u)\frac{\delta x}{2}\right]\delta y \delta z. \tag{3.43b}$$

Now, just as in our simple example above, the rate of accumulation of mass (as a result of x-direction flow) inside the infinitesimal cube must be equal to the inflow rate minus the outflow rate. Using (3.43) this is expressed as

$$\begin{aligned}\frac{\partial M_x}{\partial t} &= \left[\rho u - \frac{\partial}{\partial x}(\rho u)\frac{\delta x}{2}\right]\delta y \delta z - \left[\rho u + \frac{\partial}{\partial x}(\rho u)\frac{\delta x}{2}\right]\delta y \delta z \\ &= -\frac{\partial}{\partial x}(\rho u)\delta x \delta y \delta z \end{aligned} \tag{3.44}$$

where M_x represents the rate of mass accumulation in the cube resulting from x-direction mass flux divergence. Similar expressions representing the rates of mass accumulation in the cube resulting from y- and z-direction mass flux divergences are given by

$$\frac{\partial M_y}{\partial t} = -\frac{\partial}{\partial y}(\rho v)\delta x \delta y \delta z \quad \text{and} \quad \frac{\partial M_z}{\partial t} = -\frac{\partial}{\partial z}(\rho w)\delta x \delta y \delta z$$

so that the net rate of mass accumulation in the cube is represented as

$$\frac{\partial M}{\partial t} = -\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right]\delta x \delta y \delta z. \tag{3.45}$$

By definition, the net mass accumulation rate per unit volume is equal to the Eulerian rate of change of the density. Thus, dividing (3.45) by the volume of the cube ($\delta x \delta y \delta z$) yields

$$\frac{\partial \rho}{\partial t} = -\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] = -\nabla \cdot (\rho \vec{V}). \tag{3.46}$$

The expression above is known as the **mass divergence** form of the mass continuity equation. An alternative form of this expression arises by recalling that

$$\nabla \cdot (\rho \vec{V}) = \rho \nabla \cdot \vec{V} + \vec{V} \cdot \nabla \rho$$

so that (3.46) becomes

$$\frac{\partial \rho}{\partial t} + \vec{V} \cdot \nabla \rho + \rho \nabla \cdot \vec{V} = 0 \quad \text{or} \quad \frac{1}{\rho}\frac{d\rho}{dt} + \nabla \cdot \vec{V} = 0 \tag{3.47}$$

which is known as the **velocity divergence** form of the mass continuity equation.

This exact same relationship can be derived for a cube of fixed mass, δM, but varying dimensions δx, δy, and δz. Given that the mass in this example is fixed, then $d(\delta M)/dt = 0$ or

$$\frac{d(\rho\delta x\delta y\delta z)}{dt} = 0 = \frac{d\rho}{dt}\delta x\delta y\delta z + \rho\frac{d(\delta x)}{dt}\delta y\delta z + \rho\frac{d(\delta y)}{dt}\delta x\delta z + \rho\frac{d(\delta z)}{dt}\delta x\delta y \tag{3.48a}$$

by the chain rule. Now

$$\lim_{\delta x\to 0}\frac{d(\delta x)}{dt} = \partial u$$

with similar expressions applying for the last two time derivatives in (3.48a). Therefore, dividing both sides of (3.48a) by the volume of cube gives

$$\frac{d\rho}{dt} + \rho\frac{\partial u}{\partial x} + \rho\frac{\partial v}{\partial y} + \rho\frac{\partial w}{\partial z} = \frac{d\rho}{dt} + \rho\nabla\cdot\vec{V} = 0 \tag{3.48b}$$

which can be easily rearranged into (3.47).

It is instructive at this point to consider the implications of (3.47) for the fluid atmosphere. A fluid in which individual parcels experience no change of density following the motion (i.e. $d\rho/dt = 0$) is known as an **incompressible fluid**. Conversely, a compressible fluid is one in which the density can change along a parcel trajectory. As you might guess, the atmosphere is a compressible fluid, but for many atmospheric phenomena the compressibility is not a major physical consideration. In such cases, the mass continuity equation becomes a statement of zero velocity divergence. We will see later that choice of a different vertical coordinate will render the continuity equation in a much simpler, unapproximated form.

3.3 Conservation of Energy: The Energy Equation

The law of conservation of energy states that the sum of all energies in the universe is constant. This is a valuable piece of knowledge but there are many different kinds of energies manifest in the atmosphere including kinetic energy, potential energy, latent heat energy, and radiant energy to name a few. Of all these types, radiant energy from the Sun is the source of nearly all of the total energy in the atmosphere/ocean system. When solar radiation is absorbed at the Earth's surface and in the atmosphere it appears as internal energy, made manifest as a temperature change. Given the many other kinds of energy involved in the atmosphere/ocean system, one of the major problems in the atmospheric sciences is determining how this internal energy is converted into the other forms of energy.

We can get some insights into the nature of the energies in the atmosphere by taking the dot product of the acceleration vector, $d\vec{V}/dt$, with the velocity vector, $\vec{V}$. This operation is the mathematical equivalent of multiplying the component equations of motion (3.35a, b, and c) by their respective component velocities (u, v, and w). The resulting expressions are

$$\frac{1}{2}\frac{d(u^2)}{dt} - \frac{u^2 v \tan\phi}{a} + \frac{u^2 w}{a} = -\frac{u}{\rho}\frac{\partial p}{\partial x} + 2\Omega \sin\phi uv - 2\Omega \cos\phi uw + uF_x \tag{3.49a}$$

$$\frac{1}{2}\frac{d(v^2)}{dt} + \frac{u^2 v \tan\phi}{a} + \frac{v^2 w}{a} = -\frac{v}{\rho}\frac{\partial p}{\partial y} - 2\Omega \sin\phi uv + vF_y \tag{3.49b}$$

$$\frac{1}{2}\frac{d(w^2)}{dt} - \frac{w(u^2+v^2)}{a} = -\frac{w}{\rho}\frac{\partial p}{\partial z} - gw + 2\Omega \cos\phi uw + wF_z. \tag{3.49c}$$

Summing the component expressions (3.49) together we note that all of the Coriolis and curvature terms sum to zero resulting in

$$\frac{d}{dt}\left[\frac{(u^2+v^2+w^2)}{2}\right] = -\frac{1}{\rho}\vec{V}\cdot\nabla p - gw + \vec{V}\cdot\vec{F}. \tag{3.50}$$

The LHS term in (3.50) represents the rate of change of the total kinetic energy (per unit mass) of the flow and so is a rate of work term. The first term on the RHS of (3.50) is pressure advection divided by density. When the velocity vector is directed across isobars from high to low (low to high) pressure, (3.50) shows that kinetic energy is produced (consumed). Note that if the flow were purely geostrophic, $\vec{V}\cdot\nabla p$ would vanish. This term is often referred to as the pressure work term and describes the rate of work done by the ageostrophic flow across isobars.

By definition, $w = dz/dt$, so that $-gw$ can be rewritten as

$$-gw = -g\frac{dz}{dt} = -\frac{d\phi}{dt}$$

where ϕ is the geopotential, a measure of the work required to raise a unit mass a distance, z, above sea level. It is instructive, therefore, to rewrite (3.50) as

$$\frac{d}{dt}\left[\frac{(u^2+v^2+w^2)}{2} + \phi\right] = -\frac{1}{\rho}\vec{V}\cdot\nabla p + \vec{V}\cdot\vec{F} \tag{3.51}$$

where the LHS represents the sum of the kinetic and potential energies per unit mass of an atmospheric parcel. The last term on the RHS of (3.51) represents the energy dissipated by the action of the friction force ($\vec{F}$). Note that since $\vec{V}$ and $\vec{F}$ are almost always opposite one another, the product $\vec{V}\cdot\vec{F}$ will be negative and the total kinetic and potential energies of the parcel will decrease in the presence of friction in accord with physical intuition.

Since (3.51) is derived from the equations of motion it deals only with mechanical forms of energy and is therefore referred to as the **mechanical energy equation**. In order to include thermal energy we must include the first law of thermodynamics in the form

$$\dot{Q} = c_v \frac{dT}{dt} + p\frac{d\alpha}{dt} \tag{3.52}$$

where $\dot{Q}$ represents the diabatic heating rate, c_v is the specific heat of dry air at constant volume ($717\,\mathrm{J\,kg^{-1}\,K^{-1}}$), and α is the specific volume. This expression relates the important fact that absorption of solar radiation (represented by $\dot{Q}$) can be converted to both internal energy (in the form of a temperature increase) or mechanical energy made manifest in expansion work (represented by the expansion term, $d\alpha/dt$). By rearranging (3.51) as

$$0 = \frac{d}{dt}\left[\frac{(u^2+v^2+w^2)}{2} + \phi\right] + \frac{1}{\rho}\vec{V}\cdot\nabla p - \vec{V}\cdot\vec{F}$$

we can add zero to both sides of (3.52) to yield

$$\dot{Q} = c_v\frac{dT}{dt} + p\frac{d\alpha}{dt} + \frac{d}{dt}\left[\frac{(u^2+v^2+w^2)}{2} + \phi\right] + \frac{1}{\rho}\vec{V}\cdot\nabla p - \vec{V}\cdot\vec{F}. \tag{3.53}$$

Noting that $(1/\rho)\vec{V}\cdot\nabla p$ is equal to $\alpha(dp/dt - \partial p/\partial t)$, and that

$$p\frac{d\alpha}{dt} + \alpha\frac{dp}{dt} = \frac{d}{dt}(p\alpha),$$

we can regroup terms and rewrite (3.53) as

$$\dot{Q} = \frac{d}{dt}\left[\frac{(u^2+v^2+w^2)}{2} + \phi + c_vT + p\alpha\right] - \alpha\frac{\partial p}{\partial t} - \vec{V}\cdot\vec{F} \tag{3.54}$$

which is known as the **energy equation**. This relationship implies that if the flow is frictionless ($\vec{F}=0$), adiabatic ($\dot{Q}=0$), and steady state ($\partial p/\partial t = 0$), then the quantity

$$\frac{(u^2+v^2+w^2)}{2} + \phi + c_vT + p\alpha$$

is constant. This is a special case of Bernoulli's[7] equation for an incompressible flow in which the quantity

$$\frac{(u^2+v^2+w^2)}{2} + \phi + p\alpha = \mathit{Constant}.$$

This relationship suggests that for an atmosphere at rest, any increase in elevation results, unsurprisingly, in a decrease in the hydrostatic pressure. If the atmosphere is in motion, however, a larger pressure difference will result over the same elevation interval since the difference, in this case, is a difference in the *dynamic* pressure. For

[7] Daniel Bernoulli (1700–1782) was a Swiss mathematician and fluid dynamicist. Though from an illustrious family of mathematicians, he studied medicine at his father's insistence and discovered a means to measure blood pressure that was used until the dawn of the twentieth century. When he was 25, Catherine the Great appointed him Professor of Mathematics at the Imperial Academy of St Petersburg where Leonhard Euler became one of his first students. He developed the fluid dynamical equation that bears his name at the age of 30.

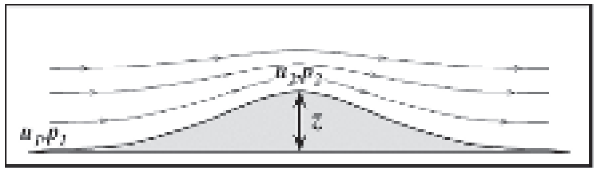

Figure 3.15 Flow over a hill illustrating the effect of dynamic pressure. Thin lines are streamlines of the flow – the closer the streamlines in the vertical, the greater the flow speed. Since $u_2 > 0$, p_2 is less than the hydrostatic pressure at height z

flow over the hill in Figure 3.15, as the air rises over the hill, the speed of the flow increases. Thus, the pressure difference between the top and the bottom of the hill $(p_2 - p_1)$ must be larger than their hydrostatic pressure difference because the wind speed is higher at the top than at the bottom of the hill $(u_2 > u_1)$.

Two additional relationships of meteorological consequence arise from further consideration of aspects of the energy equation. First, an illuminating alternative expression for the first law of thermodynamics arises from combining (3.52) and the ideal gas law. Differentiating the gas law with respect to time yields

$$p\frac{d\alpha}{dt} + \alpha\frac{dp}{dt} = R\frac{dT}{dt}. \tag{3.55a}$$

Substituting for $p d\alpha/dt$ (from (3.55a)) in (3.52), and recalling that $c_p = c_v + R$, yields

$$c_p\frac{dT}{dt} - \alpha\frac{dp}{dt} = \dot{Q}. \tag{3.55b}$$

If we then divide (3.55b) by T, noting that $\alpha/T = R/p$, we get

$$c_p\frac{d\ln T}{dt} - R\frac{d\ln p}{dt} = \frac{\dot{Q}}{T} \tag{3.55c}$$

where $\dot{Q}/T$ is known as the **entropy**. If the entropy is constant with time, then we have an **isentropic process** and, consequently,

$$c_p\frac{d\ln T}{dt} - R\frac{d\ln p}{dt} = 0. \tag{3.55d}$$

Integration of (3.55d) from a given p and T to a reference pressure, p_0, and a reference temperature, θ, defines what is known as the **potential temperature**. We begin by noting that

$$\int_T^{\theta} c_p d\ln T = \int_p^{p_0} R d\ln p$$

which yields

$$c_p(\ln\theta - \ln T) = R(\ln p_0 - \ln p).$$

Rearranging the above expression and taking anti-logs results in

$$\frac{\theta}{T} = \left(\frac{p_0}{p}\right)^{\frac{R}{c_p}} \quad \text{or} \quad \theta = T\left(\frac{p_0}{p}\right)^{\frac{R}{c_p}}, \tag{3.56}$$

known as the Poisson equation.

Physically, θ is the temperature a parcel of air would have if it were adiabatically compressed (or expanded) from its original pressure, p, to a reference pressure, p_0 (usually 1000 hPa). Every air parcel has a unique value of θ and that value is conserved for adiabatic processes (i.e. conditions in which the entropy does not change). For this reason, lines of constant θ are often referred to as **isentropes** and flow along surfaces of constant potential temperature is known as **isentropic flow**.

Finally, if we take the log differential of (3.56) with respect to height (z) we get

$$\frac{\partial \ln \theta}{\partial z} = \frac{\partial \ln T}{\partial z} + \frac{R}{c_p}\left(\frac{\partial \ln p_0}{\partial z} - \frac{\partial \ln p}{\partial z}\right). \tag{3.57a}$$

Since p_0 is a constant, its derivative is zero and (3.57a) can be rewritten as

$$\frac{1}{\theta}\frac{\partial \theta}{\partial z} = \frac{1}{T}\frac{\partial T}{\partial z} - \frac{R}{c_p p}\frac{\partial p}{dz}. \tag{3.57b}$$

Substituting for $\partial p/dz$ from the hydrostatic equation yields

$$\frac{1}{\theta}\frac{\partial \theta}{\partial z} = \frac{1}{T}\frac{\partial T}{\partial z} + \frac{R\rho g}{c_p p}. \tag{3.57c}$$

Finally, with the help of the gas law and some rearranging, (3.57c) can be written as

$$\frac{T}{\theta}\frac{\partial \theta}{\partial z} = \frac{\partial T}{\partial z} + \frac{g}{c_p} \tag{3.57d}$$

which yields an expression for the dry adiabatic lapse rate (Γ_d). If θ is constant with height (i.e. the lapse rate is dry adiabatic), then $-\partial T/\partial z = \Gamma_d = g/c_p = 9.8°\mathrm{C\,km^{-1}}$. When $\partial\theta/\partial z$ is non-zero, the lapse rate ($\Gamma = -\partial T/\partial z$) is given by

$$\Gamma = \Gamma_d - \frac{T}{\theta}\frac{\partial \theta}{\partial z}. \tag{3.58}$$

Based upon (3.58), there are three conditions for stability that can be assessed. First, when $\partial\theta/\partial z > 0$, then $\Gamma < \Gamma_d$ which corresponds to a statically stable stratification. In such an environment, a lifted parcel of dry air (which must cool at the dry adiabatic rate) will always be cooler than its new environment. Second, when $\partial\theta/\partial z = 0$, then $\Gamma = \Gamma_d$ and the stratification is said to be neutral and a lifted parcel of dry air will always have the same temperature as its new surroundings. Finally, when $\partial\theta/\partial z < 0$, then $\Gamma > \Gamma_d$ which corresponds to an absolutely unstable stratification. In such a case, a lifted parcel of dry air will always be warmer than its new surroundings and will, therefore, freely convect.

In the statically stable case just described, a lifted parcel, being colder than its environment upon being lifted, will be forced back downward to its original level once the impulse that forced it to rise is exhausted. A series of oscillations about that original level will ensue. The frequency of these buoyancy oscillations is dependent on the restoring force that compels them. In this case, the restoring force (per unit volume) is the product of gravity and the density difference between the displaced parcel and its environment.

If we let δz be the vertical displacement of an air parcel about its original level, then Newton's second law tells us that

$$\frac{F_z}{Mass} = \frac{dw}{dt} = \frac{d^2(\delta z)}{dt^2}. \tag{3.59a}$$

Letting ρ (ρ') and T (T') be the density and temperature of the environment (parcel) and assuming that the pressures of the parcel and the environment are always equal, then the restoring force (per unit volume) for a displaced parcel is given by

$$\frac{F_z}{Volume} = -(\rho' - \rho)g. \tag{3.59b}$$

Thus, the restoring force per unit mass for the displaced parcel can be written as

$$\frac{F_z}{Mass} = -\frac{(\rho' - \rho)g}{\rho'}. \tag{3.59c}$$

Employing the gas law allows this expression to be rewritten as

$$\frac{F_z}{Mass} = -\left(\frac{1}{T'} - \frac{1}{T}\right) gT' = -g\left(\frac{T - T'}{T}\right). \tag{3.59d}$$

Now we can say that $(T - T')$ is equal to $(\Gamma_d - \Gamma)\delta z$ since the dry parcel cools at the dry adiabatic lapse rate and must be compared to the environment whose temperature changes at a rate described by Γ. Therefore, the restoring force per unit mass can be written as

$$\frac{F_z}{Mass} = -\frac{g}{T}(\Gamma_d - \Gamma)\delta z \tag{3.59e}$$

so that (3.59a) becomes a second-order, ordinary differential equation

$$\frac{d^2(\delta z)}{dt^2} + \frac{g}{T}(\Gamma_d - \Gamma)\delta z = 0 \tag{3.60}$$

whose solution describes a buoyancy oscillation with period $2\pi / N$ where

$$N = \left[\frac{g}{T}(\Gamma_d - \Gamma)\right]^{1/2}$$

or, substituting from (3.58),

$$N = \left[\frac{g}{\theta}\frac{\partial\theta}{\partial z}\right]^{1/2}. \tag{3.61}$$

N is known as the **Brunt–Väisälä frequency** and has units of s^{-1}. It is clear from (3.61) that for the condition of neutrality alluded to earlier (i.e. $\partial\theta/\partial z = 0$), $N = 0$ and there is no buoyancy oscillation physically consistent with a neutral displacement. For the statically stable case (i.e. $\partial\theta/\partial z > 0$), $N > 0$ and buoyancy oscillations are observed. For the absolutely unstable case (i.e. $\partial\theta/\partial z < 0$), N is imaginary and in perturbation theory such a case corresponds to a growing disturbance. Physically, this is consistent with the fact that in an absolutely unstable stratification, a lifted parcel of dry air will always be warmer than its environment and therefore, according to (3.59), experience an upward-directed buoyancy force without interruption. It is important to note that instances of absolute instability are exceedingly rare and, even when they do exist, are very short-lived as the atmosphere mixes rapidly toward neutrality in such instances.

Selected References

Hess, *Introduction to Theoretical Meteorology*, offers an alternative perspective on accelerating reference frames.
Holton, *An Introduction to Dynamic Meteorology*, provides discussion of many of the same issues.
Brown, *Fluid Mechanics of the Atmosphere*, provides illuminating discussion of the energy equation.
Acheson, *Elementary Fluid Dynamics*, discusses many of the same issues.

Problems

3.1. Assume that air flows over a broad building 10 m high. The flow is in steady state and the density is constant ($\rho = 1.3\,\mathrm{kg\,m^{-3}}$) through this depth of the atmosphere. The observed speed at ground level is $5\,\mathrm{m\,s^{-1}}$ while on the rooftop it is $9\,\mathrm{m\,s^{-1}}$.

(a) What is the pressure difference, in hPa, between the ground and roof level?
(b) How much of this pressure difference is purely hydrostatic?
(c) What is the magnitude and direction of the non-hydrostatic pressure gradient force vector generated by these circumstances?

In all of the above, you may neglect the vertical variation in temperature.

3.2. (a) Prove that the divergence of the geostrophic wind is given by

$$\nabla \cdot \vec{V}_g = -V_g(\cot\phi/a)$$

where a = radius of the earth and ϕ is latitude.
(b) Explain why (physically) this is true. (Hint: recall that the magnitude of the Coriolis force depends on wind speed.)
(c) Calculate the divergence of the geostrophic wind at 43°N at a point where $|v_g| = 20\,\mathrm{m\,s^{-1}}$.

3.3. The perturbation ocean surface height (POSH) is defined as the height of the local ocean surface above or below mean sea level (which is 0 meters). Suppose a sophisticated satellite instrument is built that can measure the local POSH to an accuracy of 1 cm. A