We want to compute $\sigma^2 = \int_{-\infty}^{\infty} x^2 \rho(x)\, dx \Big/ \int_{-\infty}^{\infty} \rho(x)\, dx$

We have $\rho(x) = \int_{-\infty}^{\infty} \frac{\rho_0(\xi)}{\sqrt{4\pi D t}} e^{-(x-\xi)^2/4Dt} d\xi$

$$= \int_{-d_0}^{d_0} \frac{\rho_0}{\sqrt{4\pi D t}} e^{-(x-\xi^2)/4Dt} d\xi$$

We begin with

$$\int_{-\infty}^{\infty} \rho(x)\, dx = \int_{-\infty}^{\infty}\left[\int_{-d_0}^{d_0} \frac{\rho_0}{\sqrt{4\pi D t}} e^{-(x-\xi)^2/4Dt} d\xi\right] dx$$

$$= \int_{-d_0}^{d_0}\left[\int_{-\infty}^{\infty} \frac{\rho_0}{\sqrt{4\pi D t}} e^{-(x-\xi)^2/4Dt} dx\right] d\xi$$

We set $y = (x-\xi)/\sqrt{4Dt} \Rightarrow dy = \frac{dx}{\sqrt{4Dt}}$

Thus we get $\int_{-\infty}^{\infty} \rho(x)\, dx = \int_{-d_0}^{d_0}\left[\int_{-\infty}^{\infty} \frac{\rho_0}{\sqrt{\pi}} e^{-y^2} dy\right] d\xi$

$$= \int_{-d_0}^{d_0} \rho_0\, d\xi = \rho_0 \int_{-d_0}^{d_0} 1 \cdot d\xi = 2 d_0 \rho_0$$

↳ Exercise Set 3 or "Hints"

We still need to compute

$$\int_{-\infty}^{\infty} x^2 \rho(x)\, dx = \int_{-\infty}^{\infty}\left[\int_{-d_0}^{d_0} x^2 \frac{\rho_0}{\sqrt{4\pi D t}} e^{-(x-\xi)^2/4Dt} d\xi\right] dx$$

$$= \int_{-d_0}^{d_0}\left[\int_{-\infty}^{\infty} x^2 \frac{\rho_0}{\sqrt{4\pi D t}} e^{-(x-\xi)^2/4Dt} dx\right] d\xi$$ we set again $y = (x-\xi)/\sqrt{4Dt}$

$$= \int_{-d_0}^{d_0}\int_{-\infty}^{\infty} \frac{\rho_0}{\sqrt{\pi}} (\sqrt{4Dt}\, y + \xi)^2 e^{-y^2} dy\, d\xi$$

$$= \underbrace{\int_{-d_0}^{d_0}\int_{-\infty}^{\infty} \frac{\rho_0}{\sqrt{\pi}} (4Dt) y^2 e^{-y^2} dy\, d\xi}_{①} + \underbrace{\int_{-d_0}^{d_0}\int_{-\infty}^{\infty} \frac{2\rho_0}{\sqrt{\pi}} \sqrt{4Dt}\, \xi\, y\, e^{-y^2} dy\, d\xi}_{②}$$

$$+ \underbrace{\int_{-d_0}^{d_0}\int_{-\infty}^{\infty} \frac{\rho_0 \xi^2}{\sqrt{\pi}} e^{-y^2} dy\, d\xi}_{③}$$

② is zero because $y e^{-y^2}$ is an odd function

① gives $\int_{-d_0}^{d_0} (4Dt) \frac{\rho_0}{2} \, d\xi = 4 d_0 Dt \rho_0$ (for $\int_{-\infty}^{\infty} y^2 e^{-y^2} dy$ see exercise set 3)

③ gives $\int_{-d_0}^{d_0} \rho_0 \xi^2 \, d\xi = \frac{2}{3} d_0^3 \rho_0$ (for $\int_{-\infty}^{\infty} e^{-y^2} dy$ see exercise set 3)

"Hints"