Microscopic description

$$\langle x_i(n) \rangle := \frac{1}{N} \sum_{i=1}^{N} x_i(n)$$

particle i

$a_n^i = \pm 1$

n = timestep

$$x_i(n) = x_i(n-1) + a_{n-1}^i \delta$$

(1) $\langle x_i(n) \rangle = \langle x_i(n-1) \rangle + \langle a_{n-1}^i \delta \rangle$

$$= \langle x_i(n-1) \rangle + \underbrace{\langle a_{n-1}^i \rangle \delta}_{=0} = \langle x_i(n-1) \rangle \overset{\text{by recurrence}}{=} \underbrace{\langle x_i(0) \rangle}_{\text{initial condition}} = 0$$

(2) Variance $\langle (x_i(n) - \langle x_i(n) \rangle)^2 \rangle \overset{\text{exercise}}{=} \langle x_i^2(n) \rangle \underbrace{- \langle x_i(n) \rangle^2}_{=0}$

And $\langle x_i^2(n) \rangle = \frac{1}{N} \sum_{i=1}^{N} \left[x_i^2(n-1) + 2 a_{n-1}^i \delta x_i(n-1) + \underbrace{a_n^{i\,2}}_{=1} \delta^2 \right]$

$$= \langle x_i^2(n-1) \rangle + \underbrace{2 \delta \langle a_{n-1}^i x_i(n-1) \rangle}_{=0} + \delta^2$$

$=0$ (on average no privileged direction) as many i with $a^i = +1$ than i with $a^i = -1$

$n = 1$: $x_i(n-1) = 0$

$\Rightarrow \langle x_i^2(1) \rangle = \delta^2$

$\langle x_i^2(2) \rangle = \langle x_i^2(1) \rangle + \delta^2 = 2\delta^2$

$\Rightarrow$ Variance $\langle x_i^2(n) \rangle = n \delta^2 = \underbrace{\frac{\delta^2}{\tau}}_{=:2D} t$

Length associated with diffusion $\sigma(n) = \sigma(t) = \sqrt{2Dt}$