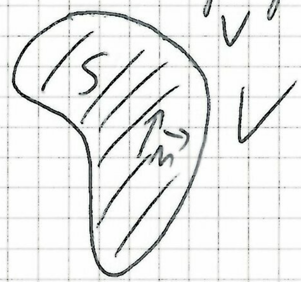
Advection equation

$$\frac{dQ}{dt} + \oiint_S \vec{j} \cdot d\vec{S} = \underbrace{\Sigma}_{\text{source / sink}} \qquad (1)$$

$\vec{j} \cdot \vec{n} > 0$ for flux going out of V



Q = amount of substance in volume V

S = closed surface (normals oriented outside)

$\vec{j}$ = flux of Q per unit area and unit of time

We use the divergence theorem and assume $\Sigma = 0$.

$$\frac{dQ}{dt} = \iiint_V \frac{\partial C}{\partial t}(x, y, z, t)\, dV \overset{(1)}{=} -\iiint_V \vec{\nabla} \cdot \vec{j}\, dV$$

$$\Rightarrow \frac{\partial C}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0$$

in the case of advection $\vec{j}_{advection} = C\vec{u}$

Thus $$\frac{\partial C}{\partial t} + \vec{\nabla} \cdot \underbrace{[C \cdot \vec{u}]}_{\text{in the presence of diffusion this term is different}} = 0$$

$$\vec{\nabla} \cdot [C \cdot \vec{u}] = (\partial_x, \partial_y, \partial_z) \begin{pmatrix} Cu \\ Cv \\ Cw \end{pmatrix} = \partial_x(Cu) + \partial_y(Cv) + \partial_z(Cw)$$

$$\vec{u} = u\vec{e}_x + v\vec{e}_y + w\vec{e}_z \qquad = [\vec{\nabla} C] \cdot \vec{u} + C[\vec{\nabla} \cdot \vec{u}]$$

For incompressible fluid $\vec{\nabla} \cdot \vec{u} = 0$ (continuity equation)

$$\Rightarrow \boxed{\frac{\partial C}{\partial t} + \vec{u} \cdot \vec{\nabla} C = 0}$$

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