



Advection Diffusion Equation (ADE)

$$\frac{\partial C}{\partial t} + \vec{u} \cdot \vec{\nabla} C = D \nabla^2 C \quad \text{linear in } \vec{u} \text{ and } \vec{C}$$

in components, the equation becomes

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} + w \frac{\partial C}{\partial z} = D \nabla^2 C$$

$$\vec{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = u \vec{e}_x + v \vec{e}_y + w \vec{e}_z$$

Comparison with the Navier-Stokes equation:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \vec{u} = \frac{\mu}{\rho} \nabla^2 \vec{u} + \dots \quad \text{Non-linear! Much more complex!}$$

$$\text{in components: } \frac{\partial \vec{u}}{\partial t} + u \frac{\partial \vec{u}}{\partial x} + v \frac{\partial \vec{u}}{\partial y} + w \frac{\partial \vec{u}}{\partial z} = \frac{\mu}{\rho} \nabla^2 \vec{u} + \dots$$

1D ADE, point source

→ examples: tracer going down a river
- last year's exam

$$\frac{\partial C}{\partial t} + V_{adv} \frac{\partial C}{\partial x} = D \frac{\partial^2 C}{\partial x^2} \quad (*)$$

$$\text{We set } \xi = x - V_{adv} t$$

$$\tau = t$$



(2)

We have

$$\frac{\partial}{\partial t} \cdot = \frac{\partial \xi}{\partial t} \frac{\partial}{\partial \xi} \cdot + \frac{\partial \tau}{\partial t} \frac{\partial}{\partial \tau} \cdot = -V_{adv} \frac{d}{d\xi} \cdot + \frac{d}{d\tau} \cdot$$

$$\frac{\partial}{\partial x} \cdot = \left(\frac{\partial x}{\partial \xi} \right) \frac{\partial}{\partial \xi} \cdot + \frac{\partial \tau}{\partial x} \frac{\partial}{\partial \tau} \cdot = \frac{\partial}{\partial \xi}$$

$$\frac{\partial^2}{\partial x^2} \cdot = \frac{\partial^2}{\partial \xi^2}$$

(x) thus becomes

$$-V_{adv} \frac{d}{d\xi} C + \frac{d}{d\tau} C + V_{adv} \frac{d}{d\xi} C = D \frac{d^2 C}{d\xi^2}$$

$$\Rightarrow \frac{dC}{d\tau} = D \frac{d^2 C}{d\xi^2} \quad D.E.$$

Solution already known $C(\xi, \tau) = \frac{M}{A\sqrt{4\pi D\tau}} e^{-\xi^2/4D\tau}$

$$\Rightarrow C(x, t) = \frac{M}{A\sqrt{4\pi D t}} e^{-(x - V_{adv} t)^2/4D t}$$

→ Transform in the other direction

I.C. $C(x, t=0) = \frac{M}{A} \delta(x)$ satisfied



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Lengthscales

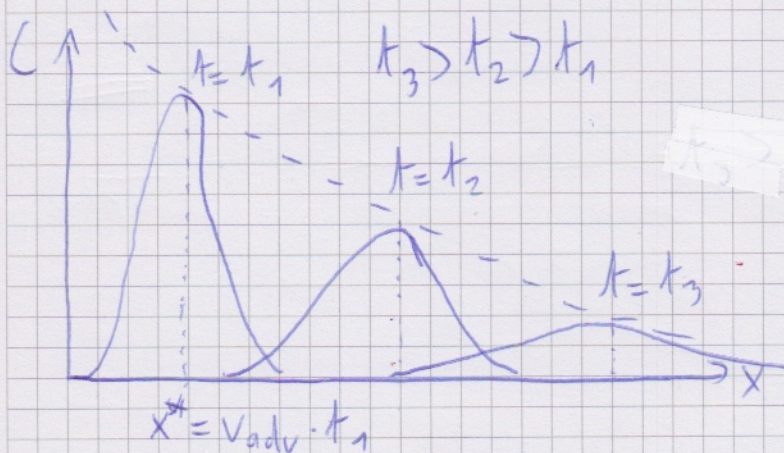
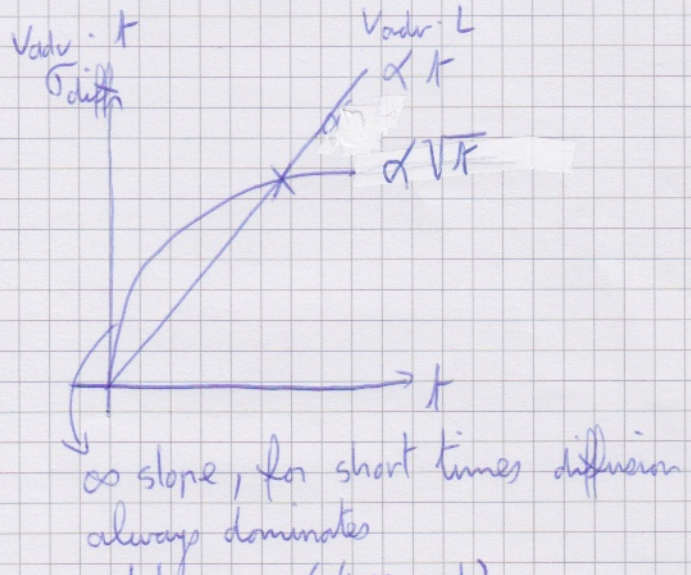
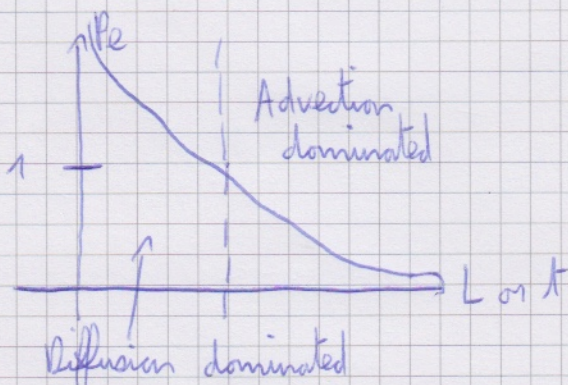
From the dimensional analysis, we know the dimensionless number

$$Pe := \frac{D}{v_{adv} L} \quad \text{"Peclet number"}$$

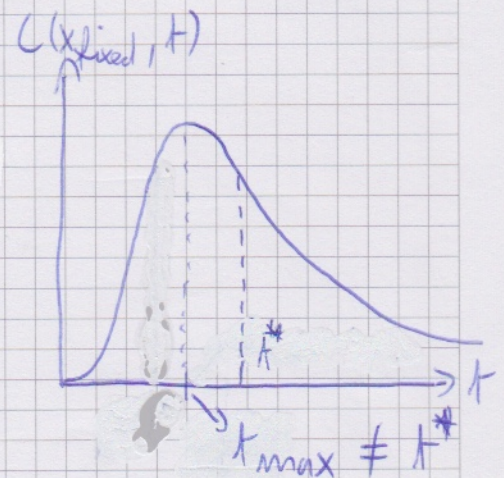
Δ sometimes the fraction is reversed in the definition of the Peclet number

The Peclet number can also be written as $Pe = \frac{D}{v_{adv} L}$ with $L = v_{adv} \cdot t$.

$L \rightarrow$ distance downstream



Max located at $x^* = v_{adv} \cdot t$, $C_{max} = \frac{M}{A \sqrt{4 \pi D t}}$



Not symmetric $t^* = \frac{x_{fixed}}{v_{adv}}$

Fixed time

Fixed position