Optimal Control (EE-736)

Summary and Wrap-Up

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Block course @ EPFL

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Content of the Course

Part I – Basics of Optimization

- Revisiting optimality conditions for Nonlinear Programs

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- Optimality conditions for Optimal Control Problems with finite-dimensional systems
- Direct and indirect solutions methods for OCPs
- Translating control tasks into OCP formulations

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- Direct and indirect solutions methods for OCPs
- Translating control tasks into OCP formulations

Part III – Nonlinear Model Predictive Control

- Principle of Predictive Control
- Stability of NMPC with and without terminal constraints
- Economic NMPC formulations
- Numerical implementation of NMPC $\rightarrow$ exercises and projects
- Simulation case studies $\rightarrow$ exercises and projects

Design Steps for Optimal Control and NMPC

4. NMPC Design

- Stability? State estimation? Robustness? ...

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- Choose a numerical strategy to obtain an (approximated) solution.
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- Does the model have a solution for any admissible control?
- Can the constraints be met? $\rightarrow$ feasibility
- Can the system outputs be observed/measured?
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1. Problem Formulation

- Model of system dynamics and constraints (physical restrictions)?
- Choose a performance criterion (objective functional).
- Choose the decision variables.

OCPs and NLPs

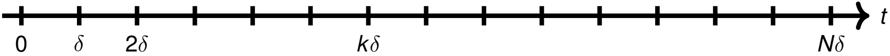
$\dot{x} = f(x(t), u(t)), x(0) = x_0$

Solution: $x(t) = x(t, x_0, u(\cdot))$

$$\xrightarrow[t = k\delta]{\text{Num. Solution of ODE}}$$

$x(k+1) = f^d(x(k), u(k)), x(0) = x_0$

Solution: $x(k) = x(k, x_0, u(\cdot))$



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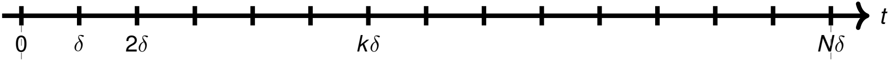

$$V_T(x_0) \doteq \min_{u(\cdot)} \int_0^T \ell(x(\tau), u(\tau))\, \mathrm{d}\tau + \varphi(x(T))$$

- ODE + initial condition
- State and input constraints
  $x(t) \in \mathbb{X} \subseteq \mathbb{R}^{n_x},\ u(t) \in \mathbb{U} \subseteq \mathbb{R}^{n_u}$
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$$\xrightarrow{\text{Discretization}}$$

$$\min_{u(\cdot)} \sum_{k=0}^{N-1} \ell(x(k), u(k)) + \varphi(x(N))$$

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Optimization $\downarrow$

Pontryagin's Maximum Principle
(PMP)

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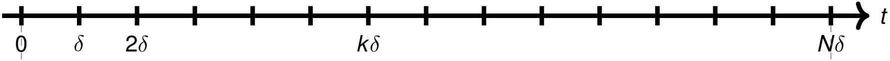

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KKT conditions of discretized OCP

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Optimization $\downarrow$ Optimization $\downarrow$

Pontryagin's Maximum Principle (PMP) $\xrightarrow{\text{Discretization}}$ KKT conditions of discretized OCP

$$V_T(x_0) \doteq \int_0^t \ell(x^\star(\tau, x_0, u^\star(\cdot)), u^\star(\tau))\,\mathrm{d}\tau \quad + V_{T-t}((x^\star(t, x_0, u^\star(\cdot)))$$

Bellman's Principle of Optimality

Meaning of Adjoints and Multipliers

PMP (w.o. state constraints)

$$H(x,u,\lambda) = \lambda_0 \ell(x,u) + \lambda^\top f(x,u) \quad (\lambda_0 \equiv 1)$$

$$\begin{aligned}
\dot{x}^\star &= H_\lambda(x^\star, u^\star, \lambda^\star) \\
&= f(x^\star, u^\star) \\
\dot{\lambda}^\star &= -H_\lambda(x^\star, u^\star, \lambda^\star) \\
&= -f_x^\top \lambda^\star - \ell_x
\end{aligned}$$

$$\forall v \in \mathbb{U}: \quad H(x^\star, v, \lambda^\star) \geq H(x^\star, u^\star, \lambda^\star)$$

$$\begin{aligned}
x^\star(0) &= x_0 \\
\lambda^\star(T) &= \varphi_x(x^\star(T)) + \ldots
\end{aligned}$$

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KKT cond. of discretized OCP

$$\begin{aligned}
L_k \doteq\ & \ell(x(k), u(k)) + \lambda(k+1)^\top (x(k+1) - f^d(x(k), u(k))) \\
& + \mu(k)^\top g(x(k), u(k))
\end{aligned}$$

$$L \doteq \lambda(0)^\top (x(0) - x_0) + \sum_{k=0}^{N-1} L_k + \varphi(x(N))$$

$$\begin{aligned}
L_\lambda = 0 \rightarrow & \quad x^\star(k+1) = f^d(x^\star(k), u^\star(k)) \\
L_x = 0 \rightarrow & \quad \lambda^\star(k) = (f_x^d)^\top \lambda^\star(k+1) + \ell_x + g_x^\top \mu(k) \\
L_u = 0 \rightarrow & \quad 0 = (f_u^d)^\top \lambda^\star(k) \quad + \ell_u + g_u^\top \mu(k)
\end{aligned}$$

$\dots$

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$\dots$

What is the meaning of $\lambda^\star$?

- Sensitivity of optimal value function w.r.t. pertubations of equality constraints (dynamics)
- OCPs and NLPs

Nonlinear Model Predictive Control

1. State measurement/estimate $x(t_k)$

2. Solve $OCP(x(t_k))$:

$$\min_{u(\cdot|t_k))} \int_{t_k}^{t_k+T} \ell(x(\tau|t_k)), u(\tau|t_k))\,\mathrm{d}\tau + V_f(x(t_k+T|t_k))$$

$$\forall \tau \in [t_k, t_k+T] : \frac{\mathrm{d}x(\tau|t_k)}{\mathrm{d}\tau} = f(x(\tau|t_k), u(\tau|t_k))$$

$$x(t_k|t_k) = x(t_k)$$

$$x(\tau|t_k) \in \mathbb{X},\;\; u(\tau|t_k) \in \mathbb{U}$$

$$x(t_k+T|t_k) \in \mathbb{X}_f$$

3. Apply $u^\star(\tau|t_k)$ for $\tau \in [t_k, t_{k+1})$

NMPC for Setpoint Stabilization

- Lower boundedness of stage cost

$$\ell(x,u) \geq \alpha(\|x-\bar{x}\|)$$

- With term. constraints and penalty ($\mathbb{X}_f$ and V_f)

$$\frac{\partial V_f}{\partial x} \cdot f(x,u) + \ell(x,u) \leq 0$$

- Without term. constraints and penalty ($\mathbb{X}_f$ and V_f)

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Economic NMPC

- Dissipation inequality

$$\frac{\partial S}{\partial x} \cdot f(x, u) \leq -\alpha(\|(x - \bar{x})\|) + \ell(x, u) - \ell(\bar{x}, \bar{u})$$

- Turnpike properties are crucial
- With term. constraints and penalty ($\mathbb{X}_f$ and V_f)
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The End

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