technische universität dortmund

EPFL

Optimal Control (EE-736)

Part II.1: Optimal Control Theory
In-Class Exercises

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Overview

In-Class Exercise – Formulation of NLPs

In-Class Exercise – Basics of Optimal Control

In-Class Exercise – Optimality Conditions

In-Class Exercise – Singular OCPs

Overview

Task I – Optimal Fishing

Setting

Consider the following model of a fish population in a habitat operated by a company:

$$\dot{x} = x(x_s - x - u), \quad x(0) = x_0 > 0 \tag{1}$$

where $x \in \mathbb{R}_0^+$ is the fish density and $u \in \mathbb{R}_0^+$ is the fishing rate. The following objective $\ell : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ models the economic operation of the company's fishing business

$$\ell(x, u) = -ax - bu + cxu$$

for given values of x and u. The parameter values are
$a = 1, b = c = 2, x_s = 5$.[1]

[1] The example is modification from E.M. Cliff **and** T.L. Vincent. "An optimal policy for a fish harvest". **in** *Journal of Optimization Theory and Applications*: 12.5 (1973), **pages** 485–496

Task I – Optimal Fishing (cont'd)

Tasks

a) Interpret the parameters a, b, c and x_s and the objective ℓ.

b) How to compute steady solutions to (5)?

c) Formulate an NLP to model optimal stationary operation of the fishing business.
Hint: Yo do not need to consider further constraints on x and u besides stationarity.

d) Solve the NLP for $x^\star, u^\star$ and $\lambda^\star$.

e) How could you check for global optimality of the solution?

Task I – Optimal Fishing **Solution**

a)
- a: cost of keeping per fish population
- b: cost of fishing per fish population
- c: income related to fish density and fishing rate
- x_s: highest sustainable fish density
- ℓ: profit

b)

$$\dot{x} = x(x_s - x - u) \stackrel{!}{=} 0 \rightarrow x = 0 \text{ or } u = x_s - x$$

c)

$$\min_{(x,u)} \; -\ell(x,u)$$
$$\text{s. t.} \;\; x(x_s - x - u) = 0$$

d) We have $f(x,u) = -\ell = ax + bu - cxu$ and $h(x,u) = x(x_s - x - u)$. Then Theorem I.3 gives

$$\begin{aligned} \nabla_x f(x^\star,u^\star) + \nabla h_x(x^\star,u^\star)\lambda^\star &= a - cu + \lambda(x_s - 2x - u) &&= 0 \\ \nabla_u f(x^\star,u^\star) + \nabla h_u(x^\star,u^\star)\lambda^\star &= b - cx - \lambda x &&= 0 \end{aligned}$$

Together with $h(x^\star,u^\star) = 0$ we get $x^\star = 2.75$, $u^\star = 2.25$, $\lambda^\star = -1.27$.

e) The problem is non-convex. No guarantee of global optimality.

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Task II – Formulation of OCPs

Given the following in-complete optimal control problem:

$$\min_{u(\cdot)} \int_{t_0}^{t_1} \ell(t, x(t), u(t))\mathrm{d}t$$

subject to

$$\dot{x}(t) = f(t, x(t), u(t)), \quad x(t_0) = x_0$$

The underlying control problem requires that the state x hits a target curve $g : [t_0, \infty) \to \mathbb{R}^{n_x}$ at time t_1.

a) Propose a constraint which models this control problem.

b) What are real-world examples of such a control problems?

c) What are reasons why the proposed constraint might lead to infeasibility?

Task II – Formulation of OCPs (cont'd)

d) Propose an objective functional which models the control problem. Can you come up with a formulation which is guaranteed to be feasible?

Task III – Formulation of OCPs

In the lecture we have shown how an OCP in Lagrange form can be rewritten as an OCP in Mayer form.

- Show that a Mayer problem can also be written in Lagrange form.
  Hint: Consider a Mayer term which only depends on the terminal state, i.e., $\phi(t_1, x(t_1))$.

Task IV – Weak and Strong Norms

Let $\mathcal{B}_\varepsilon^{\infty,1}(\bar{x})$ be an open ε-ball, centred at $\bar{x}$ and let the underlying linear space $\mathcal{X}$ be endowed with the weak norm $\|\cdot\|_{\infty,1}$. Similarly, let $\mathcal{B}_\varepsilon^{\infty}(\bar{x})$ be an open ε-ball, centred at $\bar{x}$ and the underlying linear space $\mathcal{X}$ is endowed with the strong norm $\|\cdot\|_{\infty}$.

Which of the following statements are correct?

a) $x \in \mathcal{B}_\varepsilon^{\infty}(\bar{x}) \quad \Rightarrow \quad x \in \mathcal{B}_\varepsilon^{\infty,1}(\bar{x})$ **Solution:** False

b) $x \in \mathcal{B}_\varepsilon^{\infty,1}(\bar{x}) \quad \Rightarrow \quad x \in \mathcal{B}_\varepsilon^{\infty}(\bar{x})$ **Solution:** True

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Task V – Relation of NCOs of OCPs and NLPs

Consider the following OCP

$$\min_{u(\cdot)} \int_{t_0}^{t_1} \ell(x(t), u(t))\mathrm{d}t \tag{2a}$$

subject to

$$\dot{x}(t) = f(x(t), u(t)), \quad x(t_0) = x_0 \tag{2b}$$

and the corresponding steady-state optimization problem

$$\min_{\bar{u},\bar{x}} \ell(\bar{x}, \bar{u}) \tag{3a}$$

subject to

$$0 = f(\bar{x}, \bar{u}). \tag{3b}$$

a) Formulate the NCOs of (2).
b) Formulate the NCOs of (3).
c) Show that the NCOs of (3) correspond to the NCOs of (2) evaluated at steady state.
   Hint: Use the NCO formulation based on the Hamiltonian.

Task VI – Relation of NCOs of OCPs and NLPs

Consider a second OCP

$$\min_{u(\cdot)} \int_{t_0}^{t_1} \ell(x(t), u(t))\mathrm{d}t + \bar{\lambda}^\top x(t_1) \tag{4a}$$

subject to

$$\dot{x}(t) = f(x(t), u(t)), \quad x(t_0) = x_0 \tag{4b}$$

where $\bar{\lambda}^\top \in \mathbb{R}^{n_x}$ is a constant vector.

a) Classify (4) as a Mayer, Bolza, or Lagrange problem.

b) Formulate the NCOs of (4). Denote the adjoint variable of OCP (4) as $\tilde{\lambda}$.

c) Reformulate (4) as a Lagrange problem. Derive the NCOs of (4) formulated as Lagrange problem. Denote the adjoint variable as $\hat{\lambda}$.

d) What is the relation between $\tilde{\lambda}$ and $\hat{\lambda}$?

Task VI – Relation of NCOs of OCPs and NLPs (cont'd)

e) Give a geometrical interpretation of the reformulation of (4) as a Lagrange problem.

f) Compare the adjoint equations for (4) and (2). Give an interpretation of the Mayer term $\varphi(x) = \bar{\lambda}^\top x$ in terms of the adjoint dynamics.

Task VII – Example

$$\min_{u(\cdot)} \int_0^1 \frac{1}{2} u^2(t)\mathrm{d}t$$

subject to

$$\dot{x}(t) = u(t) - x(t), \quad x(0) = 1, \; x(1) = 0$$
$$u(\cdot) \in \mathcal{C}([0,1], \mathbb{R})$$

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Task VIII – Optimal Fishing

Setting

Consider the following model of a fish population in a habitat operated by a company:

$$\dot{x} = x(x_s - x - u), \quad x(t_0) = x_0 > 0 \tag{5}$$

where $x \in \mathbb{R}_0^+$ is the fish density and $u \in [0, 5]$ is the fishing rate. The following objective $\ell : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ models the economic operation of the company's fishing business

$$\ell(x, u) = -ax - bu + cxu$$

for given values of x and u. The parameter values are $a = 1, b = c = 2, x_s = 5$.[2]

[2] The example is a modification from E.M. Cliff **and** T.L. Vincent. "An optimal policy for a fish harvest". **in** *Journal of Optimization Theory and Applications*: 12.5 (1973), **pages** 485–496

Task VIII – Optimal Fishing (cont'd)

Tasks

a) Formulate an OCP to model optimal operation of the fishing business. Hint: Observe that we already discussed the stationary (and simplified) variant of this problem in a previous task. You may neglect the state constraint $x \in \mathbb{R}_0^+$.
b) Derive the NCOs for the OCP formulated. Is the considered problem singular? Why?
c) Characterize the arcs of the optimal solution.
d) Determine the controls, states and adjoints on the singular arc.
e) Compare your result with the optimal steady state computed previously.
f) Can you conjecture a structure of the optimal input profile?
g) How could one avoid singular arcs in the OCP?

Task VIII – Optimal Fishing Solution

a) OCP reads

$$\min_{u(\cdot)} -\int_{t_0}^{t_1} \ell(x(t), u(t))\,\mathrm{d}t \tag{6a}$$

subject to

$$\dot{x}(t) = x(t)\big(x_s - x(t) - u(t)\big), \quad x(t_0) = x_0, \tag{6b}$$

$$u(\cdot) \in \hat{\mathcal{C}}([t_0, t_1], [0, 5]). \tag{6c}$$

Observe that when $x = 0$, $\dot{x} = x(x_s - x - u)$ implies $\dot{x} = 0$. Thus for any $x_0 > 0$, $x(\cdot) \geq 0$ always holds since the zero-crossing is impossible. Therefore, we can neglect the state constraint $x(\cdot) \in \mathbb{R}_0^+$.

b) The Hamiltonian function is

$$\begin{aligned} H(x, u, \lambda_0, \lambda) &= -\lambda_0 \ell(x, u) + \lambda f(x, u) \\ &= \lambda_0(ax + bu - cxu) + \lambda\big(x(x_s - x - u)\big). \end{aligned} \tag{7}$$

Task VIII – Optimal Fishing Solution

The following conditions are obtained from PMP

$$\dot{x}^\star = \quad H_\lambda = f(x^\star, u^\star), \qquad x^\star(t_0) = x_0 \tag{8a}$$

$$\dot{\lambda}^\star = -H_x = -\lambda_0^\star \ell_x(x^\star, u^\star) + \lambda^\star(t) f_x(x^\star, u^\star), \qquad \lambda^\star(t_1) = 0. \tag{8b}$$

Moreover, $H(x^\star, u, \lambda_0^\star, \lambda^\star)$ obtains its minimum on $\hat{\mathcal{C}}([t_0, t_1], [0, 5])$ at $u = u^\star$:

$$u^\star(\cdot) = \arg \min_{u(\cdot)\in\hat{\mathcal{C}}([t_0,t_1],[0,5])} H(x^\star, u, \lambda_0^\star, \lambda^\star) \tag{8c}$$

and $\lambda_0^\star$ is a constant. Hamiltonian (7) shows it is an affine function with respect to input u, which implies $H_{uu} = 0$. Therefore, (6) is said to be a singular OCP.

c) Arrange Hamiltonian (7) as

$$H = \lambda x(x_s - x) + \lambda_0 a x + \big(\lambda_0(b - cx) - \lambda x\big) u$$

The arcs of $u^\star(\cdot)$ can be characterized as

$$u^\star(\cdot) = \begin{cases} 0, & \text{if } \lambda_0(b - cx) - \lambda x > 0, \\ 5, & \text{if } \lambda_0(b - cx) - \lambda x < 0, \\ ?, & \text{if } \lambda_0(b - cx) - \lambda x = 0. \end{cases}$$

Task VIII – Optimal Fishing Solution

d) Choose $\lambda_0 = 1$ and compute $\frac{\mathrm{d}^k}{\mathrm{d}t^k} H_u$ until u appears:

$$H_u = b - cx - \lambda x \qquad\qquad = 0 \tag{9a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} H_u = cx^2 - \lambda x^2 + (a - cx_s)x = 0 \tag{9b}$$

Plug (9a) into (9b) to eliminate λ and compute the derivative with respect to t, we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} H_u = (2cx + a - b - cx_s)(x_s - x - u)x = 0, \tag{10}$$

where u appears. When (10) holds, we have either $\dot{x} = (x_s - x - u)x = 0$ or $2cx + a - b - cx_s = 0$, which both imply x is at steady state ($\dot{x} = 0$) on the singular arc and admit the same solution. Solving the NCOs on the singular arc, we get

$$x^\star = \frac{-a + b + cx_s}{2c}, \; u^\star = \frac{a - b + cx_s}{2c}, \; \lambda^\star = \frac{2bc}{-a + b + cx_s} - c.$$

Task VIII – Optimal Fishing Solution

e) The solutions are the same.

f) The structure of the optimal input can be divided into three parts: steering to the singular arc, staying on the singular arc, and leaving the singular arc.

g) Yes, we can add a strictly quadratic term in u into the stage cost to avoid singular arcs. In this fishing example, we could add εu^2 into the stage cost $-\ell$ such that it reads

$$\tilde{\ell}(x, u) = -\ell(x, u) + \varepsilon u^2.$$