technische universität dortmund

EPFL

Optimal Control (EE-736)

Part III: Nonlinear Model Predictive Control

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Block course @ EPFL

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Overview

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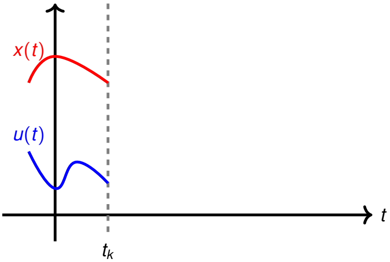
Why do we need feedback control?

Principle of Predictive Control

Predictive control $\equiv$ repeated optimal control.



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2. Solve $OCP(x(t_k))$:

$$\min_{u(\cdot|t_k)} \int_{t_k}^{t_k+T} \ell(x(\tau|t_k), u(\tau|t_k))\,\mathrm{d}\tau + V_f(x(t_k+T|t_k))$$

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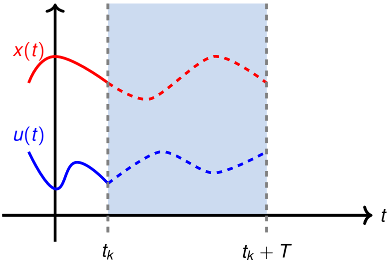

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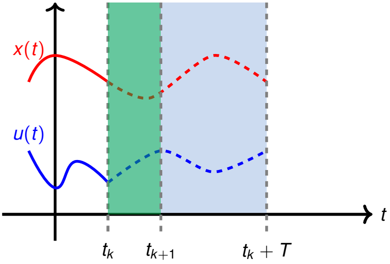

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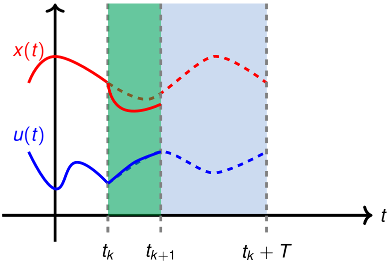

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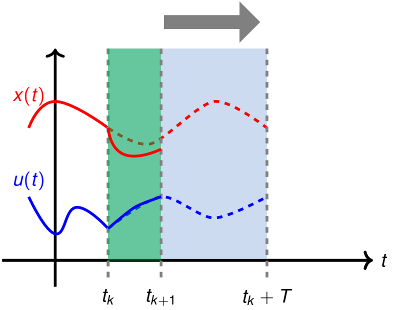

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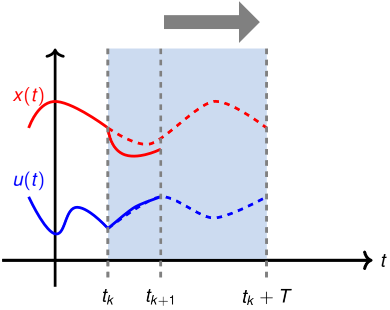

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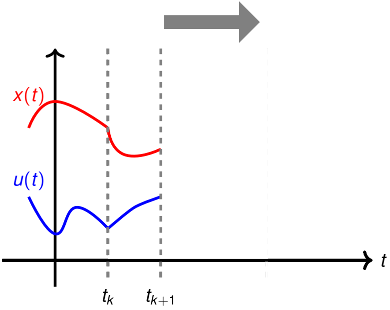

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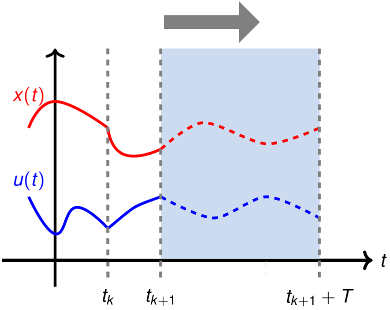

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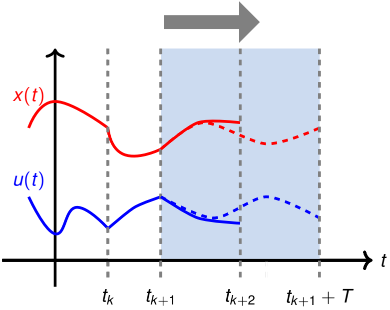

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Notions and Notation:

- MPC $=$ pred. control with linear model, convex quadratic cost and linear inequality constraints.
- NMPC $=$ pred. control with nonlinear models and/or non-quadratic stage cost / cost function.
- $x(\cdot|t_k)$ denotes predicted state trajectory with initial condition $x(t_k)$.
- $u(\cdot|t_k)$ denotes predicted input trajectory for the NMPC feedback at time t_k.

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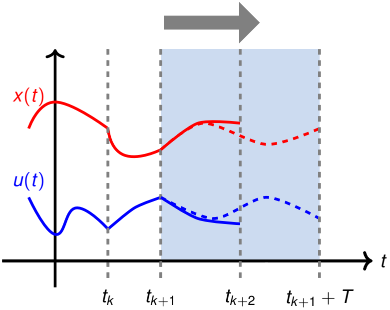

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Questions:

- How to design and implement an NMPC scheme?
- Why use an NMPC scheme?

Ingredients for Sampled-Data NMPC Design

- System model $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$
- State constraints $\mathbb{X} \subseteq \mathbb{R}^{n_x}$
- Input constraints $\mathbb{U} \subseteq \mathbb{R}^{n_u}$
- State feedback $x(t_k)$

$\rightarrow$ **Assumed to be exactly known.**

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- Stage cost $\ell : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}$
- Terminal penalty $V_f : \mathbb{R}^{n_x} \to \mathbb{R}$
- Terminal constraint $\mathbb{X}_f \subseteq \mathbb{X} \subseteq \mathbb{R}^{n_x}$
- Prediction horizon $T \in (0, \infty)$
- Sampling period $\delta = t_{k+1} - t_k \in (0, T]$

$\to$ **To be designed/chosen!**

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$\downarrow$

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Motivations for Using NMPC

Feedback Control

- Control of nonlinear MIMO systemd subject to constraints
- *Pre-requisite* = control task at hand, such as setpoint stabilization, trajectory tracking, disturbance attenuation, ...
- Choose ℓ, V_f, $\mathbb{X}_f$, δ, T according to task.

$\rightarrow$ **Next!**

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Economics / Economic Operation

- Receding horizon approximation to the solution of an infinite-horizon OCP.
- *Pre-requisite* $=$ specified performance functional:

$$J_\infty = \int_0^\infty \ell(x(\tau), u(\tau))\, \mathrm{d}\tau,$$

i.e. **the cost function ℓ is given**!

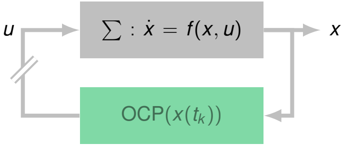
- Choose δ, T and analyze properties of NMPC loop.

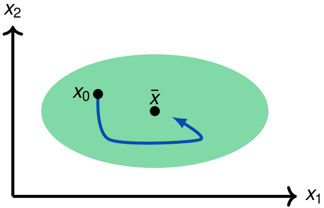
$\rightarrow$ **Later!**

Overview

Considered Control Problem – Setpoint Stabilization

- Reference = setpoint $\bar{x} \in \mathbb{X} \subseteq \mathbb{R}^{n_x}$





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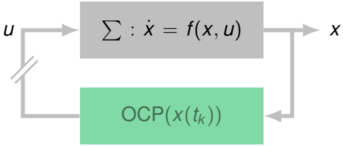

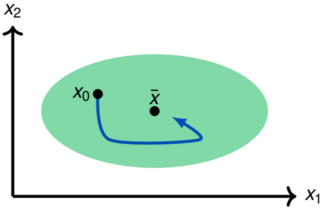

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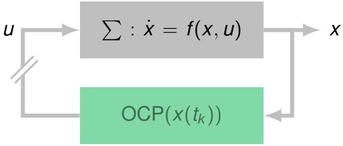
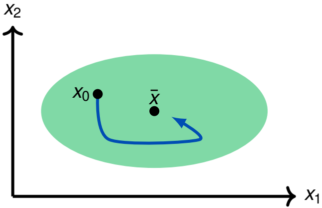
- Constraint satisfaction: $\forall t \geq 0 : u(t) \in \mathbb{U}$ and $x(t, x_0, u(\cdot)) \in \mathbb{X}$
- Stability: $\forall \varepsilon > 0 \; \exists \delta > 0$ such that

$$\|x(0) - \bar{x}\| < \delta \quad \Rightarrow \quad \|x(t, x_0, u(\cdot)) - \bar{x}\| < \varepsilon \quad \forall t \geq 0$$





NMPC Pitfall Example

$$\min_{u(\cdot|t_k)} \int_{t_k}^{t_k+T} x(\tau|t_k)^\top Q x(\tau|t_k) + u(\tau|t_k)^\top R u(\tau|t_k)\, \mathrm{d}\tau$$

subject to

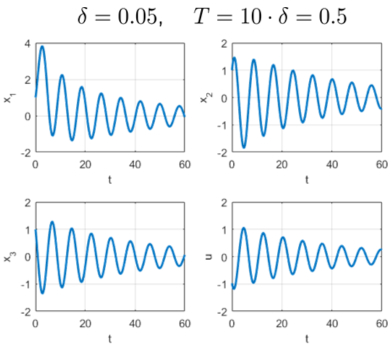
$$\forall \tau \in [t_k, t_k + T]: \ \frac{\mathrm{d}x(\tau|t_k)}{\mathrm{d}\tau} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x(\tau|t_k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(\tau|t_k),$$

$$x(t_k|t_k) = x(t_k)$$

Setting:

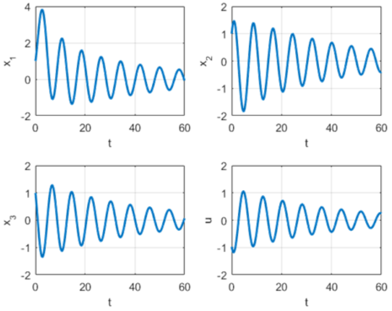
- Weight matrices: $Q = \mathrm{diag}(5, 5, 0)$, $\qquad R = 1$
- Inputs approximated as piece-wise constant
- Sampling period: $\delta = 0.05$
- Prediction horizon: $T = N \cdot \delta$, $N =$ # number of shooting intervals

NMPC Pitfall Example (cont'd)

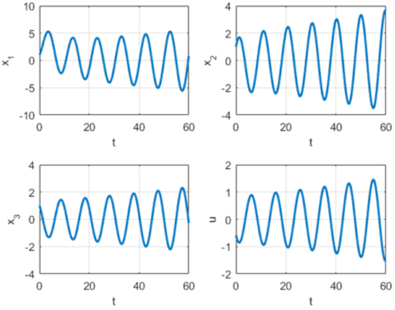


NMPC Pitfall Example (cont'd)

$\delta = 0.05, \quad T = 10 \cdot \delta = 0.5$

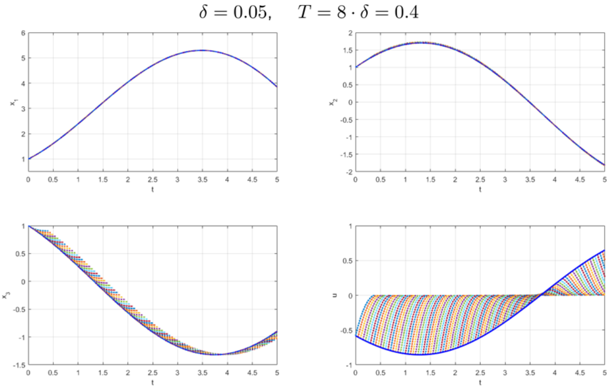


$\delta = 0.05, \quad T = 8 \cdot \delta = 0.4$



- At each sampling time t_k, the optimal input $u^\star(t_k|t_k)$ is applied.
- **For short horizons the closed loop is unstable.**

A Pitfall Example of NMPC (cont'd)



- Predicted trajectories $\neq$ closed-loop trajectories:

$$x^\star(\tau|t_k) \neq x(\tau, x(t_k), u^\star(\cdot|t_k)), \quad \tau \in (t_k + \delta, t_k + T]$$

Stability vs. Optimality in NMPC

In the engineering literature it is often assumed (tacitly and incorrectly) that a system with an optimal control law is necessarily stable.

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- Likewise in NMPC: **Optimality does not imply stability!**
- **There is a need for NMPC design procedures guaranteeing stability!**

Useful Definitions

Definition (Class-$\mathcal{K}$ function)

- *A scalar function* $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *is said to belong to class* $\mathcal{K}$*, if it is continuous, strictly increasing, and* $\alpha(0) = 0$*.*
- $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *is said to belong to class* $\mathcal{K}_\infty$*, if* $\alpha \in \mathcal{K}$ *and if it is radially unbounded, i.e.* $\alpha(s) \to \infty$ *as* $s \to \infty$*.*

C.M. Kellett. "A compendium of comparison function results". **in** *Mathematics of Control, Signals, and Systems*: 26.3 (2014), **pages** 339–374

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Definition (Stability of equilibria)

The system $\dot{x} = f(t, x)$ *with* $f(t, 0) = 0$ *is said to be uniformly (locally) stable at* $x = 0$*, if for every* $\epsilon > 0$ *there exists an* $\delta = \delta(\epsilon) > 0$*, which is independent from* t_0*, such that all solutions* $x(\cdot, t_0, x_0)$ *fulfill*

$$\|x_0\| < \delta \quad \Rightarrow \quad \|x(t, t_0, x_0)\| < \epsilon \quad \textit{for all } t \geq t_0 \geq 0.$$

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The system $\dot{x} = f(t, x)$ with $f(t, 0) = 0$ is said to be uniformly (locally) stable at $x = 0$, if for every $\epsilon > 0$ there exists an $\delta = \delta(\epsilon) > 0$, which is independent from t_0, such that all solutions $x(\cdot, t_0, x_0)$ fulfill

$$\|x_0\| < \delta \quad \Rightarrow \quad \|x(t, t_0, x_0)\| < \epsilon \quad \textit{for all } t \geq t_0 \geq 0.$$

If $x = 0$ is a uniformly stable equilibrium of $\dot{x} = f(t, x)$, if there exists a positive constant $c = c(t_0)$, and if, additionally, the solutions fulfill

i)
$$\lim_{t \to \infty} \|x(t, t_0, x_0)\| = 0 \quad \textit{for all } \|x_0\| < c,$$

ii) *and for each $\eta > 0$ there exists $T = T(\eta) > 0$ such that*
$$\|x(t, t_0, x_0)\| < \eta, \quad \textit{for all } t \geq t_0 + T(\eta), \quad \textit{for all } \|x_0\| < c,$$

then $x = 0$ is said to be uniformly (locally) asymptotically stable.

Useful Definitions (cont'd)

Lemma (Lyapunov function and stability)

Let $\beta_1, \beta_2 \in \mathcal{K}_\infty$, and $\beta_3 \in \mathcal{K}$, and the system $\dot{x} = f(t, x)$ fulfills $f(t, 0) = 0$. Consider some compact domain $\mathbb{X}$ containing $x = 0$ in its interior, and a function $V : \mathbb{R}_0^+ \times \mathbb{X} \to \mathbb{R}_0^+$ such that

$$\begin{aligned} \beta_1(\|x\|) \leq V(t, x) &\leq \quad \beta_2(\|x\|) \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) &\leq -\beta_3(\|x\|) \end{aligned}$$

holds for all $t \geq 0$ and all $x \in \mathbb{X}$. Then $x = 0$ is uniformly asympt. stable on $\mathbb{X}$.

For further details on stability definitions see:

- H.K. Khalil. *Nonlinear Systems*. 3rd. Prentice Hall, New Jersey, 2002
- E.D. Sontag. *Mathematical Control Theory: Deterministic Finite Dimensional Systems*. Springer, 1998

Nominal NMPC Setting

Here we consider an **nominal NMPC setting:**

- No plant-model mismatch, i.e., f is an exact plant model.

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- No plant-model mismatch, i.e., f is an exact plant model.
- State feedback $=$ plant state $x(t_k)$ is exactly known at time t_k.
- For all $k \in \mathbb{N}$, an optimal solution to OCP($x(t_k)$) exists and is attained.

Standing Assumptions

A1 (Steady state): Given $\bar{x}$, there exists $\bar{u} \in \mathbb{U}$, such that $0 = f(\bar{x}, \bar{u})$ and $(\bar{x}, \bar{u}) \in \mathrm{int}(\mathbb{X} \times \mathbb{U})$. W.l.o.g. we suppose $(\bar{x}, \bar{u}) = (0, 0)$.

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A2 (Lower boundedness of ℓ): There exists a class-$\mathcal{K}$ function $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, such that

$$\ell(x, u) \geq \alpha(\|x - \bar{x}\|), \quad \text{and} \quad \ell(0, 0) = 0.$$

Standing Assumptions

A3 (Absolute continuity of ODE solutions): For all $x_0 \in \mathbb{X}$, and any $u(\cdot) \in \hat{\mathcal{C}}([0, T], \mathbb{U})$, the solution $x(\cdot, x_0, u(\cdot))$ exists on $[0, T]$ and is absolutely continuous.

Definition (Absolute continuity)

A trajectory $x(\cdot)$ is said to be absolutely continuous on $[0, T]$, iff

- *$x(t)$ is almost everywhere differentiable w.r.t. t, $\dot{x}(t)$ is Lebesgue integrable, and*
- *for all $t \in [0, T]$,*

$$x(t) = x(0) + \int_0^t \dot{x}(\tau)\, \mathrm{d}\tau$$

holds.

Instantaneous Infinite-Horizon NMPC

1. State measurement/estimate $x(t_k)$

2. Solve:

$$\min_{u(\cdot|t_k)} \int_{t_k}^{t_k+\infty} \ell(x(\tau|t_k), u(\tau|t_k)) \, \mathrm{d}\tau$$

$$\text{subject to } \forall \tau \in [t_k, t_k + \infty] : \qquad (OCP_\infty(x(t_k)))$$

$$\frac{\mathrm{d}x(\tau|t_k)}{\mathrm{d}\tau} = f(x(\tau|t_k), u(\tau|t_k)), \quad x(t_k|t_k) = x(t_k)$$

$$x(\tau|t_k) \in \mathbb{X}, \;\; u(\tau|t_k) \in \mathbb{U}$$

3. Apply $u^\star(\tau|t_k)$ for $\tau \in [t_k, t_{k+1})$

- Infinite prediction horizon ($T = \infty$) and instantaneous recalculation ($\delta \doteq t_{k+1} - t_k = 0$)
- No terminal penalty ($V_f(x(t_k + T|t_k)) = 0$) and no terminal region ($\mathbb{X}_f = \mathbb{X}$)
- Value function: $V_\infty(x(t)) \doteq \displaystyle\int_t^{t+\infty} \ell(x^\star(\tau|t_k), u^\star(\tau|t_k)) \, \mathrm{d}\tau$

Stability of Instantaneous Infinite-Horizon NMPC

Theorem (Stability of instantaneous infinite-horizon NMPC)

Let Assumptions A1–A3 hold and suppose that,

(i) *for all $x \in \mathbb{X}_0$, the value function $V_\infty(x)$ is continuously differentiable and*

$$\beta_1(\|x\|) \le V_\infty(x) \le \beta_2(\|x\|), \qquad \beta_{1,2} \in \mathcal{K}_\infty$$

Then, the NMPC scheme based on $OCP_\infty(x(t_k))$ achieves local asymptotic stability of Σ at $x = 0$. The region of attraction is given by the set of initial conditions for which (i) holds.

A. Jadbabaie, J. Yu, J. Hauser. “Unconstrained receding-horizon control of nonlinear systems”. **in***IEEE Trans. Automat. Contr.*: 46.5 (2001), **pages** 776–783. DOI: 10.1109/9.920800

Instantaneous Infinite-Horizon NMPC

Proof sketch:

Instantaneous Infinite-Horizon NMPC

Overview

From Infinite to Quasi Infinite-Horizon NMPC

Idea: Truncate OCP to finite horizon and bound *cost-to-go*.

$$\int_{t_k}^{\infty} \ell(x(\tau), u(\tau))\, \mathrm{d}\tau = \int_{t_k}^{t} \ell(x(\tau), u(\tau))\, \mathrm{d}\tau + \underbrace{\int_{t}^{\infty} \ell(x(\tau), u(\tau))\, \mathrm{d}\tau}_{\text{cost-to-go}}$$

Consider

$$\frac{\partial V_f}{\partial x} f(x, u) \leq -\ell(x, u)$$

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Consider

$$\frac{\partial V_f}{\partial x} f(x, u) \leq -\ell(x, u)$$

Integrating from t to ∞ gives

$$V_f(x(\infty)) - V_f(x(t)) + \int_{t}^{\infty} \ell(x(\tau), u(\tau))\, \mathrm{d}\tau \leq 0.$$

Let $V_f(x(\infty)) = 0$, then

$$-V_f(x(t)) + \int_{t}^{\infty} \ell(x(\tau), u(\tau))\, \mathrm{d}\tau \leq 0.$$

From Infinite to Quasi Infinite-Horizon NMPC

The inequality

$$\frac{\partial V_f}{\partial x} f(x,u) \leq -\ell(x,u)$$

implies that $V_f : \mathbb{X}_f \to \mathbb{R}$, $V_f \in \mathcal{C}^1$ is an upper bound on the cost-to-go, i.e,

$$\int_t^{\infty} \ell(x(\tau), u(\tau)) \,\mathrm{d}\tau \leq V_f(x(t))$$

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Drawback: How to find global bound on cost to go?

Local Bound on Cost-To-Go

Assume that, for all $x \in \mathbb{X}_f \subseteq \mathbb{X}$, there exists a feedback $u = k(x)$

$$\frac{\partial V_f}{\partial x} f(x, k(x)) + \ell(x, k(x)) \leq 0. \tag{1}$$

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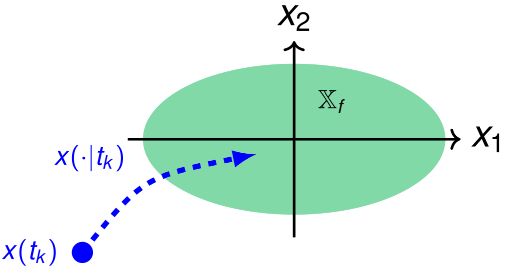

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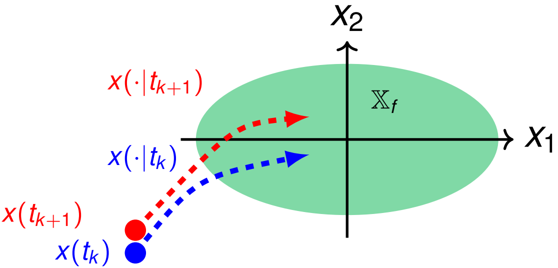

Idea: For NMPC require that all predicted trajectories end in the terminal set $\mathbb{X}_f$, whereby the local feedback $k : x \in \mathbb{X}_f \mapsto u \in \mathbb{U}$ satisfy (1).

Sampled-Data Quasi-Infinite-Horizon NMPC

1. State measurement/estimate $x(t_k)$

2. Solve:

$$\min_{u(\cdot|t_k)} \int_{t_k}^{t_k+T} \ell(x(\tau|t_k), u(\tau|t_k))\,\mathrm{d}\tau + V_f(x(t_k+T|t_k))$$

$$\text{subject to } \forall \tau \in [t_k, t_k+T]: \qquad (OCP_T^{\mathbb{X}_f}(x(t_k)))$$

$$\frac{\mathrm{d}x(\tau|t_k)}{\mathrm{d}\tau} = f(x(\tau|t_k), u(\tau|t_k)), \quad x(t_k|t_k) = x(t_k)$$

$$x(\tau|t_k) \in \mathbb{X}, \;\; u(\tau|t_k) \in \mathbb{U}$$

$$x(t_k+T|t_k) \in \mathbb{X}_f$$

3. Apply $u^\star(\tau|t_k)$ for $\tau \in [t_k, t_{k+1})$

- Finite prediction horizon ($T < \infty$) and sampled-data recalculation ($\delta \doteq t_{k+1} - t_k > 0$)
- Terminal penalty ($V_f \neq 0$) and terminal region ($\mathbb{X}_f \subseteq \mathbb{X}$)
- Value function: $V_T^{\mathbb{X}_f}(x(t_k)) \doteq \int_{t_k}^{t_k+T} \ell(x^\star(\tau|t_k), u^\star(\tau|t_k))\,\mathrm{d}\tau + V_f(x^\star(t_k+T|t_k))$

Convergence of Quasi-Infinite-Horizon NMPC

Theorem (Convergence of sampled-data NMPC)

Let Assumptions A1–A3 hold and suppose that there exist V_f, $\mathbb{X}_f \subseteq \mathbb{X}$ ($0 \in \mathbb{X}_f$), and a feedback $k : \mathbb{X}_f \to \mathbb{U}$ such that

(i) $V_f : \mathbb{X}_f \to \mathbb{R}_0^+$ *is positive semi-definite and* $V_f(0) = 0$,

(ii) *for all* $x \in \mathbb{X}_f$: $\quad \dfrac{\partial V_f}{\partial x} f(x, k(x)) + \ell(x, k(x)) \leq 0$,
and for all $t \in [0, \delta]$: $x(t, x, k(x)) \in \mathbb{X}_f$,

(iii) $OCP_T^{\mathbb{X}_f}(x(t_k))$ *is feasible at* $k = 0$.

Then,

- $OCP_T^{\mathbb{X}_f}(x(t_k))$ *is recursively feasible,*
- *the NMPC scheme based on* $OCP_T^{\mathbb{X}_f}(x(t_k))$ *achieves* $\lim_{t \to \infty} \|x(t)\| = 0$,
- *and the region of attraction is given by the set of initial conditions for which (iii) holds.*

H. Chen **and** F. Allgöwer. "A quasi-infinite horizon nonlinear model predictive control scheme with guaranteed stability". **in** *Automatica*: 34.10 (1998), **pages** 1205–1217

F. Fontes. "A General Framework to Design Stabilizing Nonlinear Model Predictive Controllers". **in** *Sys. Contr. Lett.*: 42.2 (2001), **pages** 127–143

Main Steps of NMPC Stability Proofs

- Step 1 – Recursive feasibility of the sequence $OCP_T^{\mathbb{X}_f}(x(t_k))$ for all sampling instants t_k, $k \in \mathbb{N}$

Definition (Recursive feasibility of an OCP)

In the context of NMPC, an $OCP(x(t_k))$ is said to be recusively feasible, if, for all $k \in \mathbb{N}$ its feasibility at time t_k implies its feasibility at time t_{k+1}.

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- Step 4 – Consider the value function $V_T^{\mathbb{X}_f}(x(t))$ as a *Lyapunov* function of the closed-loop system

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Step 1 – Recursive Feasibility

Idea: Construct feasible $u(\cdot)$ for $\mathrm{OCP}_T^{\mathbb{X}_f}(x(t_{k+1}))$ from solution to $OCP_T^{\mathbb{X}_f}(x(t_k))$.

Consider

$$\tilde{u}_{k+1}(t) = \begin{cases} u^\star(t|t_k) & t \in [t_{k+1}, t_k + T) \\ k(x(t)) & t \in [t_k + T, t_{k+1} + T] \end{cases} .$$

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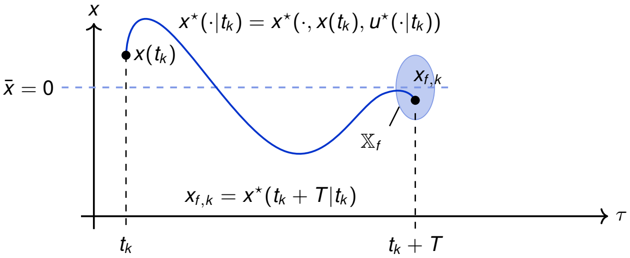

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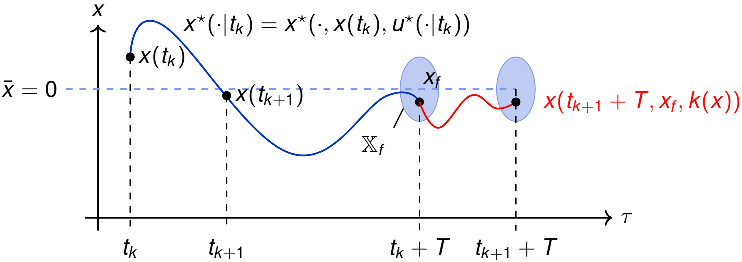

Observe that $\tilde{u}_{k+1} : [t_{k+1}, t_{k+1} + T] \to \mathbb{U}$, i.e. $\tilde{u}_{k+1}(\cdot)$ is feasible.

Step 2 – Decrease in-between two Consecutive Instants

Consider the value function

$$V_T^{\mathbb{X}_f}(x(t_k)) \doteq \int_{t_k}^{t_k+T} \ell(x^\star(\tau|t_k), u^\star(\tau|t_k))\,\mathrm{d}\tau + V_f(x^\star(t_k+T|t_k)).$$

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Hence,

$$V_T^{\mathbb{X}_f}(x(t)) \doteq V_T^{\mathbb{X}_f}(x(t_k)) - \int_{t_k}^{t} \ell(x^\star(\tau|t_k), u^\star(\tau|t_k)) \,\mathrm{d}\tau$$

is continuous and decreasing for all $t \in [t_k, t_{k+1})$.

Step 3 – Decrease from One Sampling Instant to the Next

For all $k \in \mathbb{N}$, it holds that $\quad V_T^{\mathbb{X}_f}(x(t_{k+1})) - V_T^{\mathbb{X}_f}(x(t_k)) \leq 0.$

Note

$$V_T^{\mathbb{X}_f}(x(t_{k+1})) - V_T^{\mathbb{X}_f}(x(t_k)) \leq J(x(t_{k+1}), \tilde{u}_{k+1}(\cdot)) - V_T^{\mathbb{X}_f}(x(t_k))$$

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Step 4 – Barbalat's Lemma and Convergence

Lemma (Barbalat's lemma)

Let $M : \mathbb{R}^{n_x} \to \mathbb{R}_0^+$ *be a continuous positive definite function and* $x(\cdot)$ *be an absolutely continuous function on* $\mathbb{R}$*. If* $x(\cdot) \in \mathcal{L}^\infty, \quad \dot{x}(\cdot) \in \mathcal{L}^\infty$ *and*

$$\lim_{t\to\infty} \int_0^t M(x(\tau))\,\mathrm{d}\tau < \infty$$

then

$$\lim_{t\to\infty} \|x(t)\| = 0.$$

H. Michalska **and** R.B. Vinter. "Nonlinear stabilization using discontinuous moving-horizon control". **in** *IMA Journal of Mathematical Control and Information*: 11.4 (1994), **pages** 321–340

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How to compute terminal regions and end penalties?

H. Chen **and** F. Allgöwer. "A quasi-infinite horizon nonlinear model predictive control scheme with guaranteed stability". **in** *Automatica*: 34.10 (1998), **pages** 1205–1217

F. Fontes. "A General Framework to Design Stabilizing Nonlinear Model Predictive Controllers". **in** *Sys. Contr. Lett.*: 42.2 (2001), **pages** 127–143

Overview

NMPC with Zero-Terminal Constraint

Let $\mathrm{OCP}_T^0(x(t_k))$ denote the variant of $OCP_T^{\mathbb{X}_f}(x(t_k))$ with $\mathbb{X}_f = \{0\}$, $V_f(x) = 0$.

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Corollary (Convergence with zero-terminal constraint)

Let Assumptions A1–A3 hold and suppose that the terminal penalty is $V_f(x) = 0$, the terminal region is $\mathbb{X}_f = \{0\}$, and

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D.Q. Mayne **and** H. Michalska. “Receding horizon control of nonlinear systems”. **in** *IEEE Trans. Automat. Contr.*: 35.7 (1990), **pages** 814–824. DOI: 10.1109/9.57020

S.S. Keerthi **and** E.G. Gilbert. “Optimal infinite-horizon feedback laws for a general class of constrained discrete-time systems: Stability and moving-horizon approximations”. **in** *Journal of Optimization Theory and Applications*: 57.2 (1988), **pages** 265–293

Structured Computation of Terminal Regions

Consider

$$\dot{x} = f(x, u), \quad x(0) \in \mathbb{X}_0 \tag{Σ}$$

with $f(0, 0) = 0$. Let

$$A \doteq \left.\frac{\partial f}{\partial x}\right|_{(0,0)}, \qquad B \doteq \left.\frac{\partial f}{\partial u}\right|_{(0,0)}.$$

Lemma (Nonlinear local stabilizability)

If the pair (A, B) is stabilizable—i.e., there exists $u = Kx$ such that the real parts of all eigenvalues of $A + BK$ are negative—then the feedback $u = Kx$ achieves local asymptotic stability of $x = \bar{x} = 0$ for the nonlinear system Σ.

H.K. Khalil. *Nonlinear Systems*. 3rd. Prentice Hall, New Jersey, 2002

Structured Computation of Terminal Regions

To compute a terminal region and a corresponding terminal penalty, we consider

- $\ell(x, u) = \frac{1}{2}x^\top Qx + \frac{1}{2}u^\top Ru$, with $Q \succ 0, R \succ 0$, and that
- the Jacobian linearization of $f(x, u)$ at $(0, 0)$, $(A,\ B)$, is stabilizable.

Let P be the positive definite solution of the Algebraic Riccati Equation (ARE)

$$A^\top P + PA - PBR^{-1}B^\top P + Q = 0, \quad P = P^\top \succ 0, \tag{ARE}$$

and consider the feedback $u = Kx$

$$K = -R^{-1}B^\top P.$$

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- the Jacobian linearization of $f(x,u)$ at $(0,0)$, $(A,\ B)$, is stabilizable.

Let P be the positive definite solution of the Algebraic Riccati Equation (ARE)

$$A^\top P + PA - PBR^{-1}B^\top P + Q = 0, \quad P = P^\top \succ 0, \tag{ARE}$$

and consider the feedback $u = Kx$

$$K = -R^{-1}B^\top P.$$

In Matlab the algebraic Riccati equation can be solved using lqr or care, dare commands.

Structured Computation of Terminal Regions

Furthermore, $u = Kx = -R^{-1}B^\top Px$ is the optimal solution to the following linear-quadratic OCP:

$$\begin{aligned} \min_{u(\cdot)} \quad & \int_{t_0}^{\infty} \frac{1}{2}\left(x^\top(\tau)Qx(\tau) + u^\top(\tau)Ru(\tau)\right)\, \mathrm{d}\tau \\ \text{subject to} \quad & \\ & \dot{x} = Ax + Bu, \quad x(t_0) = x_0 \end{aligned} \tag{LQR}$$

and the optimal value function of (LQR) is given by

$$V_{LQR}(x_0) = \frac{1}{2}x_0^\top Px_0.$$

B.D. Anderson **and** J.B. Moore. *Optimal Control - Linear Quadratic Methods*. Information and system science series. Prentice Hall, Englewood Cliffs, London, 1990

Structured Computation of Terminal Regions

Then, it is straightforward to show

- $u = Kx$ stabilizes the nonlinear system (Σ) locally, and
- $\frac{1}{2}x^\top Px$ is a local Lyapunov function of the nonlinear system Σ controlled by $u = Kx$.

Question: Domain of attraction of the nonlinear system (Σ) under the feedback $u = Kx$?

Structured Computation of Terminal Regions

Ellipsoidal ansatz for the terminal region

$$\mathbb{X}_f = \left\{ x \in \mathbb{R}^{n_x} \mid \tfrac{1}{2} x^\top P x \leq \rho^2 \right\}$$

with $\ell(x,u) = \frac{1}{2} x^\top Q x + \frac{1}{2} u^\top R u$ and $V_f(x) = \frac{1}{2} x^\top P x$

$$\begin{aligned} &\min_{\rho} \quad -\rho \\ &\text{subject to} \\ &\forall x \in \mathbb{X}_f: \quad Kx \in \mathbb{U} \\ &\qquad\qquad\quad \dot{V}_f(x) + \ell(x, Kx) \leq 0 \end{aligned} \tag{P_ρ}$$

Structured Computation of Terminal Regions

Ellipsoidal ansatz for the terminal region

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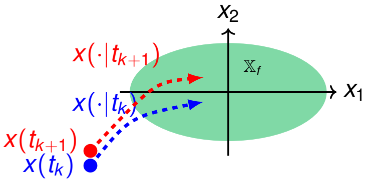
- $\max \rho$ is equivalent to $\max \; vol(\mathbb{X}_f)$ for fixed P
- Semi-infinite program $\rightarrow$ difficult to solve
- $\dot{V}_f(x) + \ell(x, Kx) \leq 0$ is a non-convex constraint
- Structural assumptions on nonlinearity of f and polytopic constraints for simplification

Structured Computation of Terminal Regions

Choice of terminal region and end penalty based on solution to linearized infinite-horizon problem (LQR):

- $\mathbb{X}_f = \left\{ x \in \mathbb{R}^{n_x} \mid \frac{1}{2} x^\top P x \leq \rho^2 \right\}$ with ρ from (P_ρ)
- $V_f(x) = \frac{1}{2} x^\top P x$
- P solves the algebraic Riccati equation (ARE)



H. Chen **and** F. Allgöwer. “A quasi-infinite horizon nonlinear model predictive control scheme with guaranteed stability”. **in** *Automatica*: 34.10 (1998), **pages** 1205–1217

Terminal regions $\mathbb{X}_f$:

- Guarantee recursive feasibility in presence of state constraints.
- Are often computed based on linearization
- Their computation is in general difficult (semi-infinite program)

Terminal penalties V_f:

- Typically constructed to be upper bounds on the cost-to-go
- Often computed based on linearization.
- Local Lyapunov functions arising from locally stabilizing feedbacks are natural candidates.
- Easiest non-trivial choice: value function of corresponding LQR problem

Food for thought

How to choose terminal penalty and terminal region if target steady state is globally asymptotically stable?

NMPC Pitfall Example – Revisited

$$\min_{u(\cdot|t_k)} \int_{t_k}^{t_k+T} \|x(\tau|t_k)\|_Q^2 + \|u(\tau|t_k)\|_R^2 \,\mathrm{d}\tau + \|x(t_k+T|t_k)\|_P^2$$

subject to

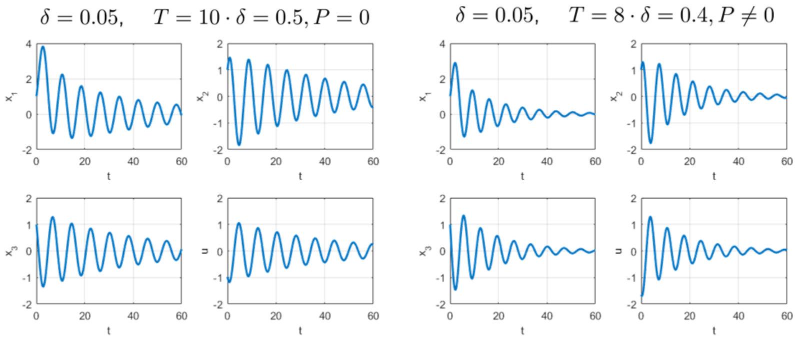
$$\forall \tau \in [t_k, t_k+T]: \; \frac{\mathrm{d}x(\tau|t_k)}{\mathrm{d}\tau} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x(\tau|t_k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(\tau|t_k),$$

$$x(t_k|t_k) = x(t_k)$$

Setting:

- Weight matrices: $Q = \mathrm{diag}(5, 5, 0)$, $\qquad R = 1$
- Inputs approximated as piece-wise constant
- Sampling period: $\delta = 0.05$
- Prediction horizon: $T = N \cdot \delta$, $N = \#$ number of shooting intervals
- $P \succ 0$ solves (ARE)

NMPC Pitfall Example – Revisited



- Long horizons or terminal penalties plus terminal regions fix stability problem.
- How to avoid terminal regions?
- How to enforce recursive feasibility without terminal regions?

Summary – NMPC with Terminal Constraints

- Terminal regions and terminal penalties can be used to guarantee stability/convergence.
- Their computation is difficult for nonlinear systems.
- A plethora of results discuss special cases and discrete-time settings:
    - D.Q. Mayne, J.B. Rawlings, C.V. Rao, P.O.M. Scokaert. “Constrained model predictive control: Stability and optimality”. **in***Automatica*: 36.6 (2000), **pages** 789–814
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Questions:

- How to avoid terminal constraints?
- How to consider other control problems than set-point stabilization?

Overview

NMPC with Replaced Terminal Constraint

1. State measurement/estimate $x(t_k)$

2. Solve:

$$\min_{u(\cdot|t_k)} \int_{t_k}^{t_k+T} \ell(x(\tau|t_k), u(\tau|t_k))\,\mathrm{d}\tau + \beta V_f(x(t_k+T|t_k))$$

subject to $\forall \tau \in [t_k, t_k+T]$: $\qquad (OCP_T^\beta(x(t_k)))$

$$\frac{\mathrm{d}x(\tau|t_k)}{\mathrm{d}\tau} = f(x(\tau|t_k), u(\tau|t_k)), \quad x(t_k|t_k) = x(t_k)$$

$$x(\tau|t_k) \in \mathbb{X}, \;\; u(\tau|t_k) \in \mathbb{U}$$

3. Apply $u^\star(\tau|t_k)$ for $\tau \in [t_k, t_{k+1})$

- Finite horizon ($T < \infty$) and sampled-data recalculation ($\delta \doteq t_{k+1} - t_k > 0$)
- Terminal penalty βV_f is $\mathcal{C}^1$ and positive definite, and **no explicit terminal region** $\mathbb{X}_f$
- Value function: $V_T^\beta(x(t_k)) \doteq \int_{t_k}^{t_k+T} \ell(x^\star(\tau|t_k), u^\star(\tau|t_k))\,\mathrm{d}\tau + \beta V_f(x^\star(t_k+T|t_k))$

Replacing the Terminal Set by a Terminal Cost

How to choose $\beta \in [0, \infty)$ such that the NMPC scheme based on $OCP_T^\beta(x(t_k))$ achieves stability?

Replacing the Terminal Set by a Terminal Cost

How to choose $\beta \in [0, \infty)$ such that the NMPC scheme based on $OCP_T^{\beta}(x(t_k))$ achieves stability?

Consider a quasi-infinite horizon NMPC scheme based on $OCP_T^{\mathbb{X}_f}(x(t_k))$.

Assumption

A4 For some $\gamma > 0$, let $\mathbb{X}_{f,\gamma} = \{x \in \mathbb{X} \mid V_f(x) \leq \gamma\}$ and V_f satisfy the quasi infinite-horizon NMPC stability conditions for specific values of $\delta > 0, T > 0$.

Replacing the Terminal Set by a Terminal Cost

Consider the quasi infinite-horizon NMPC scheme based on $OCP_T^0(x(t_k))$ with zero-terminal constraint $\mathbb{X}_f = \{0\}$ and $V_f(x) = 0$.

Let $V_T^0(x)$ be the associated optimal value function, and consider the set of feasible initial conditions

$$\Omega_T^0 = \left\{ x \in \mathbb{X} \mid V_T^0(x) < \infty \right\}.$$

Assumptions

A5 The set Ω_T^0 is compact and $0 \in \mathrm{int}(\Omega_T^0)$.

A6 There exists $\beta_2 \in \mathcal{K}_\infty$, such that $V_T^0(x) \leq \beta_2(\|x\|)$.

NMPC with replaced terminal constraint

Theorem (Convergence with replaced terminal constraint)

Let Assumptions A1–A6 hold. Then, there exists $\beta \in (0, \infty)$*, such that*

- *the NMPC scheme based on* $OCP_T^\beta(x(t_k))$ *achieves*

$$\lim_{t \to \infty} \|x(t)\| = 0,$$

- *and the region of attraction is given by* Ω_T^0*.*

Convergence with Replaced Terminal Constraint

Proof sketch:

Convergence with Replaced Terminal Constraint

D. Limon, T. Alamo, F. Salas, E.F. Camacho. “On the stability of constrained MPC without terminal constraint”. **in** *IEEE Trans. Automat. Contr.*: 51.5 (2006), **pages** 832–836

Excursion – Control Lyapunov Function

Definition (Control Lyapunov function)

Let $V : \mathbb{R}^{n_x} \to \mathbb{R}_0^+$ be positive definite and continuously differentiable on $\mathbb{R}^{n_x}$. If, for all $x \in \mathbb{R}^{n_x}$, there exists $u \in \mathbb{U}$ such that

$$\frac{\partial V}{\partial x} f(x, u) < 0, \qquad \forall x \neq 0$$

then V is said to be a control Lyapunov function.

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then V *is said to be a control Lyapunov function.*

Observation

Let $V : \mathbb{R}^{n_x} \to \mathbb{R}_0^+$ be a control Lyapunov function satisfying

$$\frac{\partial V}{\partial x} f(x, u) \leq -\ell(x, u),$$

then V can be used as terminal penalty V_f.

A. Jadbabaie, J. Yu, J. Hauser. “Unconstrained receding-horizon control of nonlinear systems”. **in** *IEEE Trans. Automat. Contr.*: 46.5 (2001), **pages** 776–783. DOI: 10.1109/9.920800

Summary – NMPC without Terminal Constraints

How to avoid terminal constraints?

1. Replace by scaled terminal penalty $\beta V_f(x)$.
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- State constraints may lead to difficulties in approaches 2.–4. (recursive feasibility).
- Considering turnpike properties allows tackling this issue ($\rightarrow$ **later**).

Overview

Stability Ingredients for Setpoint Stabilization

- Reference setpoint: $(\bar{x}, \bar{u}) \in \mathrm{int}(\mathbb{X} \times \mathbb{U})$
  $0 = f(\bar{x}, \bar{u})$
- Error variables: $x - \bar{x}$, $u - \bar{u}$
- Cost function ℓ chosen such that

$$\ell(x, u) \geq \alpha(\|x - \bar{x}\|),\ \alpha \in \mathcal{K}$$

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  $$\ell(x, u) \geq \alpha(\|x - \bar{x}\|), \;\; \alpha \in \mathcal{K}$$
- Typical choices:
  $$\ell(x, u) = \|x - \bar{x}\|_Q^2 + \|u - \bar{u}\|_R^2$$
  $$Q = Q^\top \succ 0, R = R^\top \succ 0$$
  $$V_f(x) = \|x - \bar{x}\|_P^2$$
  $$\mathbb{X}_f = \{x \in \mathbb{R}^{n_x} \mid V_f(x) \leq \gamma\}$$

1. State measurement/estimate $x(t_k)$

2. Solve $OCP(x(t_k))$:

$$\min_{u(\cdot|t_k)} \int_{t_k}^{t_k+T} \ell(x(\tau|t_k), u(\tau|t_k))\,\mathrm{d}\tau + V_f(x(t_k + T|t_k))$$
$$\forall \tau \in [t_k, t_k + T] : \frac{\mathrm{d}x(\tau|t_k)}{\mathrm{d}\tau} = f(x(\tau|t_k), u(\tau|t_k))$$
$$x(t_k|t_k) = x(t_k)$$
$$x(\tau|t_k) \in \mathbb{X}, \;\; u(\tau|t_k) \in \mathbb{U}$$
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3. Apply $u^\star(\tau|t_k)$ for $\tau \in [t_k, t_{k+1})$

Notation: $\|z\|_S^2 = z^\top S z, \quad S \succeq 0$

Stability Ingredients for Trajectory Tracking

- Reference trajectory: $\dot{r} = f(r, w)$
  $t \in [0, \infty) \to r(t) \in \operatorname{int} \mathbb{X}$,
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3. Apply $u^\star(\tau|t_k)$ for $\tau \in [t_k, t_{k+1})$

Note: NMPC for trajectory tracking leads to a time-varying problem.

T. Faulwasser. *Optimization-based Solutions to Constrained Trajectory-tracking and Path-following Problems*. Shaker, Aachen, Germany, 2013. DOI: 10.2370/9783844015942

Continuous Time vs. Discrete Time Setting?

Continuous-Time NMPC

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3. Apply $u^\star(\tau|t_k)$ for $\tau \in [t_k, t_{k+1})$

Discrete-Time NMPC

1. State measurement/estimate $x(k)$

2. Solve $OCP(x(k))$:

$$\min_{u(\cdot|k)} \sum_{i=k}^{k+N-1} \ell^d(x(i|k), u(i|k)) + V_f(x(k+N|k))^d$$

$$\begin{aligned} \forall i \in \{k, k+N-1\} : x(i+1|k) &= f^d(x(i|k), u(i|k)) \\ x(k|k) &= x(k) \\ x(i|k) \in \mathbb{X},&\ \ u(i|k) \in \mathbb{U} \\ x(k+N|k) &\in \mathbb{X}_f \end{aligned}$$

3. Apply $u^\star(k|k)$ for time k

Continuous Time vs. Discrete Time Setting?

Continuous-Time NMPC

1. State measurement/estimate $x(t_k)$

2. Solve $OCP(x(t_k))$:

$$\min_{u(\cdot|t_k)} \int_{t_k}^{t_k+T} \ell(x(\tau|t_k), u(\tau|t_k))\,\mathrm{d}\tau + V_f(x(t_k+T|t_k))$$

$$\forall \tau \in [t_k, t_k+T] : \frac{\mathrm{d}x(\tau|t_k)}{\mathrm{d}\tau} = f(x(\tau|t_k), u(\tau|t_k))$$
$$x(t_k|t_k) = x(t_k)$$
$$x(\tau|t_k) \in \mathbb{X}, \quad u(\tau|t_k) \in \mathbb{U}$$
$$x(t_k+T|t_k) \in \mathbb{X}_f$$

3. Apply $u^\star(\tau|t_k)$ for $\tau \in [t_k, t_{k+1})$

Discrete-Time NMPC

1. State measurement/estimate $x(k)$

2. Solve $OCP(x(k))$:

$$\min_{u(\cdot|k)} \sum_{i=k}^{k+N-1} \ell^d(x(i|k), u(i|k)) + V_f(x(k+N|k))^d$$

$$\forall i \in \{k, k+N-1\} : x(i+1|k) = f^d(x(i|k), u(i|k))$$
$$x(k|k) = x(k)$$
$$x(i|k) \in \mathbb{X}, \quad u(i|k) \in \mathbb{U}$$
$$x(k+N|k) \in \mathbb{X}_f$$

3. Apply $u^\star(k|k)$ for time k

Note: Previous results hold (mutatis mutandis) in discrete-time!

- L. Grüne **and** J. Pannek. *Nonlinear Model Predictive Control: Theory and Algorithms.* 2nd Edition. Communication and Control Engineering. Springer Verlag, 2017
- J.B. Rawlings, D.Q. Mayne, M. Diehl. *Model Predictive Control: Theory, Computation, and Design.* Nob Hill Publishing, Madison, WI, 2017

Summary – Advanced NMPC Formulations

Note that NMPC can be used for problems beyond setpoint stabilization.

Key steps of NMPC design:

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- Check stability and performance in simulations having formal stability proofs in mind.
- Tuning parameters (as in case of setpoint stabilization): ℓ, T, δ, V_f, $\mathbb{X}_f$

Overview

Recap – Motivations for Using NMPC

Feedback Control

- Control of nonlinear MIMO system subject to constraints.
- *Pre-requisite* = control task at hand, such as setpoint stabilization, trajectory tracking, disturbance attenuation, ...
- Choose ℓ, $V_f, \mathbb{X}_f, \delta$, T according to task.

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Economics / Economic Operation

- Receding horizon approximation to the solution of an infinite-horizon OCP.
- *Pre-requisite* $=$ specified performance functional:

$$J_\infty = \int_0^\infty \ell(x(\tau), u(\tau))\, \mathrm{d}\tau,$$

  i.e. **the cost function ℓ is given**!
- Choose δ, T and analyze properties of NMPC loop.

$\rightarrow$ **Now!**

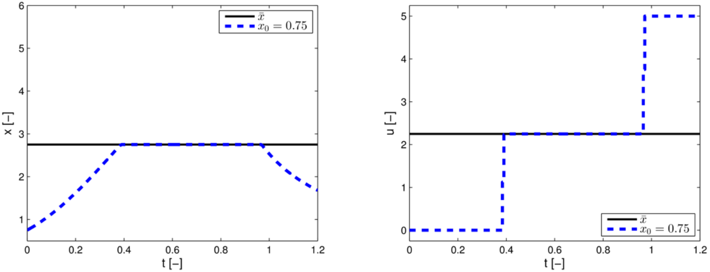
Example – Optimal Fish Harvest

$$\min_{u(\cdot)} \int_0^T ax(\tau) + bu(\tau) - cx(\tau)u(\tau) \, \mathrm{d}\tau$$

subject to

$$\dot{x} = x(\bar{x} - x - u), \quad x(0) = x_0$$
$$u(t) \in [0, u_{max}], x(t) \in (0, \infty)$$

- x fish density
- u fishing rate
- $\bar{x} = 5$ highest sustainable fish density
- $a = 1, b = c = 2, u_{max} = 5$



E.M. Cliff **and** T.L. Vincent. “An optimal policy for a fish harvest”. **in** *Journal of Optimization Theory and Applications*: 12.5 (1973), **pages** 485–496

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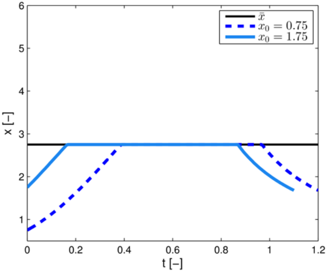

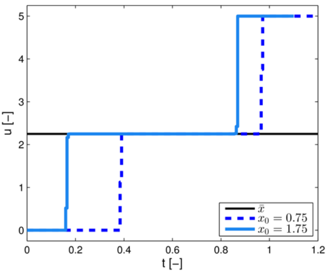

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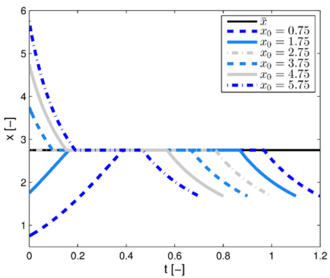

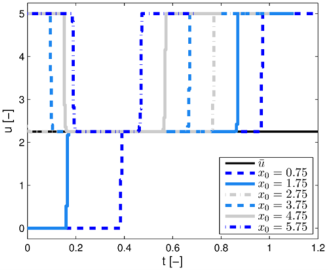

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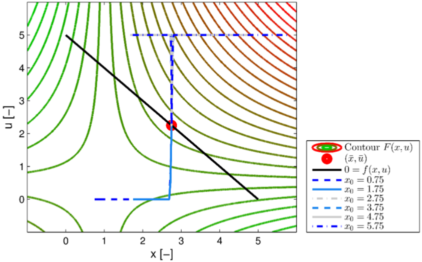
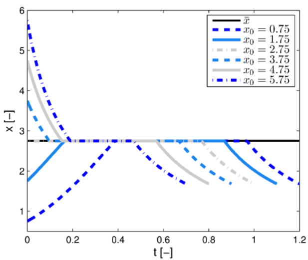
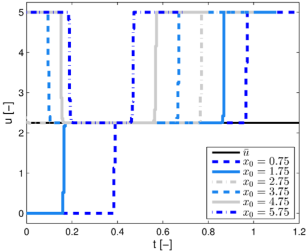






Example – Optimal Fish Harvest

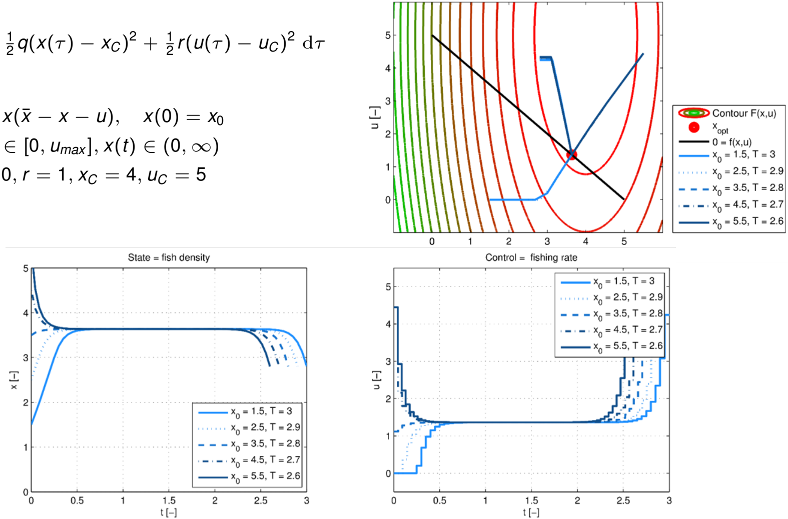
$$\min_{u(\cdot)} \quad \int_0^T \tfrac{1}{2} q(x(\tau) - x_C)^2 + \tfrac{1}{2} r(u(\tau) - u_C)^2 \,\mathrm{d}\tau$$

subject to

$$\dot{x} = x(\bar{x} - x - u), \quad x(0) = x_0$$

$$u(t) \in [0, u_{max}], x(t) \in (0, \infty)$$

$$q = 10, r = 1, x_C = 4, u_C = 5$$



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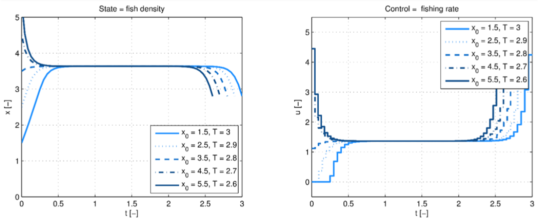

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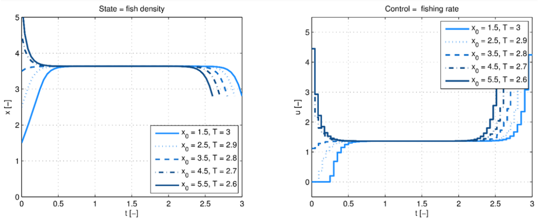

$\rightarrow$ Similar behavior for different initial conditions and horizon lengths.

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$\rightarrow$ **Turnpike Property!**

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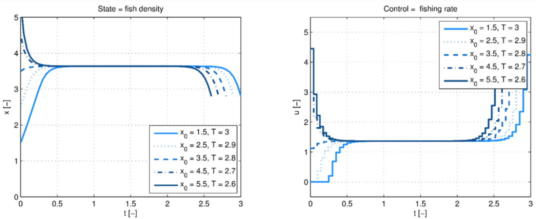

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$\rightarrow$ **Turnpike Property!**

It is exactly like a turnpike paralleled by a network of minor roads. [. . .], if origin and destination are far enough part, it will always pay to get on to the turnpike and cover distance at the best rate of travel, even if this means adding a little mileage at either end.

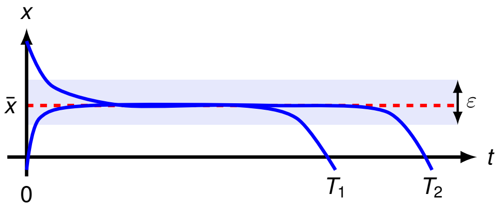
R. Dorfman, P.A. Samuelson, R.M. Solow. *Linear Programming and Economic Analysis*. McGraw-Hill, New York, 1958

Problem Setup

$$\begin{aligned} &\min_{u(\cdot)} \int_0^T \ell(x(\tau), u(\tau))\,\mathrm{d}\tau \\ &\text{subject to} \qquad\qquad\qquad\qquad\qquad\qquad (\mathrm{OCP}_T(x_0)) \\ &\Sigma: \quad \dot{x} = f(x,u), \quad x(0) = x_0 \in \mathbb{X}_0 \\ &u(\tau) \in \mathbb{U} \subset \mathbb{R}^{n_u}, x(\tau) \in \mathbb{X} \subset \mathbb{R}^{n_x} \end{aligned}$$

- Here: turnpike properties of OCPs without terminal constraints
- Definition of turnpikes?
- Verfication?
- Use for NMPC?

Definition of Turnpike Properties

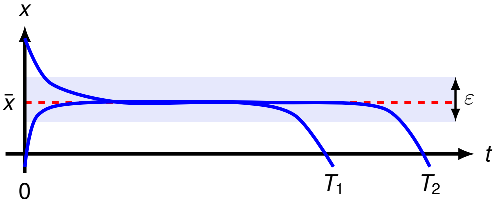


Adhoc Definition (Input-state turnpike at $\bar{z}$)

Consider $\text{OCP}_T(x_0)$ and let $z^\star(\cdot, x_0) = (x^\star(\cdot, x_0), u^\star(\cdot))$ be its optimal pairs. $\text{OCP}_T(x_0)$ is said to have an *input-state turnpike* at $\bar{z} = (\bar{x}, \bar{u})$ if, for all $x_0 \in \mathbb{X}_0$, all $T \geq 0$, and all $\varepsilon > 0$,

$$\left[\text{time } z^\star(\cdot, x_0) \text{ spends outside of } \varepsilon\text{-neighborhood of } \bar{z}\right] \leq \nu_z(\varepsilon) < \infty.$$

Definition of Turnpike Properties



Definition (Input-state turnpike)

Consider the optimal pairs $z^\star(\cdot, x_0) = (x^\star(\cdot, x_0), u^\star(\cdot))$ *of* $OCP_T(x_0)$ *and let*

$$\Theta_{\varepsilon,T} \doteq \{\tau \in [0, T] : \|z^\star(\tau, x_0) - \bar{z}\| > \varepsilon\}\,.$$

$OCP_T(x_0)$ *is said to have an* ***input-state turnpike property*** *with respect to* $\bar{z}$ *if there exists* $\nu : [0, \infty) \to [0, \infty)$ *s. t.*

$$\forall x_0 \in \mathbb{X}_0, \forall T \geq 0, \forall \varepsilon > 0 : \quad \mu[\Theta_{\varepsilon,T}] \leq \nu(\varepsilon) < \infty,$$

where $\mu[\cdot]$ *is the Lebesgue measure on the real line.*

T. Faulwasser, M. Korda, C.N. Jones, D. Bonvin. “On Turnpike and Dissipativity Properties of Continuous-Time Optimal Control Problems”.

in *Automatica*: 81 (2017), **pages** 297–304. DOI: `10.1016/j.automatica.2017.03.012`

Turnpike Properties of OCPs

Conceptual idea of turnpike properties

- Property of OCPs **with and without terminal constraints**.
- Optimal solutions approach neighborhood of a specific steady state.
- Time spend at turnpike grows with increasing horizon length T.
- Turnpike $\equiv$ property of parametric OCPs, hence we write $\text{OCP}_T(x_0)$
- If turnpike at $\bar{x}$, then for $T = \infty$, we have that

$$\lim_{t \to \infty} x^\star(t) \approx \bar{x}.$$

- Different notions for turnpikes: dichotomy in OCPs, hyper-sensitive OCPs, ...

Dissipativity – A Useful Concept

Nonlinear system

$$\Sigma : \quad \begin{aligned} \dot{x} &= f(x, u), \\ y &= h(x) \end{aligned}$$

Definition (Dissipativity)

Σ is said to be dissipative on $\mathcal{D} \subseteq \mathbb{X} \times \mathbb{U}$ if there exists a bounded storage function $S : \mathbb{X} \to \mathbb{R}_0^+$ and a supply rate $w : \mathbb{R}^{n_y} \times \mathbb{U} \to \mathbb{R}$ such that

$$\frac{\partial S}{\partial x} f(x, u) \leq w(y, u)$$

holds for all $(x, u) \in \mathcal{D} \subseteq \mathbb{X} \times \mathbb{U}$.

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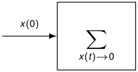
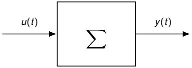
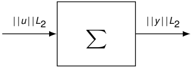
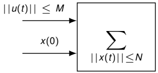
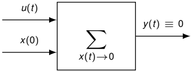
Interpretation:

$$\text{Stored Energy} \quad \leq \int \text{Supplied Power } \mathrm{d}\tau$$

Dissipativity – A Useful Concept

System Property	Supply Rate	Diagram
Asymptotic Stability	$-\alpha(\|x\|)$	
Passivity	$u^\top y$	
L_2-Gain	$\gamma^2\|u\|^2 - \|y\|^2$	
Input-to-state Stability	$-\alpha(\|x\|) + \varsigma(\|u\|)$	
Minimum Phase Property	$[y, \dot{y}, \dots, y^{(r)}]^\top \rho(x, u)$	

C. Ebenbauer, T. Raff, F. Allgöwer. "Dissipation inequalities in systems theory: An introduction and recent results". **in***R. Jeltsch and G. Wanner (ed.), Invited Lectures of the International Congress on Industrial and Applied Mathematics 2007*: 2009, **pages** 23–42

A Dissipativity Notion for Turnpikes

Definition (Strict dissipativity w.r.t. $(\bar{x}, \bar{u})$)

Σ is strictly dissipative with respect to the steady state pair $(\bar{x}, \bar{u})$ if there exists a bounded storage function $S : \mathbb{X} \to \mathbb{R}_0^+$ and $\alpha \in \mathcal{K}$ such that

$$\frac{\partial S}{\partial x} f(x, u) \leq -\alpha(\|(x, u) - (\bar{x}, \bar{u})\|) + \ell(x, u) - \ell(\bar{x}, \bar{u}) \qquad \text{(DI)}$$

holds for all $(x, u) \in \mathcal{D} \subseteq \mathbb{X} \times \mathbb{U}$.

Implications of Dissipativity

Lemma (Strict dissipativity w.r.t. $(\bar{x}, \bar{u}) \Rightarrow$ optimality of $(\bar{x}, \bar{u})$)

If Σ is strictly dissipative w.r.t. to $(\bar{x}, \bar{u})$, then $(\bar{x}, \bar{u})$is an optimal solution of the Steady State Optimization (SOP)

$$\begin{aligned} \min_{(\bar{x}, \bar{u})} \;\; & \ell(\bar{x}, \bar{u}) \\ \textit{subject to} \;\; & \\ 0 &= f(\bar{x}, \bar{u}) \\ (\bar{x}, \bar{u}) &\in \mathbb{X} \times \mathbb{U}. \end{aligned} \tag{SOP}$$

Implications of Dissipativity

Theorem (Strict dissipativity $\Rightarrow$ turnpike)

Suppose that

- *from all $x_0 \in \mathbb{X}_0$ the optimal steady state $\bar{x}^\star$ is reachable in some finite time $T_{\bar{x}^\star}$,*
- *Σ is strictly dissipative w.r.t. to $(\bar{x}^\star, \bar{u}^\star)$.*

Then the optimal pairs $z^\star(\cdot, x_0)$ of $OCP_T(x_0)$ have a turnpike property with respect to the steady state pair $(\bar{x}^\star, \bar{u}^\star)$.

T. Faulwasser, M. Korda, C.N. Jones, D. Bonvin. "On Turnpike and Dissipativity Properties of Continuous-Time Optimal Control Problems".

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Implications of Dissipativity (cont'd)

Definition (Optimal operation at steady state)

Σ is said to be optimally operated at $(\bar{x}^\star, \bar{u}^\star)$, if for all $x_0 \in \mathbb{X}_0$ and any infinite-horizon admissible pair $(x(\cdot), u(\cdot))$

$$\liminf_{T\to\infty} \frac{1}{T}\int_0^T \ell(x(\tau, x_0, u(\cdot)), u(\tau))\,\mathrm{d}\tau \geq \ell(\bar{x}^\star, \bar{u}^\star).$$

Implications of Dissipativity (cont'd)

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Theorem (Dissipativity $\Rightarrow$ optimal operation at steady state)

Suppose that Σ is strictly dissipative on $\mathbb{X} \times \mathbb{U}$ w.r.t. to $(\bar{x}^\star, \bar{u}^\star)$, then Σ is optimally operated at the steady state $(\bar{x}^\star, \bar{u}^\star)$.

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Implications of Turnpike Properties

Σ is strictly dissipative w.r.t. $(\bar{x}^\star, \bar{u}^\star)$.

Further implications and proofs can be found in:

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T. Faulwasser **and** C.M. Kellett. "On Continuous-Time Infinite Horizon Optimal Control – Dissipativity, Stability and Transversality". **in***Automatica*: 134 (2021), **page** 109907. DOI: 10.1016/j.automatica.2021.109907

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Σ is strictly dissipative w.r.t. $(\bar{x}^\star, \bar{u}^\star)$.

Σ is optimally operated at $(\bar{x}^\star, \bar{u}^\star)$.

The optimal solutions of $\text{OCP}_T(x_0)$ have a turnpike at $(\bar{x}^\star, \bar{u}^\star)$.

Further implications and proofs can be found in:

T. Faulwasser, M. Korda, C.N. Jones, D. Bonvin. "On Turnpike and Dissipativity Properties of Continuous-Time Optimal Control Problems". **in***Automatica*: 81 (2017), **pages** 297–304. DOI: `10.1016/j.automatica.2017.03.012`

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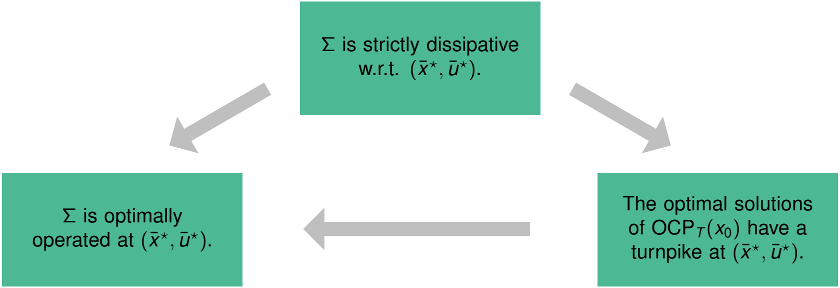
Implications of Turnpike Properties



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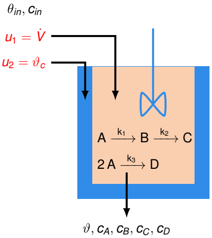
Example – Chemical Reactor

Van de Vusse Reactor

$$A \xrightarrow{k_1} B \xrightarrow{k_2} C, \qquad 2A \xrightarrow{k_3} D$$



Dynamics (partial model) b

$$\dot{c}_A = r_A(c_A, \vartheta) + (c_{in} - c_A)u_1$$

$$\dot{c}_B = r_B(c_A, c_B, \vartheta) - c_B u_1$$

$$\dot{\vartheta} = h(c_A, c_B, \vartheta) + \alpha(u_2 - \vartheta) + (\vartheta_{in} - \vartheta)u_1,$$

$$r_A(c_A, \vartheta) = -k_1(\vartheta)c_A - 2k_3(\vartheta)c_A^2$$

$$r_B(c_A, c_B, \vartheta) = \quad k_1(\vartheta)c_A - k_2(\vartheta)c_B$$

$$h(c_A, c_B, \vartheta) = -\delta\Big(k_1(\vartheta)c_A\Delta H_{AB} + k_2(\vartheta)c_B\Delta H_{BC} + 2k_3(\vartheta)c_A^2\Delta H_{AD}\Big)$$

$$k_i(\vartheta) = \quad k_{i0} \exp\frac{-E_i}{\vartheta + \vartheta_0}, \qquad i = 1, 2, 3.$$

Constraints

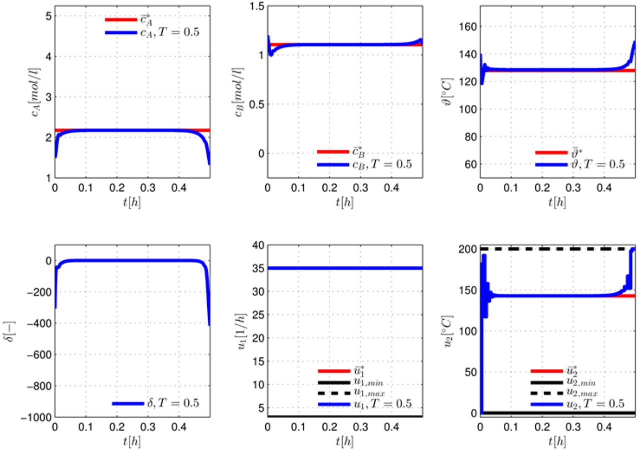
$$c_A \in [0, 6]\,\frac{mol}{l} \qquad c_B \in [0, 4]\,\frac{mol}{l} \qquad \vartheta \in [70, 150]^\circ C$$

$$u_1 \in [3, 35]\,\frac{1}{h} \qquad u_2 \in [0, 200]^\circ C.$$

Objective = maximize produced amount of B

$$J_T(x_0, u(\cdot)) = \int_0^T -\beta c_B(\tau)u_1(\tau)\,\mathrm{d}\tau, \qquad \beta > 0$$

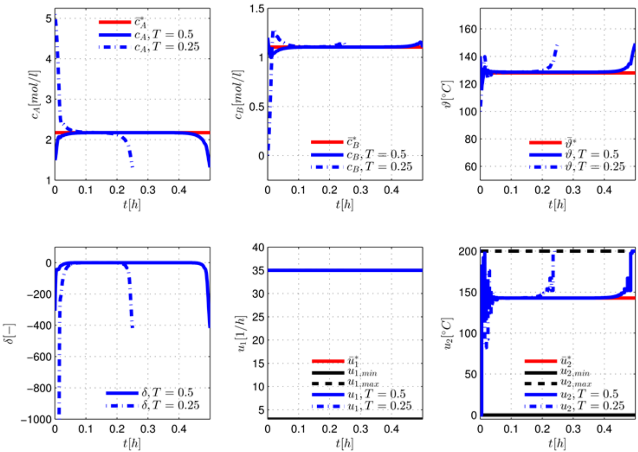
CSTR Example



Checking dissipation inequality (DI) (for a pre-computed storage function S) via:

$$\delta = \frac{\partial S}{\partial x} f(x,u) + \bar{\alpha}\|x - \bar{x}^\star\|^2 - \ell(x,u) + \ell(\bar{x}^\star, \bar{u}^\star), \quad \delta \leq 0$$

CSTR Example



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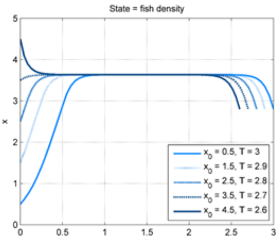
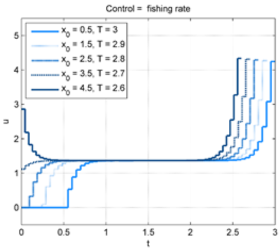
Summary – Turnpike Properties in OCPs

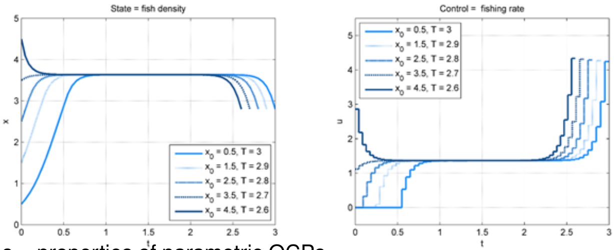




- Turnpikes = properties of parametric OCPs.
- Turnpike steady state $\bar{x}$ = best reachable steady state.
- Turnpikes occur in many OCPs.
- Strict dissipativity + reachability = sufficient condition.

How to use turnpikes in OCPs for NMPC?

Summary – Turnpike Properties in OCPs



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- Turnpikes occur in many OCPs.
- Strict dissipativity + reachability = sufficient condition.

How to use turnpikes in OCPs for NMPC?

- Turnpike steady-state pair is often a good initial guess for numerical solution.
- **T**urnpikes can be used to design NMPC schemes.

Economic and Stabilizing NMPC

NMPC for setpoint stabilization

- Based on $OCP_T^{\mathbb{X}_f}(x(t_k))$

$$\min_{u(\cdot|t_k))} \int_{t_k}^{t_k+T} \ell(x(\tau|t_k)), u(\tau|t_k))\,\mathrm{d}\tau + V_f(x(t_k+T|t_k))$$

$$\forall \tau \in [t_k, t_k+T]: \frac{\mathrm{d}x(\tau|t_k)}{\mathrm{d}\tau} = f(x(\tau|t_k), u(\tau|t_k))$$

$$x(t_k|t_k) = x(t_k)$$

$$x(\tau|t_k) \in \mathbb{X},\ u(\tau|t_k) \in \mathbb{U}$$

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- Main assumption $\ell(x,u) \geq \alpha(\|x - \bar{x}\|),\ \alpha \in \mathcal{K},\ \bar{x}$ set-point to be stabilized

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- Main assumption $\ell(x,u) \geq \alpha(\|x-\bar{x}\|),\ \alpha \in \mathcal{K},\ \bar{x}$ set-point to be stabilized

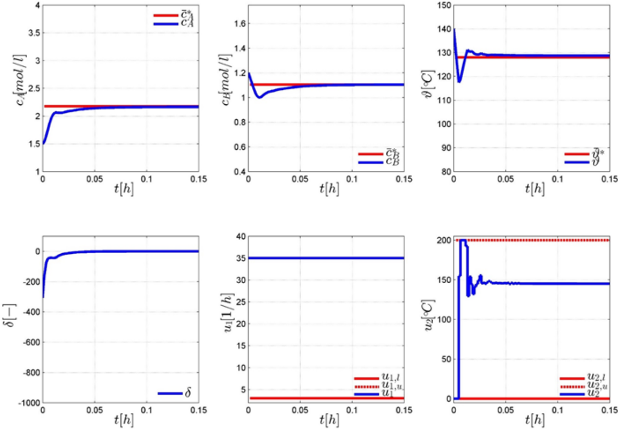
Economic NMPC

- No lower boundedness of $\ell \quad \rightarrow \quad$ generalized (economic) NMPC formulations
- Proofs of convergence to the optimal steady-state often based on the dissipation inequality (DI)

$$\frac{\partial S}{\partial x} f(x,u) \leq -\alpha(\|(x,u)-(\bar{x},\bar{u})\|) + \ell(x,u) - \ell(\bar{x},\bar{u})$$

Example – Chemical Reactor

Same OCP as before (Slide 85), solved in receding horizon fashion



Prediction horizon $T = 0.1\text{h}$, sampling period $\delta = 1.7 \cdot 10^{-3}\text{h}$

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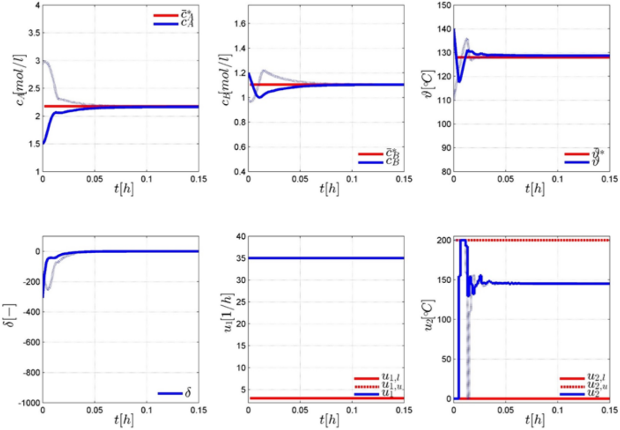
Same OCP as before (Slide 85), solved in receding horizon fashion



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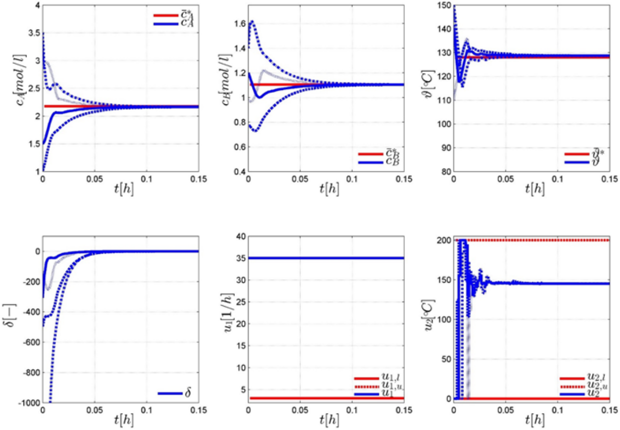

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Overview

Adjusting the Standing NMPC Assumptions

A1 from Slide 18 is considered implicitly. A2 is adjusted. A3 remains.

Adjusting the Standing NMPC Assumptions

A1 from Slide 18 is considered implicitly. A2 is adjusted. A3 remains.

Assumptions

A2' (Strict dissipativity): There exists a bounded storage function S and $\alpha \in \mathcal{K}$ such that

$$\frac{\partial S}{\partial x} f(x,u) \leq -\alpha(\|(x,u)-(\bar{x},\bar{u})\|) + \ell(x,u) - \ell(\bar{x},\bar{u})$$

holds for all $(x,u) \in \mathcal{D} \subseteq \mathbb{X} \times \mathbb{U}$. Moreover, $0 = f(\bar{x},\bar{u})$ and $(\bar{x},\bar{u}) \in \mathrm{int}(\mathbb{X} \times \mathbb{U})$.

W.l.o.g. consider $(\bar{x},\bar{u}) = (0,0)$ and $\ell(\bar{x},\bar{u}) = 0$.

Adjusting the Standing NMPC Assumptions

A1 from Slide 18 is considered implicitly. A2 is adjusted. A3 remains.

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W.l.o.g. consider $(\bar{x},\bar{u}) = (0,0)$ and $\ell(\bar{x},\bar{u}) = 0$.

A3 (Absolute continuity of ODE solutions): For all $x_0 \in \mathbb{X}$, and any $u(\cdot) \in \hat{\mathcal{C}}([0,T],\mathbb{U})$, the solution $x(\cdot, x_0, u(\cdot))$ exists and is absolutely continuous.

Economic NMPC with Zero-Terminal Constraint

1. State measurement/estimate $x(t_k)$

2. Solve:

$$\min_{u(\cdot|t_k)} \int_{t_k}^{t_k+T} \ell(x(\tau|t_k), u(\tau|t_k))\,\mathrm{d}\tau$$

$$\text{subject to } \forall \tau \in [t_k, t_k+T]: \qquad (OCP_T^{eco,\bar{x}^\star}(x(t_k)))$$

$$\frac{\mathrm{d}x(\tau|t_k)}{\mathrm{d}\tau} = f(x(\tau|t_k), u(\tau|t_k)), \quad x(t_k|t_k) = x(t_k)$$

$$x(\tau|t_k) \in \mathbb{X}, \; u(\tau|t_k) \in \mathbb{U}$$

$$x(t_k + T|t_k) = \bar{x}^\star$$

3. Apply $u^\star(\tau|t_k)$ for $\tau \in [t_k, t_{k+1})$

- Finite horizon ($T < \infty$) and sampled-data recalculation ($\delta \doteq t_{k+1} - t_k > 0$)
- No terminal penalty ($V_f = 0$) but terminal constraint at $\bar{x}^\star$

Economic NMPC with Terminal Constraint

Theorem (Convergence of eco. NMPC with terminal constraint)

Let Assumptions A2' and A3 hold and suppose that

(i) $OCP_T^{eco,\bar{x}^\star}(x(t_k))$ *is feasible at* $k = 0$.

Then,

- $OCP_T^{eco,\bar{x}^\star}(x(t_k))$ *is recursively feasible,*
- *the NMPC scheme based on* $OCP_T^{eco,\bar{x}^\star}(x(t_k))$ *achieves*

$$\lim_{t\to\infty} \|x(t)\| = \bar{x}^\star,$$

- *and the region of attraction is given by the set of initial conditions for which* $OCP_T^{eco,\bar{x}^\star}(x(t_k))$ *is feasible.*

D. Angeli, R. Amrit, J.B. Rawlings. "On Average Performance and Stability of Economic Model Predictive Control". **in***IEEE Trans. Automat. Contr.*: 57.7 (2012), **pages** 1615–1626. ISSN: 0018-9286. DOI: 10.1109/TAC.2011.2179349

M. Diehl, R. Amrit, J.B. Rawlings. "A Lyapunov function for economic optimizing model predictive control". **in***IEEE Trans. Automat. Contr.*: 56.3 (2011), **pages** 703–707

Proof Sketch

Proof Sketch

Overview

Economic NMPC without Terminal Constraints

1. State measurement/estimate $x(t_k)$

2. Solve:

$$\min_{u(\cdot|t_k)} \int_{t_k}^{t_k+T} \ell(x(\tau|t_k), u(\tau|t_k))\,\mathrm{d}\tau$$

$$\text{subject to } \forall \tau \in [t_k, t_k+T]: \qquad (OCP_T^{eco}(x(t_k)))$$

$$\frac{\mathrm{d}x(\tau|t_k)}{\mathrm{d}\tau} = f(x(\tau|t_k), u(\tau|t_k)), \quad x(t_k|t_k) = x(t_k)$$

$$x(\tau|t_k) \in \mathbb{X},\ \ u(\tau|t_k) \in \mathbb{U}$$

3. Apply $u^\star(\tau|t_k)$ for $\tau \in [t_k, t_{k+1})$

- Finite horizon ($T < \infty$) and sampled-data recalculation ($\delta \doteq t_{k+1} - t_k > 0$)
- No terminal penalty ($V_f = 0$) and no terminal region $\mathbb{X}_f = \mathbb{X}$

Economic NMPC without Terminal Constraints

1. State measurement/estimate $x(t_k)$

2. Solve:

$$\min_{u(\cdot|t_k))} \quad {\color{red} l(x(t_k))} + \int_{t_k}^{t_k+T} \ell(x(\tau|t_k), u(\tau|t_k))\, \mathrm{d}\tau$$

$$\text{subject to } \forall \tau \in [t_k, t_k+T]: \qquad (OCP_T^{eco,l}(x(t_k)))$$

$$\frac{\mathrm{d}x(\tau|t_k)}{\mathrm{d}\tau} = f(x(\tau|t_k), u(\tau|t_k)), \quad x(t_k|t_k) = x(t_k)$$

$$x(\tau|t_k) \in \mathbb{X}, \; u(\tau|t_k) \in \mathbb{U}$$

3. Apply $u^\star(\tau|t_k)$ for $\tau \in [t_k, t_{k+1})$

- Finite horizon ($T < \infty$) and sampled-data recalculation ($\delta \doteq t_{k+1} - t_k > 0$)
- Penalty on the initial condition ${\color{red} l(x(t_k))}$
- As ${\color{red} l(x(t_k))}$ does not affect optimal solutions, **we temporarily set $l \equiv 0$.**

Practical Convergence of Economic NMPC

Theorem (Practical convergence)

Suppose that, for all $x_0 \in \mathbb{X}_0$, Assumptions A2', A3 hold, and

- *the $(\bar{x}, \bar{u})$ turnpike is reachable exponentially fast;*
- $(\bar{x}, \bar{u}) \in \mathrm{int}(\mathbb{X} \times \mathbb{U})$ *with* $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial u}\right)\Big|_{(\bar{x},\bar{u})}$ *controllable, and*
- *the penalty on the initial condition is $I(x(t_k)) = S(x(t_k))$, with $S(x)$ satisfying* (DI).

Then, there exists $\delta > 0$ and $T < \infty$ such that, for all $x_0 \in \mathbb{X}_0$,

(i) $OCP_T^{eco,I}(x(t_k))$ ***is recursively feasible****; and*

(ii) ***NMPC tracks the optimal steady state $\bar{x} = \bar{x}^\star$:***

$$\lim_{t\to\infty} \mathrm{d}\big(\mathcal{B}_\rho(\bar{x}),\ x(t)\big) = 0.$$

with $\mathrm{d}\big(\mathcal{B}_\rho(\bar{x}),\ x(t)\big) \doteq \min_{z \in \mathcal{B}_\rho(\bar{x})} \|x - z\|$.

Proof Outline

Main steps of NMPC stability proofs:

- Step 0 – Turnpike in a sequence of OCPs
- Step 1 – Recursive feasibility of the sequence OCPs
- Step 2 – Decrease of the value function inbetween sampling two instants
- Step 3 – Decrease of the value function from one sampling instant to the next
- Step 4 – Consider the value function as a *Lyapunov* function

Step 0 – Turnpikes in a Sequence of OCPs?

Let $OCP_T^{eco,l}(x(t_k))$ have an input-state turnpike at $(\bar{x}, \bar{u})$, and consider $x(t_k) = x_0$

$$x_\delta \doteq x^\star(\delta, x_0, u^\star(\cdot, x_0)).$$

Does $\mathrm{OCP}_T^{eco,l}(x_\delta)$ have the same turnpike?

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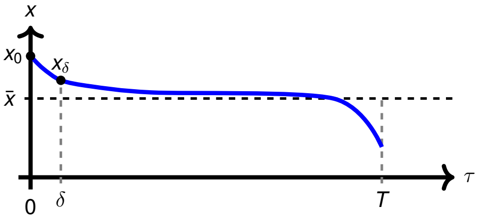

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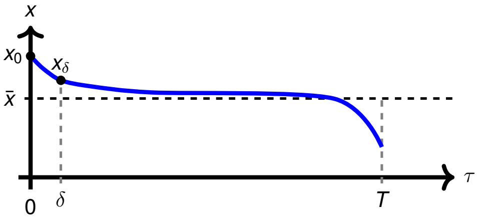
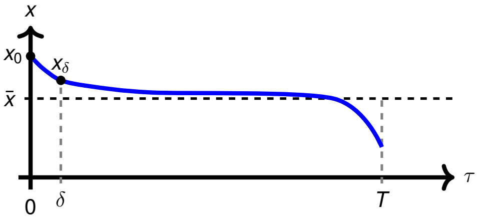


For sufficiently large T, controllability of the linearization of Σ at $(\bar{x}, \bar{u})$ implies existence of $\hat{\delta} > 0$, such that, for all $\delta \in [0, \hat{\delta})$, $\mathrm{OCP}_T^{eco,l}(x_\delta)$ has a turnpike at $(\bar{x},\ \bar{u})$.

Step 1 – Recursive Feasibility

Task: Construct admissible input $u(\cdot, x_\delta)$ from $u^\star(\cdot, x_0)$.



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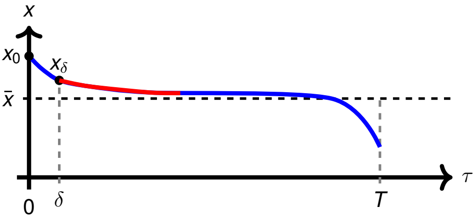

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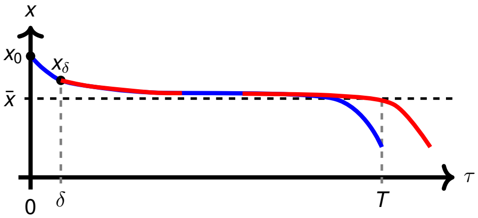

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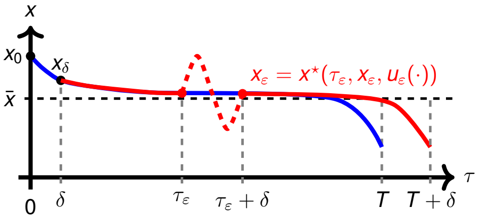

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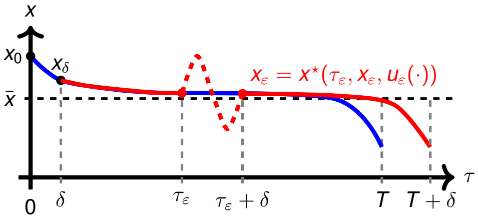

$$u(\tau, x_\delta) = \begin{cases} u^\star(\tau + \delta, x_0) & \tau \in [0, \tau_\varepsilon - \delta) \\ u_\varepsilon(\tau, x_\varepsilon) & \tau \in [\tau_\varepsilon - \delta, \tau_\varepsilon) \\ u^\star(\tau, x_0) & \tau \in [\tau_\varepsilon, T] \end{cases}$$

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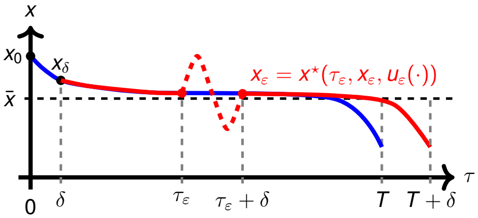

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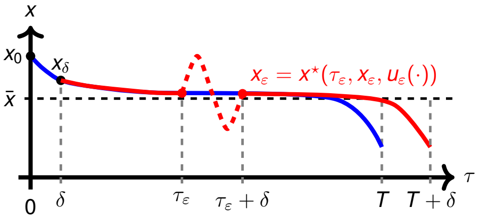

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→ Controllability of linarization at $(\bar{x}, \bar{u})$ guarantees existence of $u_\varepsilon(\cdot)$ close to $(\bar{x}, \bar{u})$.

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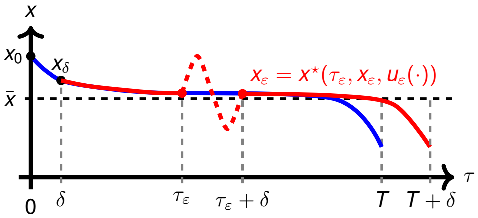

$$u(\tau, x_\delta) = \begin{cases} u^\star(\tau + \delta, x_0) & \tau \in [0, \tau_\varepsilon - \delta) \\ u_\varepsilon(\tau, x_\varepsilon) & \tau \in [\tau_\varepsilon - \delta, \tau_\varepsilon) \\ u^\star(\tau, x_0) & \tau \in [\tau_\varepsilon, T] \end{cases}$$

$\rightarrow$ Controllability of linarization at $(\bar{x}, \bar{u})$ guarantees existence of $u_\varepsilon(\cdot)$ close to $(\bar{x}, \bar{u})$.

Implication: The sequence $OCP_T^{eco,I}(x(t_k))$, with $t_k = t_{k-1} + \delta$, is recursively feasible for suitable choices of T and δ.

Step 2 – Penalty on the Initial Condition?

$$\min_{u(\cdot|t_k)} \quad I(x(t_k)) + \int_{t_k}^{t_k+T} \ell(x(\tau|t_k), u(\tau|t_k)) \, \mathrm{d}\tau$$

$$\text{subject to } \forall \tau \in [t_k, t_k + T] : \qquad (OCP_T^{eco,I}(x(t_k)))$$

$$\frac{\mathrm{d}x(\tau|t_k)}{\mathrm{d}\tau} = f(x(\tau|t_k), u(\tau|t_k)), \quad x(t_k|t_k) = x(t_k)$$

$$x(\tau|t_k) \in \mathbb{X}, \; u(\tau|t_k) \in \mathbb{U}$$

- Penalty on initial condition = storage function: $\quad I(x(t_k)) \doteq S(x(t_k))$.

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- Penalty on initial condition = storage function: $I(x(t_k)) \doteq S(x(t_k))$.
- Optimal value function:

$$V_T^S(x(t_k)) = S(x(t_k)) + \int_{t_k}^{t_k+T} \ell(x^\star(\tau|t_k), u^\star(\tau|t_k))\,\mathrm{d}\tau$$

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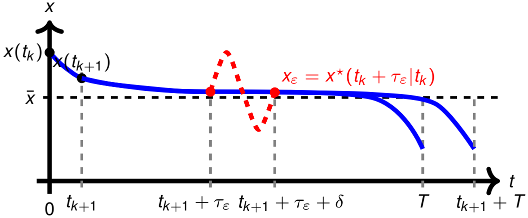
Value function decrease?

$$V_T^S(x(t_{k+1})) - V_T^S(x(t_k)) \le \underbrace{J(x(t_{k+1}), u(\cdot, x(t_{k+1}))) - V_T^S(x(t_k))}_{\doteq\,\Delta}$$

Step 3 – Value Function Decrease

$$\Delta \doteq J(x(t_{k+1}), u(\cdot, x(t_{k+1}))) - V_T^S(x(t_k))$$

$$\Delta = \quad S(x(t_{k+1})) - S(x(t_k)) - \int_{t_k}^{t_{k+1}} \ell(x^\star(\tau|t_k)), u^\star(\tau|t_k))\, \mathrm{d}\tau + \int_{t_{k+1}+\tau_\varepsilon}^{t_{k+1}+\tau_\varepsilon+\delta} \ell(\cdot)\, \mathrm{d}\tau$$



Step 3 – Value Function Decrease (cont'd)

Integral dissipation inequality:

$$
\begin{aligned}
S(x(t_{k+1})) - S(x(t_k)) - \int_{t_k}^{t_{k+1}} \ell(x^\star(\tau|t_k), u^\star(\tau|t_k))\,\mathrm{d}\tau \\
\leq \int_{t_k}^{t_{k+1}} -\alpha(\|(x^\star(\tau|t_k), u^\star(\tau|t_k)) - (\bar{x}, \bar{u})\|) - \ell(\bar{x}, \bar{u})\ \mathrm{d}\tau
\end{aligned}
$$

Step 3 – Value Function Decrease

$$\Delta \leq \int_{t_k}^{t_{k+1}} -\alpha(\|(x^\star(\tau), u^\star(\tau)) - (\bar{x}, \bar{u})\|) d\tau + \underbrace{\int_{t_{k+1}+\tau_\varepsilon}^{t_{k+1}+\tau_\varepsilon+\delta} \ell(\cdot) - \ell(\bar{x}, \bar{u}) \, \mathrm{d}\tau}_{\leq \sigma < \infty}$$

$\Rightarrow$ From t_k to t_{k+1} the optimal solution moves towards a neighborhood of $\bar{x}$.

Practical Convergence of Economic NMPC

Theorem III.7 (Practical convergence of economic NMPC)

Suppose that, for all $x_0 \in \mathbb{X}_0$, Assumptions A2', A3 hold, and

- the $(\bar{x},\, \bar{u})$ turnpike is reachable exponentially fast;
- $(\bar{x},\, \bar{u}) \in \mathrm{int}(\mathbb{X} \times \mathbb{U})$ with $\left.\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial u}\right)\right|_{(\bar{x},\bar{u})}$ controllable, and
- the penalty on the initial condition is $I(x(t_k)) = S(x(t_k))$, with $S(x)$ satisfying (DI).

Then, there exists $\delta > 0$ and $T < \infty$ such that, for all $x_0 \in \mathbb{X}_0$,

(i) $OCP_T^{eco,I}(x(t_k))$ **is recursively feasible**; and

(ii) **NMPC tracks the optimal steady state** $\bar{x} = \bar{x}^\star$:

$$\lim_{t\to\infty} \mathrm{d}\big(\mathcal{B}_\rho(\bar{x}),\, x(t)\big) = 0.$$

with $\mathrm{d}\big(\mathcal{B}_\rho(\bar{x}),\, x(t)\big) \doteq \min_{z\in\mathcal{B}_\rho(\bar{x})} \|x - z\|$.

- Expression for radius ρ of neighborhood $\mathcal{B}_\rho(\bar{x})$ see, T. Faulwasser **and** D. Bonvin. "On the Design of Economic NMPC based on Approximate Turnpike Properties". **in** *Proc. of 54th IEEE Conference on Decision and Control*: Osaka, Japan, 2015, **pages** 4964 –4970. DOI: `10.1109/CDC.2015.7402995`.
- Choosing large T and small δ, ρ can be made arbitrarily small.
- The penalty $I(x(t_k)) = S(x(t_k))$ can be dropped and the result still holds.

Summary – Economic NMPC

- Dissipativity enables stability proofs for economic NMPC with terminal constraints.
- Dissipativity implies the turnpike property.
- Turnpikes are helpful to show convergence in NMPC.
- **Recursive feasibility** can be shown **without any terminal constraint**.
- Controllability of turnpike implies recursive feasibility (long horizons).
- **Conditions apply to economic and non-economic NMPC schemes.**
- "Automatic tracking" of optimal steady state. Turnpike properties are natural candidates to design NMPC schemes.

Overview

Comparison of Stability Proofs

- NMPC for setpoint stabilization with terminal penalty relies on the following Lyapunov inequality

$$V_f(x(t))\Big|_{t_k}^{t_{k+1}} + \int_{t_k}^{t_{k+1}} \ell(x(\tau), u(\tau))\, \mathrm{d}\tau \leq 0. \tag{LI}$$

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- NMPC based on turnpikes relies on the dissipation inequality (DI) which, for $\ell(\bar{x}, \bar{u}) = 0$, can be written as

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- Can the turnpike concept be applied to NMPC for setpoint stabilization?

Main Points

- Stability of NMPC can be enforced via appropriate OCP formulation
- $\exists$ formulations with and without terminal constraints
- In applications formulations without terminal constraints are often preferred
- Here: time invariant problem formulations mainly

Summary – NMPC

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- Stability of NMPC can be enforced via appropriate OCP formulation
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Open Problems and Ongoing Research

- Real-time applicable robust NMPC?
- Stochastic economic NMPC?
- Distributed approaches?
- ...

The End.

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