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Nonlinear Model Predictive Control

A Primer on CasADi

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Version WS21/22.I

Overview

Basics on Algorithmic Differentiation via CasADi

Solving NLPs via CasADI and IPOPT

Solving Optimal Control Problems

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CasADi basics

Recall from lecture: derivatives can be computed

- By hand (**time consuming and error prone**)
- Via symbolic differentiation **(typically slow)**
- Via finite differences **(introduces numerical errors)**
- Via *algorithmic differentiation*

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CasADi is an open-source tool for **algorithmic differentiation** in combination with **state-of-the-art NLP solvers** and focus on optimal control

`https://web.casadi.org/`

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`https://web.casadi.org/`

Features:

- Usable from C++, Python, MATLAB $\rightarrow$ usable from multiple languages
- A basic symbolic framework
- Optistack interface for easy optimal control
- C-code generation

Installation

- Download CasADi from `https://web.casadi.org/get/`
- Add the extracted folder to your MATLAB path

CasADi for Algorithmic Differentiation

We use MATLAB for our demonstration, python syntax is very similar
Example: Compute the Jacobian of $g : \mathbb{R}^2 \to \mathbb{R}^2$

$$g(x) = \begin{pmatrix} \sin(x_1)\cos(3x_2) \\ e^{-x_1} \end{pmatrix}$$

$$\nabla g(x) = \begin{pmatrix} \cos(x_1)\cos(3x_2) & -3\sin(x_1)\sin(3x_2) \\ -e^{-x_1} & 0 \end{pmatrix}$$

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Step 1: Create CasADi symbolic variables (SX or MX)

```
x = SX.sym('x',2); % creates a 2-dimension symbolic vector
```

Step 2: Set up your symbolic expressions

```
g = [   sin(x(1))*cos(3*x(2));
        exp(-x(1))             ];
```

CasADi for Algorithmic Differentiation

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```
dg = jacobian(g,x)
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CasADi for Algorithmic Differentiation

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```
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```

Output:

```
dg =
@1=3, @2=(@1*x_1),
[[(cos(@2)*cos(x_0)), (-(sin(x_0)*(@1*sin(@2))))],
[(-exp((-x_0))), 00 ]]
```

(special CasADi syntax with @-expressions as placeholders)

How to evaluate symbolic expressions?

- convert into function objects

```
dgFun = Function('dg',{x},{dg})
```

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```
dgFun = dg:(i0[2])->(o0[2x2,3nz]) SXFunction
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(support/range dimension and #structural zeros)

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dgFun = dg:(i0[2])->(o0[2x2,3nz]) SXFunction
```

(support/range dimension and #structural zeros)

Get $\nabla g((1,\ 2)^\top)$:

```
dgFun([1 2]')
```

Output:

```
ans = [[0.518782, 0.70536], [-0.367879, 00]]
```

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Solving NLPs

CasADi NLP interface:

$$\begin{aligned} \underset{x}{\text{minimize:}} \quad & f(x,p) \\ \text{subject to:} \quad & x_{\text{lb}} \leq x \leq x_{\text{ub}} \\ & g_{\text{lb}} \leq g(x,p) \leq g_{\text{ub}} \end{aligned}$$

x : decision vector
p : parameters

Why extra bounds x_{lb}, x_{ub}? $\rightarrow$ more efficient numerical handling

Example: Solve

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & (1-x_1)^2 + 0.2(x_2 - x_1^2)^2 \\ \text{subject to} \quad & \frac{p^2}{4} \leq (x_1+0.5)^2 + x_2^2 \leq p^2 \\ & x_1 \geq 0 \end{aligned}$$

Solving NLPs

Step 1: Set up your decision variables (+ parameters)

```
x = SX.sym('x',2); p = SX.sym('p',1);
```

Step 2: Set up objective f, constraints g, bounds $x_{\mathrm{lb}}, x_{\mathrm{ub}}, g_{\mathrm{lb}}, g_{\mathrm{ub}}$, ...

```
f = (1-x(1))^2 + 0.2*(x(2)-x(1)^2)^2;
g = [ p^2/4 - (x(1)+0.5)^2 + x(2)^2;
      (x(1)+0.5)^2 + x(2)^2 - p^2 ];

lbx = [0; -inf];
ubx = inf;
lbg = -inf;
ubg = 0;
```

Solving NLPs

Step 3: Set up a NLP solver and solve (with zero initial conditions)

```
nlp = struct('x',x, 'f',f, 'g',g, 'p',p);
S = nlpsol('S', 'ipopt', nlp)
sol = S('x0', zeros(2,1),...
        'lam_g0', zeros(2,1),...
        'lam_x0', zeros(2,1),...
        'p',      1.25,...
        'lbx',    lbx,...
        'ubx',    ubx,...
        'lbg',    lbg, ...
        'ubg',    ubg);
```

Solving NLPs

Output:

```
This is Ipopt version 3.12.3, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt
documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:        4
Number of nonzeros in Lagrangian Hessian.............:        3

Total number of variables............................:        2
variables with only lower bounds:                             1
variables with lower and upper bounds:                        0
variables with only upper bounds:                             0
Total number of equality constraints.................:        0
Total number of inequality constraints...............:        2
inequality constraints with only lower bounds:                0
inequality constraints with lower and upper bounds:           0
inequality constraints with only upper bounds:                2
```

Output (cont'd):

```
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du
0  9.8010002e-001 1.31e-001 9.87e-001  -1.0 0.00e+000    -  0.00e+0
1  6.6794608e-001 0.00e+000 1.01e+001  -1.0 1.76e-001    -  1.19e-0
2  5.7298195e-001 0.00e+000 1.89e+000  -1.0 1.13e-001    -  4.64e-0
3  5.6860369e-001 0.00e+000 8.02e-001  -1.0 8.02e-003   2.0 1.00e+0
4  4.0551835e-001 0.00e+000 5.56e-001  -1.7 1.77e-001    -  5.29e-0
5  4.0688356e-001 0.00e+000 4.28e-001  -1.7 1.28e-002   1.5 1.00e+0
6  1.1751232e-001 0.00e+000 3.91e-001  -1.7 5.45e-001    -  6.60e-0
7  7.8213074e-002 7.65e-002 1.84e-001  -1.7 5.18e-001    -  1.00e+0
8  9.3051090e-002 0.00e+000 1.21e-002  -2.5 7.02e-002    -  1.00e+0
9  9.0600802e-002 0.00e+000 8.76e-005  -3.8 1.87e-002    -  1.00e+0
10 9.0512237e-002 0.00e+000 4.34e-007  -5.7 8.62e-004    -  1.00e+0
11 9.0510219e-002 0.00e+000 1.15e-010  -8.6 1.19e-005    -  1.00e+0

Number of Iterations....: 11
```

Solving NLPs

Output (cont'd):

```
                                   (scaled)                 (unscaled)
Objective...............:  9.051021857953676e-002   9.051021857953
Dual infeasibility......:  1.147774236764931e-010   1.147774236764
Constraint violation....:  0.000000000000000e+000   0.000000000000
Complementarity.........:  2.566280058381725e-009   2.566280058381
Overall NLP error.......:  2.566280058381725e-009   2.566280058381


Number of objective function evaluations             = 12
Number of objective gradient evaluations             = 12
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 12
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 12
Number of Lagrangian Hessian evaluations             = 11
Total CPU secs in IPOPT (w/o function evaluations)   = 0.024
Total CPU secs in NLP function evaluations           = 0.000
```

Solving NLPs

Output (cont'd):

```
EXIT: Optimal Solution Found.
S            :   t_proc      (avg)   t_wall      (avg)    n_eval
nlp_f        |        0 (       0)        0 (       0)        12
nlp_g        |        0 (       0)        0 (       0)        12
nlp_grad     |        0 (       0)        0 (       0)         1
nlp_grad_f   |   1.00ms ( 76.92us) 999.00us ( 76.85us)        13
nlp_hess_l   |        0 (       0)        0 (       0)        11
nlp_jac_g    |        0 (       0)        0 (       0)        13
total        |  31.00ms ( 31.00ms)  31.43ms ( 31.43ms)         1
```

- NLP solved successfully
- Solved within milliseconds
- Time spent in sensitivity evaluations negligible (due to AD)
- Most time spent in factoring the KKT system

Solving NLPs

Pitfalls: If converged to a point of local infeasibility or slow convergence

- Try different guess x_0
- Reformulate objective/constraints (e.g. eleminate $\sqrt{\cdot}$)
- Re-scale problem

Output: Solution struct

```
sol =
struct with fields:
f:      [1x1 casadi.DM]
g:      [2x1 casadi.DM]
lam_g:  [2x1 casadi.DM]
lam_p:  [1x1 casadi.DM]
lam_x:  [2x1 casadi.DM]
x:      [2x1 casadi.DM]
```

`x:`	$x^\star$	`lam_g:`	multipliers for g
`f:`	$f(x^\star)$	`lam_p:`	multipliers for p
`g:`	$g(x^\star)$	`lam_x:`	multipliers for $x_{\mathrm{lb}}, x_{\mathrm{ub}}$

Solving NLPs

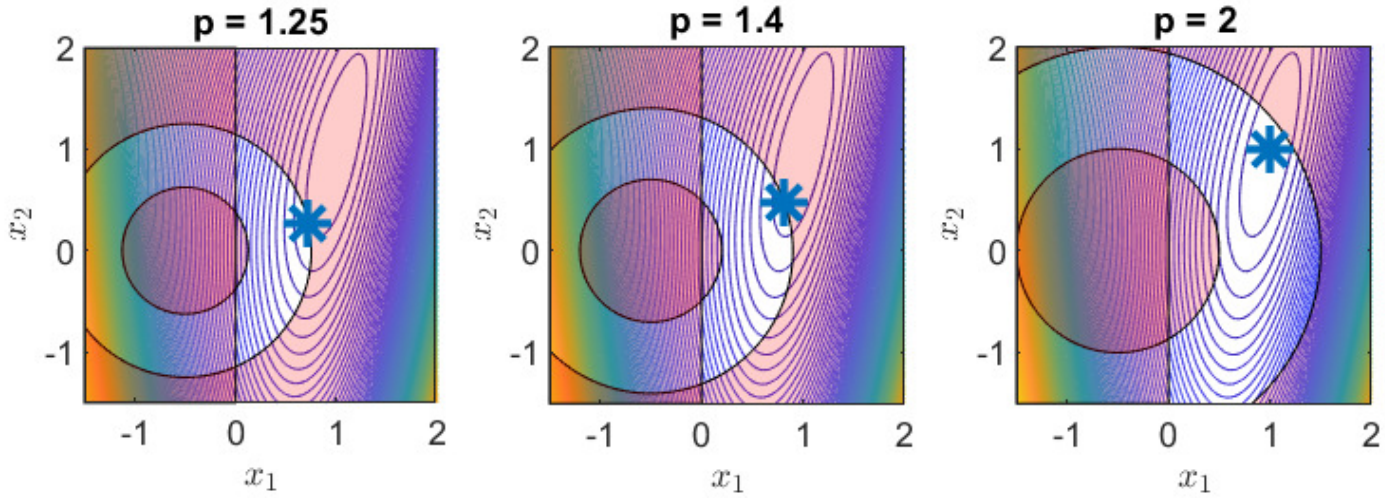
Why parameterized NLPs?

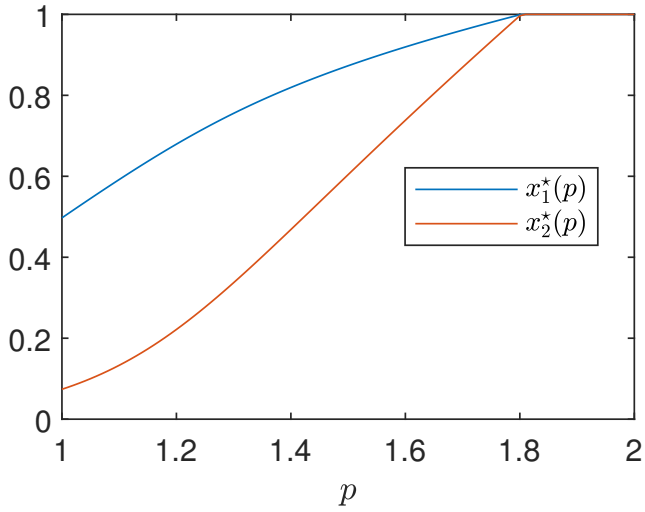
- NLP construction only once
- very relevant for MPC (changing $x(0)$)

Example: Approximate the mapping $x^\star(p)$ numerically for $p \in [1,\ 2]$

```
X = [];
pVec = 1:0.01:2;
for p = pVec
      sol = S('x0', zeros(2,1),...
              'lam_g0', zeros(2,1),...
              'lam_x0', zeros(2,1),...
              'p',      p,...
              'lbx',    lbx,...
              'ubx',    ubx,...
              'lbg',    lbg, ...
              'ubg',    ubg);
      X = [X full(sol.x)];
end
```

Solving NLPs





Source: https://web.casadi.org/blog/nlp_sens/

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Solving OCPs

Two possibilities in CasADi:

- Formulate and solve OCP via the (low-level) symbolic framework introduced above
- Formulate and solve OCP via the `Opti` syntax $\rightarrow$ simplified problem setup (now)

Example: A simple minimum-time problem

- Very simple 1d model of a car

$$\begin{pmatrix} \dot{p} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v \\ u - v \end{pmatrix}$$

- Throttle is control input u
- Arrive at destination as fast as possible
- Time-varying speed-limit $L(p) = 1 - \sin(2\pi p)/2$

Source: https://web.casadi.org/blog/ocp/

Solving OCPs

Formulate the corresponding OCP

$$
\begin{aligned}
&\underset{x(\cdot),\,u(\cdot),\,T}{\text{minimize}} && T \\
&\text{subject to} && \dot{x}(t) = f(x(t), u(t)) \quad t \in [0, T], && \text{dynamic constraints} \\
& && p(0) = 0, && \text{start at position 0} \\
& && v(0) = 0, && \text{start with zero speed} \\
& && p(T) = 1, && \text{the finish line is at position 1} \\
& && 0 \leq u(t) \leq 1, && \text{path constraint: throttle is limited} \\
& && v(t) \leq L(p(t)). && \text{speed limit varying along the track}
\end{aligned}
$$

Step 1: Construct `opti` variables:

```
opti = casadi.Opti();          % optimization problem
X    = opti.variable(2,N+1);   % state trajectory
U    = opti.variable(1,N);     % control trajectory (throttle)
T    = opti.variable();        % final time
```

Source: https://web.casadi.org/blog/ocp/

Solving OCPs

Step 2: Construct objective and integrate dynamics and RK4 integrator (direct multiple shooting):

```
opti.minimize(T);                  % minimize time

f = @(x,u) [x(2);u-x(2)];          % dx/dt = f(x,u)

dt = T/N;
for k=1:N                          % loop over control intervals
      k1 = f(X(:,k),          U(:,k));
      k2 = f(X(:,k)+dt/2*k1, U(:,k));
      k3 = f(X(:,k)+dt/2*k2, U(:,k));
      k4 = f(X(:,k)+dt*k3,   U(:,k));
      x_next = X(:,k) + dt/6*(k1+2*k2+2*k3+k4);

      opti.subject_to(X(:,k+1)==x_next);  % close the gaps
end
```

Source: https://web.casadi.org/blog/ocp/

Solving OCPs

Step 3: Add path and boundary constraints:

```
pos   = X(1,:);
speed = X(2,:);

% -- path constraints ------
limit = @(pos) 1-sin(2*pi*pos)/2;
opti.subject_to(speed<=limit(pos)); % track speed limit
opti.subject_to(0<=U<=1);           % control is limited

% -- boundary conditions ----
opti.subject_to(pos(1)==0);   % start at position 0 ...
opti.subject_to(speed(1)==0); % ... from stand-still
opti.subject_to(pos(N+1)==1); % finish line at position 1

% -- misc. constraints  -----
opti.subject_to(T>=0);        % Time must be positive
```

Source: https://web.casadi.org/blog/ocp/

Solving OCPs

Step 3: Initialize solver and solve OCP:

```
% -- initial values for solver --
opti.set_initial(speed, 1);
opti.set_initial(T, 1);

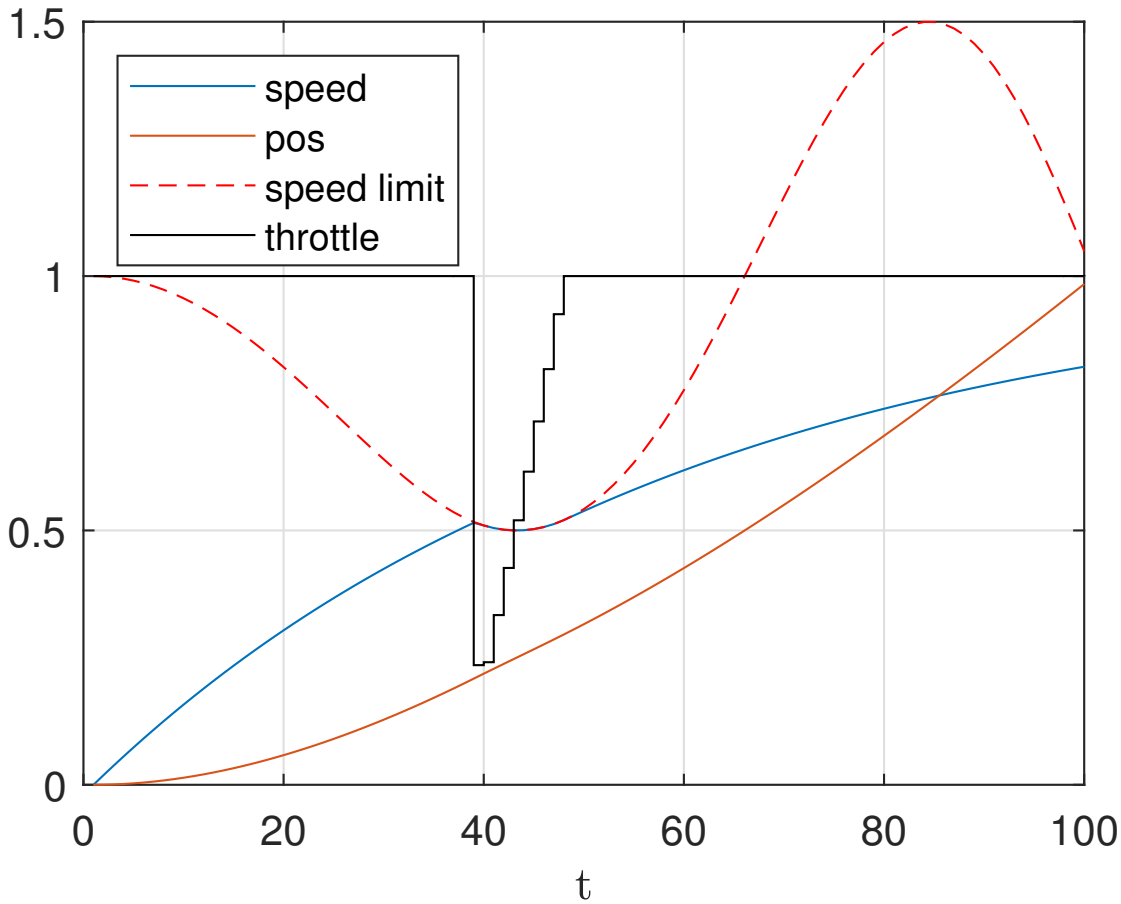
% -- solve NLP                  ---
opti.solver('ipopt'); % set numerical backend
sol = opti.solve();   % actual solve
```

Output:

- IPOPT requires 42 iterations and 216ms (Intel Corei5 10.Gen)

Source: https://web.casadi.org/blog/ocp/

Solving OCPs



For MPC:

- Put the above OCP into an MPC loop
- Change initial condition in each sampling instance

Source: https://web.casadi.org/blog/ocp/