Problem Set #1: Nonlinear Programming

Analytic Solutions to NLPs

Exercise 1

Consider the constraints

$$\begin{aligned} x_1 &\geq 0 \\ x_2 &\geq 0 \\ x_2 - (x_1 - 1)^2 &\geq 0 \end{aligned}$$

in $\mathbb{R}^2$. Sketch the feasible region. Furthermore, show that the point $x_1 = 1, x_2 = 0$ is feasible but not regular.

Exercise 2

Find a solution to the problem

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{R}^2} \quad & 2x_1^2 + 2x_1x_2 + x_2^2 - 10x_1 - 10x_2 \\ s.t. \quad & x_1^2 + x_2^2 \leq 5 \\ & 3x_1 + x_2 \leq 6. \end{aligned}$$

Exercise 3

Find a solution to the problem

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{R}^2} \quad & x_1 \\ s.t. \quad & x_2 = 0 \\ & \mathbf{x} \in \mathcal{X} := \{\mathbf{x} \mid x_1^2 \leq x_2\}. \end{aligned}$$

Exercise 4

Consider the following NLP:

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{R}^2} \quad & (x_1 - 3)^2 + (x_2 - 3)^2 \\ s.t. \quad & 4x_1^2 + 9x_2^2 \leq 36 \\ & x_1^2 + 3x_2 = 3 \\ & \mathbf{x} \in \mathcal{X} := \{\mathbf{x} \mid x_1 \geq -1\}. \end{aligned}$$

a) Sketch the feasible region and the contours of the objective function, then identify the optimum graphically.

b) Using this graphical information, determine the minimum precisely based on first-order necessary conditions. Are the second order conditions of optimality satisfied at this point?

c) Repeat by replacing $\min$ by $\max$.

Exercise 5*

Given a cardboard of area A to make a rectangular box. What is the maximum volume that can be attained?

a) Reformulate the problem as an unconstrained one, then find the candidate minima. *Hint: Eliminate* x_3 *and show that* $x_1 = x_2$ *must hold at an optimal solution.*

b) Solve the problem directly by using the method of Lagrange multipliers. *Hint: Show that* $x_1 = x_2 = x_3$ *must hold at an optimal solution.*

Numerical Solutions to NLPs

Exercise 6: Computation of Function Derivatives

Consider the *Speelpenning's* function $f : \mathbb{R}^n \to \mathbb{R}$

$$f(\mathbf{x}) = \prod_{i=1}^{n} x_i.$$

The aim is to compute the gradient of f at a point $\mathbf{x}$. Consider $n = 10$ and $x = \pi + ones(n,1)$.

a) Write a Matlab-function to compute the gradient using forward, backward and central finite differences.

b) Write a Matlab-function to compute the gradient using the *imaginary trick* in Matlab.

c) Repeat the gradient computation via algorithmic differentiation using the CasADi package. The package can be obtained from `https://github.com/casadi/casadi/wiki`. The installations instructions are available on `https://github.com/casadi/casadi/wiki/InstallationInstructions`.

d) Finally, compare the results a) and b) in terms of accuracy with the ones from c).

Exercise 7: `fmincon` Basics

We want to find (analytically and numerically) the maxima of

$$\begin{aligned} \max_{\mathbf{x} \in \mathbb{R}^2} \quad & f(\mathbf{x}) \\ s.t. \quad & h(\mathbf{x}) = 0 \end{aligned}$$

where

$$f(\mathbf{x}) = e^{-x_1^2 - x_2^2}, \qquad h(\mathbf{x}) = x_2 + x_1^2 - 1 = 0.$$

a) Check first if the problem is concave. *Hint: Note that we can solve the equality constraint for* x_2 *and plug it into* $f(\mathbf{x})$*. Plot the resulting function in 3D using* x_1 *as the free parameter.*

b) Calculate the Lagrangian, state the first-order necessary conditions of optimality and find candidate solutions.

c) How could you exclude minima from the appearing solutions?

d) Solve the problem using `fmincon` from Matlab's Optimization Toolbox using as initial guesses $[0, 0]$, $[-1, -1]$ and $[1, 1]$. Try also $[100, 1000]$ as initial guess.

Exercise 8*: Interpretation of Lagrange Multipliers

Consider the following minimization problem

$$\begin{aligned}
\min_{\mathbf{x}\in\mathbb{R}^3} \quad & x_1^2 + x_1x_2 + 2x_2^2 - 6x_1 - 2x_2 - 12x_3 \\
s.t. \quad & g_1(\mathbf{x}) = 2x_1^2 + x_2^2 \leq 15 \\
& g_2(\mathbf{x}) = x_1 - 2x_2 - x_3 \geq -3 \\
& x_1, x_2, x_3 \geq 0.
\end{aligned}$$

a) Find an optimal solution to this problem using `fmincon` from Matlab's Optimization Toolbox. *Hints:*

   - Set the solution tolerance, the function tolerance and the constraint tolerance in `fmincon` to 10^{-7};
   - Use the inital guess $\mathbf{x}_{guess} = (1,\ 1,\ 1)^T$;
   - Make sure that the constraint $g_2 : \mathbb{R}^3 \to \mathbb{R}$ is treated as a linear constraint.
   - First solve the problem by finite-difference gradient approximation for objective and constraints. Second, resolve the problem by providing explicit expressions for the gradients of objective and constraints (ask `fmincon` to check the gradients before running the optimization);
   - Make sure that the solver terminated succesfully in each case.

b) Repeat the optimization with a different initial guess, for instance $\mathbf{x}_{guess} = (14,\ 0,\ 0)^T$. Does `fmincon` converge to the same solution? Could this be expected?

c) Get the values of the Lagrange multipliers at $\mathbf{x}^\star$, as well as the gradients of objective function and constraints. Check whether the optimal solution is a regular point and that it satisfies the KKT conditions.

d) Consider the pertubed problem

$$\begin{aligned}
\min_{\mathbf{x}\in\mathbb{R}^3} \quad & x_1^2 + x_1x_2 + 2x_2^2 - 6x_1 - 2x_2 - 12x_3 \\
s.t. \quad & g_1(\mathbf{x}) = 2x_1^2 + x_2^2 \leq \theta \\
& g_2(\mathbf{x}) = x_1 - 2x_2 - x_3 \geq -3 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}$$

   where θ is the pertubation parameter in $g_1 : \mathbb{R}^3 \to \mathbb{R}$. Solve the pertubed problem for N equally spaced values of $\theta \in [0, 30]$. Use a resolution of 0.5 for θ. Lets denote the optimal solution for some value of θ as $\xi^\star(\theta)$ and the Lagrange multiplier associated with $g_1 : \mathbb{R}^3 \to \mathbb{R}$ as $\omega^\star(\theta)$.

   d1) Plot the objective $f(\xi^\star(\theta))$ versus θ and estimate the slope at $\theta = 15$. What does the corresponding value $\frac{\partial f(\xi^\star(\theta))}{\partial\theta}$ represent?

   d2) Plot $\omega^\star(\theta)$ versus θ. Comment this plot and, in particular, explain the behavior at $\theta = 0$. What is the slope of $f(\xi^\star(\theta))$ at $\theta = 0$?