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Nonlinear Model Predictive Control – Theory and Applications

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Preface

This script is based on lectures notes of courses given over the span of seven years at EPFL, KIT, and TU Dortmund. The script is currently under continuous development and further material will be added during the course of the semester. Comments and feedback are very welcome and should be addressed to the author via email (timm.faulwasser@ieee.org).

Status indicator of these notes

Version index: Draft γ.2

- **Part I:** Under construction
- **Part II:** Almost finished, except missing examples and the tracking part.
- **Part III:** Under construction
- **Part IV:** Under construction
- **Appendices:** Under construction
- **Index:** Under construction

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Part I

Optimal Control and Dissipativity — Connected on a Turnpike

Getting Started with Optimal Control

Consider a dynamical system

$$\frac{\mathrm{d}x(t)}{\mathrm{d}\tau} = f(x(t), u(t)), \quad x(t_0) = x_0 \tag{Σ}$$

given as an Ordinary Differential Equation (ODE), whereby $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$ is the vectorfield describing the dynamics, $x \in \mathbb{R}^{n_x}$ is the state variable, and $u \in \mathbb{R}^{n_u}$ is the input (or control) variable. Moreover, x_0 is the initial condition and $\tau \in \mathbb{R}_0^+$ refers to the time variable. In many applications it is of interest to compute control inputs $u : [t_0, t_1] \to \mathbb{R}^{n_u}$ such as to achieve a specific goal. For the sake of simplicity, consider the problem of steering the state x from $x(t_0) = x_0$ to a target set

$$\mathbb{X}_f \subset \mathbb{R}^{n_x}$$

during some given time interval $[t_0, t_1]$. Frequently, such a transition of the system state shall be conducted while constraints on states and inputs have to be satisfied. Such constraints be denoted in a set-based fashion as

$$u(t) \in \mathbb{U} \quad \text{and} \quad x(t) \in \mathbb{X}$$

where $\mathbb{U} \subseteq \mathbb{R}^{n_u}$ and $\mathbb{X} \subseteq \mathbb{R}^{n_x}$ are given closed sets and the constraints are required to hold for all $t \in [t_0, t_1]$. Alternatively, and with only minor loss of generality, we may assume that the constraints on input and state variables are modelled by inequality constraints, i.e., they read

$$g(x, u) \leq 0$$

where $g : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_g}$, where n_g refers to the number of inequality constraints. Constraints of this particular form are referred to as mixed input-state path constraints. Likewise, the terminal constraint can be written via

The term *path constraint* refers to the fact that the constraint shall hold along the solution path, i.e., on $[t, t_1]$.

$$\mathbb{X}_f \doteq \{x \in \mathbb{R}^{n_x} \mid \psi(x) \leq 0\}.$$

For the statement of optimality conditions, one typically prefers the notation using inequality descriptions of the constraints. However, the analysis of predictive control schemes in later parts of this book will benefit from and rely on the set-based notation.

The final element needed to state an Optimal Control Problem (OCP) is a criterion, which provides a means of ordering different candidate solutions with respect to their performance. In optimal control one usually considers functionals $J : u(\cdot) \mapsto \mathbb{R}$ of the following form

$$J(u(\cdot)) \doteq \int_{t_0}^{t_1} \ell(x(\tau), u(\tau)) \,\mathrm{d}\tau + \phi(x(t_1)),$$

where $\ell : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}$ is called Lagrange function (or stage cost) and $\phi : \mathbb{R}^{n_x} \to \mathbb{R}$ is denoted as Mayer term (or terminal penalty/cost).

Here we give the names common in the optimal control literature (Lagrange term and Mayer term) and in the predictive control literature (stage cost and terminal cost).

Combining the elements above we obtain an OCP in the following prototypical form:

$$
\begin{aligned}
\min_{u(\cdot)\in\hat{\mathcal{C}}[t_0,t_1]^{n_u}} & \int_{t_0}^{t_1} \ell(x(\tau),u(\tau))\,\mathrm{d}\tau + \phi(x(t_1)) \\
\text{subject to } \forall \tau \in [t_0,t_1] : & \\
\frac{\mathrm{d}x(\tau)}{\mathrm{d}\tau} & = f(x(\tau),u(\tau)), \quad x(t_0) = x_0 \\
g(x(\tau),u(\tau)) & \leq 0 \\
\psi(x(t_1)) & \leq 0.
\end{aligned}
\tag{OCP}
$$

Problems of this type have been extensively studied since the 1700s[1] and they arise in many fields of science and engineering.

Given an OCP the following questions arise naturally:

- How to solve (OCP) either analytically or numerically?
- How to certify that a given input $u(\cdot) \in \hat{\mathcal{C}}[t_0,t_1]^{n_u}$ is indeed an optimal solution? In which function space shall we search for an optimal solution?

The former question points towards numerical optimization and numerical optimal control. Good introductory texts on these topics are available open source.[2] The later question points us toward the need for (necessary and sufficient) optimality conditions.

Necessary Conditions of Optimality

Classic Calculus of Variations allows to state optimality conditions for (OCP) without major difficulties whenever the mixed input-state path constraints are dropped.[3] Moreover, the 20th century has witnessed tremendous progress in answering the question for optimality conditions of OCPs with constraints on inputs and states. Specifically, the Pontryagin Maximum Principle (PMP) and the Hamilton-Jacobi-Bellman-Equation (HJBE) provide complementary answers have been provided. Subsequently, we will briefly recall first the PMP before—using simplifying assumptions—we turn to the HJBE.

Our introductory exposition above has avoided giving mathematical details. Turning towards the input signals $u : [t_0,t_1] \to \mathbb{R}^{n_u}$ it is necessary to be more specific. Here we consider that $u(\cdot) \in \mathcal{L}_\infty([t_0,t_1],\mathbb{R}^{n_u})$, i.e., we consider measurable control functions which are essentially bounded on $[t_0,t_1]$.[4]

We begin with a statement of our assumptions. San

Assumptions: mixed input state constraints of degree 1, sufficiently smooth f, g.

Statement:

[1] H.J. Sussmann and J.C. Willems. "300 years of optimal control: from the brachystochrone to the maximum principle". In: *IEEE Control Systems* 17.3 (1997), pp. 32–44; H.J. Pesch. "Carathéodory's royal road of the calculus of variations: Missed exits to the maximum principle of optimal control theory". In: *Numerical Algebra, Control & Optimization* 3.1 (2013), pp. 161–173.

[2] B. Chachuat. *Nonlinear and Dynamic Optimization: From Theory to Practice*. EPFL, 2009. URL: `https://infoscience.epfl.ch/record/111939/files/Chachuat_07(IC32).pdf`; J.B. Rawlings, D.Q. Mayne, and M. Diehl. *Model Predictive Control: Theory, Computation, and Design*. Nob Hill Publishing, Madison, WI, 2017.

[3] Chachuat, *Nonlinear and Dynamic Optimization: From Theory to Practice*; D. Liberzon. *Calculus of Variations and Optimal Control Theory: A Concise Introduction*. Princeton University Press, 2012.

[4] M. Gerdts. *Optimal control of ODEs and DAEs*. Walter de Gruyter, 2011.

Define optimal value function Recall role of adjoint Recall optimal value function Recall principle of optimality

Recall HJBE

Maximum Principle

HJBE

Dissipativity and Optimal Control

Dissipation Inequalities in Systems and Control

Dissipativity of OCPs

Turnpike Properties

Part II

Predictive Control for Stabilization

From Optimal Control to Predictive Control

- give a motivation, turn From Optimal Control to Predictive Control into the motto for this part
- Put quote from Lee-Markus
- Ref to Propoi
- References on time-line

Problem Formulation

Next, we turn towards predictive control for a specific and prototypical control problem—i.e., set-point stabilization. To this end, we consider the system

$$\dot{x} = f(x, u), \quad x(0) = x_0 \tag{Σ}$$

subject to input and state constraints $\mathbb{U} \subseteq \mathbb{R}^{n_u}, \mathbb{X} \subseteq \mathbb{R}^{n_x}$. A plethora of real-world control problems can be cast as *set-point stabilization*.

A reference set-point $\bar{x}$ corresponds to a controlled equilibrium of (Σ), i.e. there exists a not necessarily unique $\bar{u} \in \mathbb{U}$ such that $0 = f(\bar{x}, \bar{u})$. Cases in which $\bar{u}$ does not exists are tricky, as $\bar{x}$ cannot be stabilized.

Problem 1 (Constrained set-point stabilization).
Consider system (Σ) *and a reference setpoint* $\bar{x} \in \mathbb{X} \subseteq \mathbb{R}^{n_x}$*, design a feedback* $k : \; x \mapsto u$ *such that the following is achieved:*

(i) asymptotic convergence: $\lim_{t\to\infty} x(t, x_0, k(x(\cdot))) = \bar{x}, \quad \forall x_0 \in \mathbb{X}_0$

(ii) Lyapunov stability: $\forall \varepsilon > 0 \; \exists \delta > 0$ *such that*

$$\|x(0) - \bar{x}\| < \delta \quad \Rightarrow \quad \|x(t, x_0, k(x(\cdot))\| < \varepsilon \quad \forall t \geq 0.$$

(iii) constraint satisfaction: $\forall t \geq 0 : u(t) \in \mathbb{U}$ *and* $x(t, x_0, k(x(\cdot)) \in \mathbb{X}$

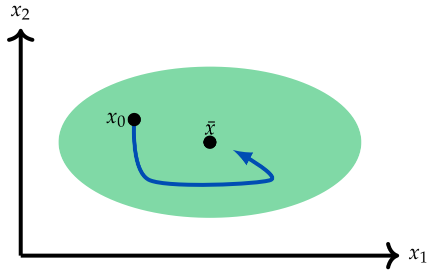


Figure 1: Set-point stabilization.

Notice that Problem 1 includes three items: (i) asymptotic convergence, (ii) Lyapunov stability, and (iii) constraint satisfaction. Indeed it turns out that the major strength of NMPC is its ability to account for all three items for nonlinear systems with multiple inputs.

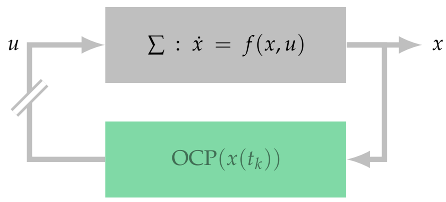


Figure 2: Basic NMPC scheme.

To highlight that our feedback strategy is based on the receding-horizon solution of a specific OCP, we write $\mathrm{OCP}(x(t_k))$, cf. Figure 2.

Mathematically and in full generality, $\mathrm{OCP}(x(t_k))$ reads

$$\begin{aligned}
&\min_{u(\cdot|t_k)} \int_{t_k}^{t_k+T} \ell(x(\tau|t_k), u(\tau|t)) \, \mathrm{d}\tau + V_{\mathrm{f}}(x(t_k + T|t_k)) \\
&\text{subject to } \forall \tau \in [t_k, t_k + T] : \qquad\qquad (OCP(x(t_k))) \\
&\quad \frac{\mathrm{d}x(\tau|t_k)}{\mathrm{d}\tau} = f(x(\tau|t_k), u(\tau|t_k)) \\
&\quad x(t_k|t_k) = x(t_k) \\
&\quad x(\tau|t_k) \in \mathbb{X}, \; u(\tau|t_k) \in \mathbb{U}, \\
&x(t_k + T|t_k) \in \mathbb{X}_{\mathrm{f}}.
\end{aligned}$$

In designing an NMPC scheme one has to specify all ingredients of the underlying $\mathrm{OCP}(x(t_k))$ (system model, constraints, objective) and one has to decide NMPC specific aspects (prediction horizon, sampling period). Hence it makes sense to be clear about which items and data are assumed to be given and what are our actual degrees of freedom for controller design. Subsequently, we will consider the following elements to be known/available:

- an exact system model $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$
- a description of the state constraints $\mathbb{X} \subseteq \mathbb{R}^{n_x}$
- a description of the input constraints $\mathbb{U} \subseteq \mathbb{R}^{n_u}$
- for all sampling instants $k \in \mathbb{N}$ state feedback is available $x(t_k)$.

Such a setting is called *nominal*, because one requires an exact plant/system model (hence no plant-model mismatch) and assumes perfect state measurements.

Unless otherwise stated our further considerations are built upon these base-line assumptions.

However, as is clear from the previous chapters, even if the data above is given, there remain choices in specifying an OCP. Hence, the considered degrees of freedom for NMPC design are:

- the stage cost $\ell : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}$
- the terminal penalty $V_\mathrm{f} : \mathbb{R}^{n_x} \to \mathbb{R}$ (if any),
- the terminal constraint $\mathbb{X}_\mathrm{f} \subseteq \mathbb{X} \subseteq \mathbb{R}^{n_x}$ (if any),
- the prediction horizon $T \in \mathbb{R}$, and
- the sampling period $\delta \doteq t_{k+1} - t_k \in (0, T]$.

Why bother about NMPC design?

At this point it is fair to ask, why one should bother about how to design an NMPC scheme? On the positive side, the fact that NMPC allows consideration of constraints for nonlinear multiple input systems subject to constraints is clearly of relevance in many applications. Moreover, there is promise that via a suitable choice of the stage cost ℓ one can as well improve control performance.

We will investigate the choice of generic cost functions in Part III.

Naively, one might expect that as NMPC is built upon receding-horizon optimal control, the core design challenge is merely of numerical nature and not a system theoretic one. However, the next example shows that this is not the case.

Example 1 (Pitfall example). *Consider a predictive control scheme based on the following linear-quadratic (LQ) OCP*

$$\min_{u(\cdot|t_k)} \int_{t_k}^{t_k+T} \|x(\tau|t_k)\|_Q^2 + \|u(\tau|t_k)\|_R^2 \,\mathrm{d}\tau$$

subject to

$$\forall \tau \in [t_k, t_k+T]: \ \frac{\mathrm{d}x(\tau|t_k)}{\mathrm{d}\tau} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x(\tau|t_k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(\tau|t_k),$$

$$x(t_k|t_k) = x(t_k).$$

Receding-horizon optimal control is a synonym for model predictive control, which stress the under-lying principle of solving OCP on receding-horizons.

$\|x(\tau|t_k)\|_Q^2 \doteq x(\tau|t_k)^\top Q x(\tau|t_k)$.

The weighting matrices are

$$Q = \operatorname{diag}\begin{bmatrix}5 & 5 & 0\end{bmatrix} \quad \textit{and} \quad R = 1,$$

the prediction horizon and the sampling period are $T = N \cdot \delta$ and $\delta = 0.05$.

Closed-loop simulation results are depicted in Figure 3. As one can see in Figure 3 for $T = 10 \cdot \delta$, the closed loop is appears to be stable. In contrast for $T = 8 \cdot \delta$ the closed-loop trajectories are increasing oscillations indicating instability, cf. Figure 3.

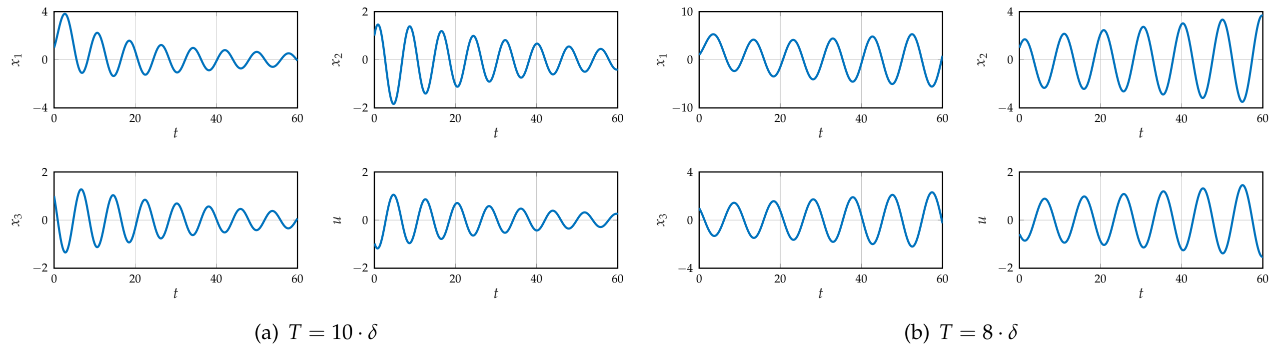


Figure 3: Closed-loop trajectories for different horizons in Example 1.

Note that this behavior is not due to any suboptimality or lack of convergence in the optimization. Indeed after suitable discretization, due to strict convexity in u and due to the linear dynamics, the considered LQ OCP can be solved to global optimality with high accuracy. Indeed, one could easily compute the analytic solution for the optimal control via standard LQ theory. Yet, the length of the horizon influences the closed-loop stability properties.

Figure 4 illustrates the main reason for the lack of closed-loop stability. It shows the open-loop predictions (in color) along side the closed-loop trajectories (in blue). As one can see the parts of the open-loop input predictions, which are not applied to the system, differ substantially from the closed-loop trajectories. This implies also a difference between predicted and closed-loop state trajectories. It is this mismatch between prediction and closed loop which—despite the prediction model being an exact copy of the plant—in this specific example leads to instability.

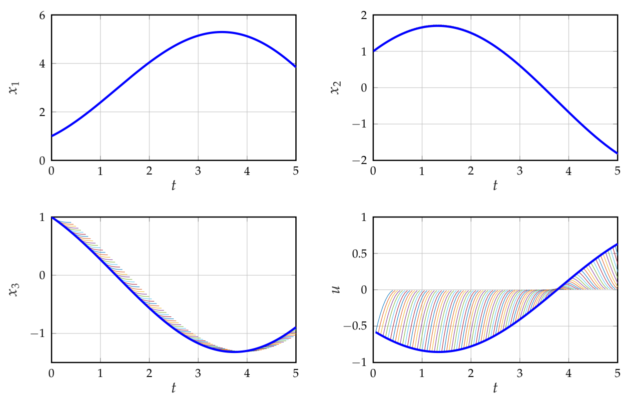


Figure 4: Closed-loop trajectories and open-loop predictions for $T = 8 \cdot \delta$ in Example 1.

The lessons learned from the above example are evident:

- *Optimality does not imply stability!*
- *There is a clear need for stability analysis of NMPC, which in turn also gives design guidelines!*

In the development of optimal control, the fact that optimality and stability do not necessarily coincide had already been observed as early as 1960. Indeed Rudolf E. Kalman (1930-2016) phrased it as follows:

> *In the engineering literature it is often assumed (tacitly and incorrectly) that a system with an optimal control law is necessarily stable.*

R.E. Kalman. "Contributions to the theory of optimal control". In: *Bol. Soc. Mat. Mexicana* 5.2 (1960), pp. 102–119

Standing Assumptions

As the previous example has illustrated, in the analysis of NMPC schemes, one has to distinguish between the open-loop prediction of input and state trajectories and their closed-loop counterparts. Hence, for all $\tau \in [t_k, t_k + T]$ the notation

$$x(\tau|t_k) \quad \text{and} \quad u(\tau|t_k)$$

refers to the values of controls and states *predicted* by the controller given the state feedback information at time t_k, i.e. $x(t_k)$.

In contrast the closed-loop system reads

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x(t), u^{MPC}(t)),\ x(0) = x_0. \tag{Σ_{cl}}$$

whereby $u^{MPC} : \mathbb{R}_0^+ \to \mathbb{U}$ denotes the control signal generated by the NMPC scheme, i.e.

$$u^{MPC}(t) \doteq u(t|t_k) \quad \text{with } k = \max\{k \in \mathbb{N} \mid t_k \leq t\}. \tag{1a}$$

Observe that due to the switch from $u(t|t_k)$ to $u(t|t_{k+1})$ at $t = t_{k+1}$ the considered closed-loop dynamics under the sampled-data NMPC feedback are *hybrid*, i.e. subject to switches.

Hence the state trajectory generated by the NMPC scheme can formally be written as

$$x^{MPC}(t) \doteq x(t, x_0; u^{MPC}(\cdot)). \tag{1b}$$

For the sake of readability and whenever no confusion can arise, we simply drop the superscript $(\cdot)^{MPC}$.

Before analysing specific NMPC schemes we detail our standing technical assumptions.

As mentioned, we consider a nominal setting—i.e. the model f is assumed to be an exact representation of the considered plant/system and exact state feedback is available at all sampling instant $k \in \mathbb{N}$. However, from a mathematics point of view it can be tricky to verify existence of optimal solutions to OCPs as one has to consider a suitable class of input functions and convexity assumptions on ℓ, cf. Part I.[5] Hence, in the MPC analysis we assume that, whenever we need to solve an OCP, a (globally) optimal solution exists and is attained. Moreover, we suppose that the following conditions are satisfied.

[5] E.B. Lee and L. Markus. *Foundations of Optimal Control Theory*. The SIAM Series in Applied Mathematics. John Wiley & Sons New York, London, Sydney, 1967, XYZ.

Standing Assumptions (For stabilization problems)**.**

A1 *(Steady state): Given $\bar{x}$, there exists $\bar{u} \in \mathbb{U}$, such that $0 = f(\bar{x}, \bar{u})$ and $(\bar{x}, \bar{u}) \in \operatorname{int}(\mathbb{X} \times \mathbb{U})$. W.l.o.g. we suppose $(\bar{x}, \bar{u}) = (0, 0)$.*

A2 *(Lower boundedness of ℓ): There exists a class-$\mathcal{K}$ function α : $\mathbb{R}_0^+ \to \mathbb{R}_0^+$, such that for all $(x, \bar{u}) \in \mathbb{X} \times \mathbb{U}$,*

$$\ell(x, u) \geq \alpha(\|x - \bar{x}\|), \quad \text{and} \quad \ell(\bar{x}, \bar{u}) = \ell(0, 0) = 0$$

holds.

A3 *(Absolute continuity of ODE solutions): For all $x_0 \in \mathbb{X}$, and any $u(\cdot) \in \hat{\mathcal{C}}([0, T], \mathbb{U})$, the solution $x(\cdot, x_0, u(\cdot))$ exists on $[0, T]$ and is absolutely continuous.*

Definition 1 (Class-$\mathcal{K}$ function)**.**

- *A scalar function $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $\mathcal{K}$, if it is continuous, strictly increasing, and $\alpha(0) = 0$.*
- *$\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $\mathcal{K}_\infty$, if $\alpha \in \mathcal{K}$ and if it is radially unbounded, i.e. $\alpha(s) \to \infty$ as $s \to \infty$.*

Definition 2 (Absolute continuity)**.** *A trajectory $x(\cdot)$ is said to be absolutely continuous on $[t_0, t_1]$, iff*

- *$x(t)$ is almost everywhere differentiable w.r.t. t,*
- *$\dot{x}(t)$ is Lebesgue integrable, and*
- *for all $t \in [0, T]$*

$$x(t) = x(0) + \int_0^t \dot{x}(\tau) \, \mathrm{d}\tau$$

holds.

Assumption **A1** ensures that the considered set-point stabilization problem is well-posed, i.e. it guarantees that $\bar{x}$ is indeed a controlled equilibrium of f. Assumption **A2** implies that whenever one computes controls u such that the corresponding state x minimizes the cost functional—i.e. it minimizes the integral of ℓ—one actually forces the state trajectory to move towards $\bar{x}$. Put differently, **A2** ensures that the stage cost ℓ reflects the considered stabilization problem. It will become evident later that in many stability proofs Assumption **A2** plays a fundamental role.

Finally, Assumption **A3** is of mostly technically nature as it enables certain steps in later proofs. In essence, it gives that despite potential jumps of the input generated by the NMPC controller, the state trajectory $x(\cdot)$ admits a bounded time derivative almost everywhere.

With slight lack of precision, one can say that an absolutely continuous function can be computed by integrating its time derivative.

Infinite-Horizon NMPC

We begin our investigations on stability of NMPC with the infinite horizon case ($T = \infty$) combined with instantaneous recalculation ($\delta \doteq t_{k+1} - t_k = 0$). While OCPs with infinite horizon are typically not easy to solve numerically—yet alone in instantaneous settings—the analysis of this scheme will provide helpful insights.

Specifically, we consider NMPC based on

$$\begin{aligned}
&\min_{u(\cdot|t_k)} \int_t^{t+\infty} \ell(x(\tau|t), u(\tau|t)) \, \mathrm{d}\tau \\
&\text{subject to } \forall \tau \in [t, t+\infty] : \qquad\qquad (OCP_\infty(x(t))) \\
&\frac{\mathrm{d}x(\tau|t)}{\mathrm{d}\tau} = f(x(\tau|t), u(\tau|t)) \\
&x(t|t) = x(t) \\
&x(\tau|t) \in \mathbb{X}, \ u(\tau|t) \in \mathbb{U}.
\end{aligned}$$

To simplify distinguishing different NMPC schemes from one another, we encode decisive information in the OCP labeling. The label $OCP_\infty(x(t))$ indicates the infinite horizon (subscript $\cdot_\infty$) and instantaneous recalculation (use of the argument t instead of t_k.

In further schemes, and whenever applicable, we will also indicate terminal constraints etc. (superscripts). Note that also the notation style for optimal value functions follows the same conventions.

Due to the instantaneous recalculation, we have dropped the subscript k and write t instead of t_k in the OCP above. Recall that in Part I we have introduced the *optimal value function* of an OCP. In case of $OCP_\infty(x(t))$ the optimal value function $V_\infty : \mathbb{X} \subseteq \mathbb{R}^{n_x} \to \mathbb{R}$ is given by

$$V_\infty(x(t)) \doteq \int_t^{t+\infty} \ell(x^\star(\tau|t), u^\star(\tau|t))\, \mathrm{d}\tau. \qquad (2)$$

The following stability result was presented by Jadbabaie et al.[6]

[6] A. Jadbabaie, J. Yu, and J. Hauser. "Unconstrained receding-horizon control of nonlinear systems". In: *IEEE Trans. Automat. Contr.* 46.5 (2001), pp. 776–783.

Theorem 1 (Stability of instantaneous infinite-horizon NMPC). *Let Assumptions **A1**–**A3** hold and suppose that,*

(i) for all $x \in \mathbb{X}_0$, the value function $V_\infty(x)$ is continuously differentiable and

$$\beta_1(\|x\|) \leq V_\infty(x) \leq \beta_2(\|x\|), \qquad \beta_{1,2} \in \mathcal{K}_\infty$$

Then, the NMPC scheme based on $OCP_\infty(x(t))$ achieves local asymptotic stability of (Σ_{cl}) at $x = 0$. The region of attraction is given by the set of initial conditions for which (i) holds.

Proof. A general question which arises in the the analysis of NMPC scheme is the issue of recursive feasibility, i.e. the question of whether or not the feasibility of $OCP_\infty(x(t))$ at k implies its feasibility at $k+1$.

Definition 3 (Recursive feasibility). *An NMPC scheme based on $OCP(x(t_k))$ is said to be recursively feasible if the feasibility of $OCP(x(t_k))$ at sampling instant k implies its feasibility at $k+1$.*

Consider the optimal open-loop input $u(\cdot|t_0)$ computed at time instant t_0 which is valid for all $t \in [t_0, \infty)$. Since $OCP_\infty(x(t))$ considers an infinite prediction horizon, Bellman's principle of optimality implies that the truncation of $u(\cdot|t_0)$ to $[t_0 + \delta, \infty)$ constitutes an optimal open-loop input for $OCP_\infty(x(t))$ with initial condition $x(t_0 + \delta, x(t_0), u(\cdot|t_0))$. Hence feasibility of $OCP_\infty(x(t))$ at instant t_0 implies its feasibility for all $t \geq t_0$.

Without loss of generality we write V_∞ as

$$V_\infty(x(t)) \doteq \int_t^{\infty} \ell(x^\star(\tau|t), u^\star(\tau|t))\, \mathrm{d}\tau.$$

Using Lemma 6 we arrive at

$$\begin{aligned} \frac{\mathrm{d}V_\infty}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_t^{\infty} \ell(x^\star(\tau|t), u^\star(\tau|t))\, \mathrm{d}\tau \\ = -\frac{\mathrm{d}}{\mathrm{d}t}\int_\infty^{t} \ell(x^\star(\tau|t), u^\star(\tau|t))\, \mathrm{d}\tau \\ = -\ell(x^\star(t|t), u^\star(t|t)). \end{aligned}$$

Using Assumption **A2** we obtain that

$$\frac{\mathrm{d}V_\infty}{\mathrm{d}t} = \nabla V_\infty^\top \cdot f(x,u) = -\ell(x,u) \overset{\mathbf{A2}}{\leq} -\alpha(\|x - \bar{x}|).$$

In view of Lemma 5 the assertion of local asymptotic stability of (Σ_{cl}) at $x = 0$ follows. $\square$

Some comments on the previous derivations are in order.

- The proof above already establishes a structure that we will revisit a couple of times: First verify recursive feasibility; second apply Lyapunov-like arguments to establish convergence or stability. Notice that without recursive feasibility, applying any NMPC scheme would be hazardous as at any instant t_k the optimization could potentially break down due to loss of feasibility.

- While the considered instantaneous infinite horizon scheme is clearly not directly applicable, it is worth to be noted that the optimal value function V_∞ turned out to be suitable Lyapunov function for the closed-loop system. Indeed the vast majority of stability results for NMPC will resort to use the value function as the standard candidate Lyapunov function.
- Moreover, we remark that in the presence of state constraints the differentiability assumption on V_∞ is quite strong. Hence we will aim to avoid it for further schemes.

normalToDo: ref to sufficient conditions for differentiable V

Finally, note that the consideration of an infinite-horizon in NMPC is mostly conceptual. Indeed, the solution of infinite-horizon OCPs is in general challenging. While, under certain conditions[7] one can approximate the solution numerically, it is clear that from an engineering and application perspective that one is tempted to consider finite horizons $T < \infty$ and a non-vanishing sampling period δ.

[7] Peter Kunkel and Oskar von dem Hagen. “Numerical solution of infinite-horizon optimal-control problems”. In: *Computational Economics* 16.3 (2000), pp. 189–205.

NMPC with Terminal Constraints

The insights obtained in the previous section motivate to analyze how one may avoid the infinite horizon in $OCP_\infty(x(t))$.

From Infinite to Finite Horizons in NMPC

To this end, consider the objective function from $OCP_\infty(x(t))$, where, for the sake of readability, we drop the notion of predicted input and state trajectories, and split the time horizon

$$\int_0^\infty \ell(x(\tau),u(\tau))\, \mathrm{d}\tau = \int_0^t \ell(x(\tau),u(\tau))\, \mathrm{d}\tau + \underbrace{\int_t^\infty \ell(x(\tau),u(\tau))\, \mathrm{d}\tau}_{\text{cost-to-go}}.$$

The second integral is referred to as *cost-to-go*.[8]

[8] A.E. Bryson and Y.-C. Ho. *Applied Optimal Control*. Ginn and Company, Waltham, Massachusetts, 1969.

Consider a function $V_\mathrm{f} : \mathbb{X} \to \mathbb{R}_0^+$ which satisfies

$$\underbrace{\nabla V_\mathrm{f}^\top \cdot f(x,u)}_{=\dot{V}_\mathrm{f}(x)} + \ell(x,u) \le 0. \tag{3}$$

Integrating the inequality from t_1 to t_2 yields

$$V_\mathrm{f}(x(t))\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \ell(x(\tau),u(\tau))\, \mathrm{d}\tau \le 0.$$

Setting $t_1 = t$ and taking the limit $t_2 \to \infty$ we obtain

$$\int_t^\infty \ell(x(\tau),u(\tau))\, \mathrm{d}\tau \le V_\mathrm{f}(x(t)) - \lim_{t_2\to\infty} V_\mathrm{f}(x(t_2)).$$

As we consider the problem of stabilizing the set-point $\bar{x} = 0$, it is reasonable to suppose $\lim\limits_{t_2\to\infty} V_\mathrm{f}(x(t_2)) = 0$. Due to the lower boundedness of ℓ (Assumption **A2**) this implies that, eventually, the set-point $\bar{x} = 0$ is attained ($V_\mathrm{f}(0) = 0$).

Moreover, we obtain from these considerations that

$$\int_t^\infty \ell(x(\tau),u(\tau))\, \mathrm{d}\tau \le V_\mathrm{f}(x(t)), \tag{4}$$

i.e. the function V_f is an upper bound on the cost-to-go which is tight at $x = 0$, i.e. $V_\mathrm{f}(x = 0) = 0$. Likewise, one may say that V_f is an upper bound on the infinite-horizon value function $V_\infty(x(t))$.

Observe that considering $u = K(x)$ and Assumption **A2**, the differential inequality (3) can also be interpreted as a non-linear Lyapunov inequality, cf. Lemma 5. A point to be addressed later.

The main message of this derivation is that, modulo technical assumptions introduced later, the differential inequality (3) means that V_f is an upper bound on the cost-to-go (4).

Quasi-Infinite Horizon NMPC

The derivation above motivates to replace the infinite-horizon scheme from $OCP_\infty(x(t))$ with a scheme wherein a finite prediction horizon $T \in \mathbb{R}$ and a terminal penalty V_f (= upper bound on the cost-to-go) are used.

The next OCP specifies such an NMPC scheme

$$\begin{aligned} \min_{u(\cdot|t_k)} & \int_{t_k}^{t_k+T} \ell(x(\tau|t_k), u(\tau|t_k))\, \mathrm{d}\tau + V_\mathrm{f}(x(t_k+T|t_k)) \\ & \text{subject to } \forall \tau \in [t_k, t_k+T]: \qquad\qquad (OCP_T^{\mathbb{X}_\mathrm{f}}(x(t_k))) \\ & \frac{\mathrm{d}x(\tau|t_k)}{\mathrm{d}\tau} = f(x(\tau|t_k), u(\tau|t_k)) \\ & x(t|t_k) = x(t_k) \\ & x(\tau|t_k) \in \mathbb{X},\ u(\tau|t_k) \in \mathbb{U}, \\ & x(t_k+T|t_k) \in \mathbb{X}_\mathrm{f}. \end{aligned}$$

The rationale of considering the terminal penalty V_f is for it to act as an upper bound on the cost-to-go. As such, such bounds are not easy to attain, i.e. one will usually only be able to find local bounds. Hence the terminal constraint $x(t_k+T|t_k) \in \mathbb{X}_\mathrm{f} \subseteq \mathbb{X}$ can be understood as a subset of the state constraints on which V_f is indeed an upper bound on the infinite-horizon value function V_∞.

Moreover, it is worth noting that in real-world applications $OCP_T^{\mathbb{X}_\mathrm{f}}(x(t_k))$ is used with a non-vanishing recalculation time, i.e.

$$\delta \doteq t_{k+1} - t_k > 0.$$

Put differently, $OCP_T^{\mathbb{X}_\mathrm{f}}(x(t_k))$ specifies a *sampled-data NMPC scheme*, which will require some extra care in the stability analysis. The next theorem summarizes the convergence properties.

Theorem 2 (Convergence of sampled-data NMPC)**.**
*Let Assumptions **A1**–**A3** hold and suppose that there exist V_f, $\mathbb{X}_\mathrm{f} \subseteq \mathbb{X}$ ($0 \in \mathbb{X}_\mathrm{f}$), and a feedback $k : \mathbb{X}_\mathrm{f} \to \mathbb{U}$ such that*

(i) $V_\mathrm{f} : \mathbb{X}_\mathrm{f} \to \mathbb{R}_0^+$ *is positive semi-definite and* $V_\mathrm{f}(0) = 0$,

(ii) for all $x \in \mathbb{X}_\mathrm{f}$:

$$\nabla V_\mathrm{f}^\top \cdot f(x, k(x)) + \ell(x, k(x)) \leq 0, \tag{5}$$

and for all $t \in [0, \delta]:\ x(t, x, k(x)) \in \mathbb{X}_\mathrm{f}$,

(iii) $OCP_T^{\mathbb{X}_\mathrm{f}}(x(t_k))$ *is feasible at* $k = 0$.

Then,

- $OCP_T^{\mathbb{X}_\mathrm{f}}(x(t_k))$ *is recursively feasible,*
- *the NMPC scheme based on* $OCP_T^{\mathbb{X}_\mathrm{f}}(x(t_k))$ *achieves* $\lim_{t\to\infty} \|x(t)\| = 0$ *for* (Σ_{cl})*, and*
- *the region of attraction is given by the set of initial conditions for which (iii) holds.*

Proof of Theorem 2

In the literature Theorem 2 appeared in numerous papers in different variants. Hence we provide a detailed proof. Actually, most MPC stability proofs based on terminal constraints follow a certain blueprint like structure:

We comment on bibliographic references at the end of this chapter.

- Step 1 – Recursive feasibility of the sequence $OCP_T^{\mathbb{X}_f}(x(t_k))$ for all sampling instants t_k, $k \in \mathbb{N}$.
- Step 2 – Decrease of the value function $V_T^{\mathbb{X}_f}(x(t))$ in between two sampling instants t_{k+1} and t_k.
- Step 3 – Decrease of the value function $V_T^{\mathbb{X}_f}(x(t))$ from one sampling instant to the next.
- Step 4 – Consider the value function $V_T^{\mathbb{X}_f}(x(t))$ as a *Lyapunov*-like function of the closed-loop system.

Step 1 – Recursive Feasibility

The main idea to establish recursive feasibility is to construct a suboptimal but feasible input for $OCP_T^{\mathbb{X}_f}(x(t_k))$ at time instant $k+1$ from the solution to $OCP_T^{\mathbb{X}_f}(x(t_k))$.

To this end consider

$$\tilde{u}_{k+1}(\tau) = \begin{cases} u^\star(\tau|t_k) & \tau \in [t_{k+1}, t_k + T) \\ k(x(\tau)) & \tau \in [t_k + T, t_{k+1} + T] \end{cases} \tag{6}$$

which is composed of two parts: on $[t_{k+1}, t_k + T)$ the *old* optimal input $u^\star(\tau|t_k)$ is used, while on $[t_k + T, t_{k+1} + T]$ the terminal control law $k(x(\tau)$ is considered. As the predicted optimal input $u^\star(\tau|t_k)$ has to satisfy the terminal constraint $x(t_k + T|t_k) \in \mathbb{X}_f$, it follows that $x_f \doteq x^\star(t_k + T|t_k) \in \mathbb{X}_f$. Observe that appending the optimal input by the terminal control law $k : x \mapsto u$ implies $x(t_{k+1} + T, x_f, k(x)) \in \mathbb{X}_f$. Hence the construction (6) does not jeopardize feasibility, see also the sketch in.

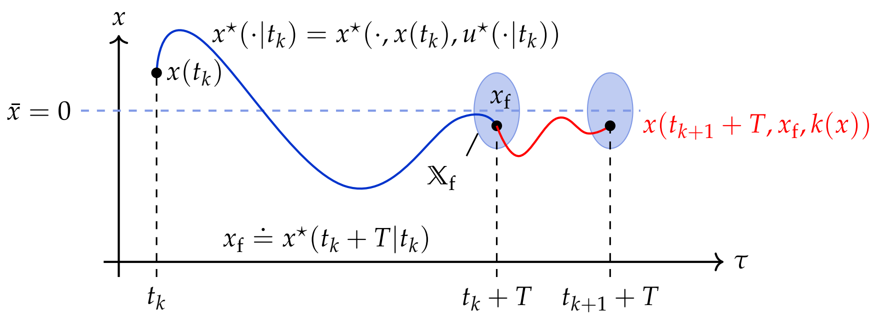


Figure 5: Construction of feasible initial guesses.

Step 2 – Decrease of the Value Function on $[t_k, t_{k+1})$

Consider the value function

$$V_T^{\mathbb{X}_f}(x(t_k)) \doteq \int_{t_k}^{t_k+T} \ell(x^\star(\tau|t_k), u^\star(\tau|t_k))\, \mathrm{d}\tau + V_f(x^\star(t_k + T|t_k)).$$

As we consider the nominal case (no plant-model mismatch), we have

$$x(t) = x^\star(t|t_k), \qquad \forall t \in [t_k, t_{k+1}].$$

Due to Assumption **A2** (lower boundedness of ℓ) we have that

$$V_T^{\mathbb{X}_\mathrm{f}}(x(t)) \doteq V_T^{\mathbb{X}_\mathrm{f}}(x(t_k)) - \int_{t_k}^{t} \ell(x^\star(\tau|t_k), u^\star(\tau|t_k))\,\mathrm{d}\tau$$

is continuous and decreasing for all $t \in [t_k, t_{k+1})$.

Step 3 – Decrease of the Value Function from t_k to t_{k+1}

Lemma 1. *Let the conditions of Theorem 2 hold. For all $k \in \mathbb{N}$, it holds that*

$$V_T^{\mathbb{X}_\mathrm{f}}(x(t_{k+1})) - V_T^{\mathbb{X}_\mathrm{f}}(x(t_k)) \le 0.$$

Proof. We use the feasible but suboptimal input from (6). Hence we have that

$$V_T^{\mathbb{X}_\mathrm{f}}(x(t_{k+1})) - V_T^{\mathbb{X}_\mathrm{f}}(x(t_k)) \le J(x(t_{k+1}), \tilde{u}_{k+1}) - V_T^{\mathbb{X}_\mathrm{f}}(x(t_k)).$$

As $J(x(t_{k+1}), \tilde{u}_{k+1}(\cdot)$ is the evaluation of the objective of ($OCP_T^{\mathbb{X}_\mathrm{f}}(x(t_k))$) for the input $\tilde{u}_{k+1}$ from (6). It reads

$$J(x(t_{k+1}), \tilde{u}_{k+1}) = \int_{t_{k+1}}^{t_{k+1}+T} \ell(\tilde{x}_{k+1}(\tau), \tilde{u}_{k+1}(\tau))\,\mathrm{d}\tau + V_\mathrm{f}(\tilde{x}_{k+1}(t_{k+1}+T))$$

where the short hand $\tilde{x}_{k+1}(\tau) \doteq x(\tau, x(t_k), \tilde{u}_{k+1})$ is used. Due to the specific construction (6) and due the consideration of the nominal case, we have that

$$\tilde{x}_{k+1}(\tau) = x(\tau, x(t_k), \tilde{u}_{k+1}) \equiv x^\star(\tau|t_k) \text{ for all } \tau \in [t_{k+1}, t_k + T].$$

This allows writing $J(x(t_{k+1}), \tilde{u}_{k+1}) - V_T^{\mathbb{X}_\mathrm{f}}(x(t_k))$ as

$$J(x(t_{k+1}), \tilde{u}_{k+1}) - V_T^{\mathbb{X}_\mathrm{f}}(x(t_k)) = \int_{t_{k+1}}^{t_k+T} \ell(x^\star(\tau|t_k), u^\star(\tau|t_k))\,\mathrm{d}\tau +$$
$$\int_{t_k+T}^{t_{k+1}+T} \ell(\tilde{x}_{k+1}(\tau), \tilde{u}_{k+1}(\tau))\,\mathrm{d}\tau + V_\mathrm{f}(\tilde{x}_{k+1}(t_{k+1}+T)) -$$
$$\int_{t_k}^{t_{k+1}} \ell(x^\star(\tau|t_k), u^\star(\tau|t_k))\,\mathrm{d}\tau - \int_{t_{k+1}}^{t_k+T} \ell(x^\star(\tau|t_k), u^\star(\tau|t_k))\,\mathrm{d}\tau - V_\mathrm{f}(x^\star(t_k+T|t_k)).$$

Observe that the color coding matches the one of Figure 5.

As the blue integrals cancel each other we arrive at

$$J(x(t_{k+1}), \tilde{u}_{k+1}) - V_T^{\mathbb{X}_\mathrm{f}}(x(t_k)) \le \int_{t_k+T}^{t_{k+1}+T} \ell(\tilde{x}_{k+1}(\tau), \tilde{u}_{k+1}(\tau))\,\mathrm{d}\tau + V_\mathrm{f}(\tilde{x}_{k+1}(t_{k+1}+T)) - V_\mathrm{f}(x^\star(t_k+T|t_k)),$$

where dropping $\int_{t_k}^{t_{k+1}} \ell(x^\star(\tau|t_k), u^\star(\tau|t_k))\,\mathrm{d}\tau \ge 0$ yields the inequality.

Finally, recall $x_\mathrm{f} \doteq x^\star(t_k+T|t_k)$. Hence integrating (5) from $t_k + T$ to $t_{k+1} + T$ gives

$$V_\mathrm{f}(\tilde{x}_{k+1(\tau)})\Big|_{\tau=t_k+T}^{\tau=t_{k+1}+T} + \int_{t_k+T}^{t_{k+1}+T} \ell(\tilde{x}_{k+1}(\tau), \tilde{u}_{k+1}(\tau))\,\mathrm{d}\tau \le 0.$$

As $\tilde{x}_{k+1}(t_k+T) = x^\star(t_k+T|t_k)$ we have

$$\int_{t_{k+T}}^{t_{k+1}+T} \ell(\tilde{x}_{k+1}(\tau), \tilde{u}_{k+1}(\tau))\, \mathrm{d}\tau + V_\mathrm{f}(\tilde{x}_{k+1}(t_{k+1}+T)) - V_\mathrm{f}(x^\star(t_k+T|t_k)) =$$
$$V_\mathrm{f}(\tilde{x}_{k+1(\tau)})\Big|_{\tau=t_k+T}^{\tau=t_{k+1}+T} + \int_{t_k+T}^{t_{k+1}+T} \ell(\tilde{x}_{k+1}(\tau), \tilde{u}_{k+1}(\tau)) \leq 0.$$

Hence we conclude that $J(x(t_{k+1}), \tilde{u}_{k+1}) - V_T^{\mathbb{X}_\mathrm{f}}(x(t_k)) \leq 0$. □

Step 4 – Convergence

So far we have shown the following properties, see also Figure 6:

- The optimal value function $V_T^{\mathbb{X}_\mathrm{f}}(x(t_k))$ decreases in-between two sampling instants (Step 2), i.e. the decay on $[t_k, t_{k+1})$.
- It also decreases from t_k to t_{k+1} (Step 3).

It remains to show that this decay can be bounded from above in an appropriate manner, cf. Figure 6.

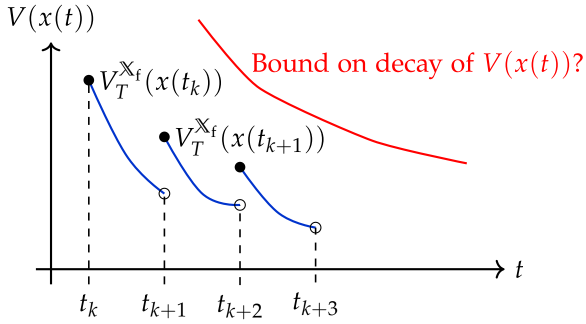


Figure 6: Sketch of results of Step 2 and Step 3, outlook on Step 4.

To this end, let the closed-loop trajectory generated by the MPC scheme ($OCP_T^{\mathbb{X}_\mathrm{f}}(x(t_k))$) be $x(\cdot)$. Then

$$V^{MPC}(x(t)) \doteq V_T^{\mathbb{X}_\mathrm{f}}(x(t_k)) - \int_{t_k}^{t} \ell(x^\star(\tau|t_k), u^\star(\tau|t_k))\, \mathrm{d}\tau, \quad k = \max\{k \in \mathbb{N} \mid t_k \leq t\}$$

is our candidate Lyapunov-like function. Due to the exclusion of plant-model mismatch, we obtain for $t = t_N$

Actually verifying that $V(x(t))$ is a Lyapunov function requires additional assumptions, cf. Theorem 3.

$$V^{MPC}(x(t_N)) = V_T^{\mathbb{X}_\mathrm{f}}(x(t_0)) - \sum_{k=0}^{N} \int_{t_k}^{t_{k+1}} \ell(x^\star(\tau|t_k), u^\star(\tau|t_k))\, \mathrm{d}\tau$$
$$= V_T^{\mathbb{X}_\mathrm{f}}(x(t_0)) - \int_{t_0}^{t_N} \ell(x(\tau), u(\tau))\, \mathrm{d}\tau. \tag{7}$$

Feasibility of $OCP_T^{\mathbb{X}_\mathrm{f}}(x(t_k))$ at time t_0 implies that $V_T^{\mathbb{X}_\mathrm{f}}(x(t_0)) < \infty$. Moreover, Assumption **A2** and the condition on the terminal penalty $V_\mathrm{f}(x(t_k+T|t_k))$ in Theorem 2 imply that $V(x(t)) > 0$ for all $x(t) \neq 0$. Hence (7) implies that

$$\lim_{t\to\infty} \int_{t_0}^{t} \ell(x(\tau), u(\tau))\, \mathrm{d}\tau < \underbrace{V_T^{\mathbb{X}_\mathrm{f}}(x(t_0))}_{=V^{MPC}(x(t_0))} < \infty.$$

Assumption A2 gives

$$\lim_{t\to\infty}\int_{t_0}^{t}\alpha(\|x(\tau)\|)\,\mathrm{d}\tau \le \lim_{t\to\infty}\int_{t_0}^{t}\ell(x(\tau),u(\tau))\,\mathrm{d}\tau < \infty.$$

Finally, invoking Barbalat´s Lemma directly gives

$$\lim_{t\to\infty}\|x(t)\| = 0.$$

This completes the proof of Theorem 2. □

A few comments are in order. Recall that the classic definition of Lyapunov stability requires

$$\forall \varepsilon > 0\ \exists \delta > 0 \mid \|x_0\| < \delta \quad \Rightarrow \quad \|x(t,t_0,x_0)\| < \varepsilon \quad \forall t \ge t_0 \ge 0, \quad (8)$$

where $x(\cdot,t_0,x_0)$ refers to the closed-loop system trajectory (Σ_{cl}). Note, however, that in the proof above we only establish asymptotic convergence but not the $\varepsilon - \delta$ argument (8). The reason for this slightly weaker result is that in Step 4 we invoked Barbalat´s Lemma. Hence it is justified to ask for how to close the gap to asymptotic (Lyapunov) stability?

Yet, at the same time, the proof above also has a number of advantages: It does not require local properties like stabilizability of the Jacobian linearization at the required set-point. Moreover, the terminal feedback law $k : \mathbb{X}_\mathrm{f} \mapsto \mathbb{U}$ does not need to be continuous. One should also note that the terminal feedback law $u = k(x)$ is actually never applied to the system. Instead, its purpose is to enable the construction of a feasible initial guess in Step 1. This allows to apply the above proof also to systems which cannot be stabilized by continuous feedback. Indeed in the version above we have followed the proof given by Fontes.[9]

Besides the issue of Lyapunov stability, another imperative question is how one shall compute terminal regions and penalties. We will comment on both items next.

Lemma 2 (Barbalat's Lemma)**.** *Let $M : \mathbb{R}^{n_x} \to \mathbb{R}_0^+$ be a continuous positive definite function and $x(\cdot)$ be an absolutely continuous function on $\mathbb{R}$. If $x(\cdot) \in \mathcal{L}^\infty$, $\dot{x}(\cdot) \in \mathcal{L}^\infty$ and*

$$\lim_{t\to\infty}\int_0^t M(x(\tau))\,\mathrm{d}\tau < \infty$$

then $\lim_{t\to\infty}\|x(t)\| = 0$.

The proof of Lemma 2 can be found in H. Michalska and R.B. Vinter. "Nonlinear stabilization using discontinuous moving-horizon control". In: *IMA Journal of Mathematical Control and Information* 11.4 (1994), pp. 321–340.

See Definition 7 in the Appendix for the full statement of Lyapunov stability.

Moreover, observe the structural similarity between the crucial stability condition (7) and the decay requirement for Lyapunov functions. The classic requirements for Lyapunov functions are stated in Lemma 5 in the Appendix.

[9] F. Fontes. "A General Framework to Design Stabilizing Nonlinear Model Predictive Controllers". In: *Sys. Contr. Lett.* 42.2 (2001), pp. 127–143.

Closing the Stability Gap

To close the gap from the asymptotic convergence to asymptotic stability, we consider the following assumption. Consider the set of feasible initial conditions for $OCP_T^{\mathbb{X}_\mathrm{f}}(x(t_k))$

$$\Omega_T^{\mathbb{X}_\mathrm{f}} = \left\{ x \in \mathbb{R}^{n_x} \mid V_T^{\mathbb{X}_\mathrm{f}}(x) < \infty \right\}. \quad (9)$$

Assumption (Bounds on the value function)**.**

A4 *There exist $\beta_1, \beta_2 \in \mathcal{K}_\infty$ and a set $\mathcal{D}$ such that, for all $x \in \mathcal{D} \subseteq \Omega_T^{\mathbb{X}_\mathrm{f}}$, it holds that*

$$\beta_1(\|x\|) \le V_T^{\mathbb{X}_\mathrm{f}}(x) \le \beta_2(\|x\|). \quad (10)$$

The upper bound $V_T^{\mathbb{X}_f}(x) \leq \beta_2(\|x\|)$ can be regarded as a (weak) reachability condition as it requires that the cost of reaching the terminal set $\mathbb{X}_f$ is not excessive. The lower bound $\beta_1(\|x\|) \leq V_T^{\mathbb{X}_f}(x)$ can be understood as a continuity requirement on $V_T^{\mathbb{X}_f}$. This assumption enables the next result which addresses the stability gap.

If the terminal penalty $V_f(x) \leq \beta_2(\|x\|)$ and the state constraints are compact, one can also show that on $\Omega_T^{\mathbb{X}_f}$ the upper bound exists.

Notice that in a discrete-time setting the lower bound on $V_T^{\mathbb{X}_f}$ is directly implied by Assumption A2. In contrast, in a continuous-time setting it is not easily derived without further technial assumptions.

Theorem 3 (Closed-loop stability of sampled-data NMPC).
Let Assumptions A1–A4 hold and suppose that there exist V_f, $\mathbb{X}_f \subseteq \mathbb{X}$ ($0 \in \mathbb{X}_f$), and a feedback $k : \mathbb{X}_f \to \mathbb{U}$ such that

(i) $V_f : \mathbb{X}_f \to \mathbb{R}_0^+$ is positive semi-definite and $V_f(0) = 0$,

(ii) for all $x \in \mathbb{X}_f$:

$$\nabla V_f^\top \cdot f(x, k(x)) + \ell(x, k(x)) \leq 0, \tag{11}$$

and for all $t \in [0, \delta]$: $x(t, x, k(x)) \in \mathbb{X}_f$,

(iii) $OCP_T^{\mathbb{X}_f}(x(t_k))$ is feasible at $k = 0$.

Then,

- *$OCP_T^{\mathbb{X}_f}(x(t_k))$ is recursively feasible,*
- *the NMPC scheme based on $OCP_T^{\mathbb{X}_f}(x(t_k))$ achieves local asymptotic stability of (Σ_{cl}) at $x = 0$, and*
- *the region of attraction is given by $\mathcal{D} \subseteq \Omega_T^{\mathbb{X}_f}$.*

Proof. Observe that the conditions of the proposition are a strengthened version of the ones from Theorem 2. Hence the proof of the recursive feasibility statement and the proof of asymptotic convergence follow the proof of Theorem 2. It remains to establish the $\varepsilon - \delta$ argument (8).

Given ε, choose $r \in (0, \varepsilon]$ such that

$$\mathcal{B}_r(0) = \{x \in \mathbb{R}^{n_x} \mid \|x\| \leq r\} \subset \mathcal{D}$$

with $\mathcal{D}$ from Assumption A4. Let $\alpha = \min_{x \in \partial\mathcal{B}_r} V^{MPC}(x)$ with V^{MPC} from (7). Note that Assumption A4 gives that $\alpha > 0$. Consider $\gamma \in (0, \alpha)$ and let

$$\Omega_\gamma = \{x \in \mathcal{B}_r(0) \mid V^{MPC}(x) \leq \gamma\}.$$

Then Ω_γ is in the interior of $\mathcal{B}_r(0)$, since $\gamma < \alpha = \min_{x \in \partial\mathcal{B}_r} V(x)$. Note that (7) implies

$$V^{MPC}(x(t)) \leq \underbrace{V_T^{\mathbb{X}_f}(x(t_0))}_{V^{MPC}(x(t_0))} \leq \gamma.$$

In turn this shows that any solution starting in Ω_γ stays in Ω_γ, i.e. the set is positive invariant. Moreover, Ω_γ is compact and any solution with $x(0) \in \Omega_\gamma$ is unique and exists for all times. Note that the upper bound in (10) implies that there exists $\delta > 0$ such that

$$\|x\| \leq \delta \quad \Rightarrow \quad V^{MPC}(x) \leq \gamma.$$

Then $\mathcal{B}_\delta(0) \subset \Omega_\gamma \subset \mathcal{B}_r(0)$, cf. Figure 7.

Due to the positive invariance of Ω_γ we have that

$$\|x(t_0)\| \leq \delta \Rightarrow \|x(t)\| < r \leq \varepsilon, \quad \forall t \geq t_0.$$

This concludes the proof. $\square$

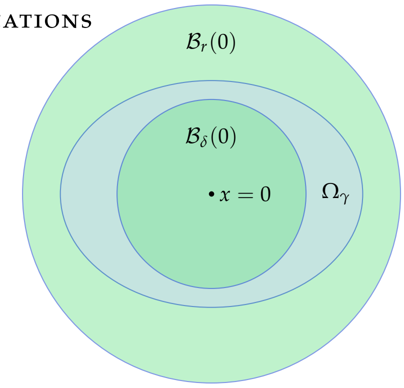


Figure 7: Sketch of $\mathcal{B}_\delta(0), \Omega_\gamma$ and $\mathcal{B}_r(0)$.

The above result has addressed the stability gap. Yet, it did not shed light on the question of how to choose the terminal region and the terminal penalty. Next, we present a classical result which certifies a particularly easy choice for $\mathbb{X}_\mathrm{f}$ and V_f. To this end, consider the variant of $OCP_T^{\mathbb{X}_\mathrm{f}}(x(t_k))$ with $\mathbb{X}_\mathrm{f} = \{0\}, V_\mathrm{f}(x) = 0$, i.e.

$$\begin{aligned}
&\min_{u(\cdot|t_k)} \int_{t_k}^{t_k+T} \ell(x(\tau|t_k), u(\tau|t_k))\,\mathrm{d}\tau \\
&\quad\text{subject to } \forall \tau \in [t_k, t_k+T]: \qquad\qquad (OCP_T^0(x(t_k))) \\
&\quad \frac{\mathrm{d}x(\tau|t_k)}{\mathrm{d}\tau} = f(x(\tau|t_k), u(\tau|t_k)) \\
&\qquad x(t|t_k) = x(t_k) \\
&\qquad x(\tau|t_k) \in \mathbb{X},\ u(\tau|t_k) \in \mathbb{U}, \\
&x(t_k+T|t_k) = 0,
\end{aligned}$$

and let the counterpart to $\Omega_T^{\mathbb{X}_\mathrm{f}}$ from (9), i.e. the set of feasible initial conditions to $OCP_T^{\mathbb{X}_\mathrm{f}}(x(t_k))$, be denoted as

$$\Omega_T^0 = \{x \in \mathbb{R}^{n_x} \mid V_T^0(x) < \infty\}, \tag{12}$$

where $V_T^0(x)$ is the optimal value function of $OCP_T^0(x(t_k))$. This setting gives rise to the following corollary to Theorems 2 and 3.

Corollary 1 (Stability with zero-terminal constraint)**.**

*Let **A1**–**A4** hold and suppose that the terminal penalty is $V_\mathrm{f}(x) = 0$, the terminal region is $\mathbb{X}_\mathrm{f} = \{0\}$, and*

(iii) $OCP_T^0(x(t_k))$ is feasible at $k = 0$.

Then,

- *$OCP_T^0(x(t_k))$ is recursively feasible,*
- *the NMPC scheme based on $OCP_T^0(x(t_k))$ achieves local asymptotic stability of (Σ_{cl}) at $x = 0$, and*
- *the region of attraction is given by $\mathcal{D} \subseteq \Omega_T^0$.*

This corollary illustrates the easiest route—from a design point of view—to stability guarantees in NMPC as the choice $\mathbb{X}_\mathrm{f} = \{0\}, V_\mathrm{f}(x) = 0$ does not require any further computations. However, it also comes at a cost as the prediction horizon has to be sufficiently long as otherwise $OCP_T^0(x(t_k))$ will be infeasible. Indeed any feasible solution to $OCP_T^0(x(t_k))$ is also feasible in $OCP_T^{\mathbb{X}_\mathrm{f}}(x(t_k))$ but not vice versa. We also remark without further elaboration that if Assumption **A4** is revoked then the setting of Corollary 1 yields asymptotic convergence.

Computation of Terminal Constraints and Penalties

So far, we have leveraged upper bounds on the cost-to-go as terminal penalty V_f. Yet, besides the trivial choice used in Corollary 1, we did not discuss how to actually choose or compute V_f and $\mathbb{X}_\mathrm{f}$.

First we consider a special case in which

- the system (Σ) is not subject to state constraints, i.e. $\mathbb{X} = \mathbb{R}^{n_x}$, and
- it is globally asymptotically stable, i.e. it satisfies the conditions of Lemma 5 and in particular there exists a Lyapunov function $V : \mathbb{R}^{n_x} \to \mathbb{R}_0^+$

$$\nabla V^\top \cdot f(x, 0) + \ell(x, 0) < 0, \quad \forall x \neq 0.$$

In this case, the choice $V_\mathrm{f} = V$ is immediate and for any choice of T and δ will satisfy the conditions of Theorem 2. We remark that in the vast majority of applications it is unrealistic to assume to global asymptotic stability of the system in question—yet alone knowledge of a global Lyapunov function. However, in this rare situation the application of MPC will still be beneficial as one may expect a performance improvement.

Notice that in case of state constraints $\mathbb{X} \subset \mathbb{R}^{n_x}$ one still has to check for recursive feasibility or make sure that the state constraints are shrunk to a level set of the Lyapunov function, i.e. $\tilde{\mathbb{X}} = \{x \in \mathbb{R}^{n_x} \mid V(x) \leq \rho^2\} \subseteq \mathbb{X}$.

These considerations motivate to investigate structured approaches to the design of $\mathbb{X}_\mathrm{f}$ and V_f. We tailor our setting and require

$$\ell(x, u) = \tfrac{1}{2} x^\top Q x + \tfrac{1}{2} u^\top R u = \tfrac{1}{2}\|x\|_Q^2 + \tfrac{1}{2}\|u\|_R^2, \tag{13}$$

with $Q \succeq 0, R \succ 0$, and that the Jacobian linearization of $f(x, u)$ at $(0, 0)$—i.e., the pair (A, B)—is stabilizable, while (A, C) with $C^\top C = Q$ is detectable.

Consider

$$\Sigma : \quad \dot{x} = f(x, u), \quad x(0) \in \mathbb{X}_0$$

with $f(0, 0) = 0$. Let

$$A \doteq \left.\frac{\partial f}{\partial x}\right|_{(0,0)}, \qquad B \doteq \left.\frac{\partial f}{\partial u}\right|_{(0,0)}.$$

Lemma 3 (Local stabilizability)**.** *If the pair (A, B) is stabilizable, i.e., there exists $u = Kx$ such that the real parts of all eigenvalues of $A + BK$ are negative, then the feedback $u = Kx$ achieves local asymptotic stability of $x = \bar{x} = 0$ for the nonlinear system Σ.*

As we have seen in Part I, the following linear-quadratic OCP

$$\begin{aligned} &\min_{u(\cdot)} \int_{t_0}^{\infty} \tfrac{1}{2}\left(\|x(\tau)\|_Q^2 + \|u(\tau)\|_R^2\right) \,\mathrm{d}\tau \\ &\text{subject to} \\ &\dot{x} = Ax + Bu, \quad x(t_0) = x_0 \end{aligned} \tag{LQR$_\infty$}$$

can be regared as a local approximation to $OCP_\infty(x(t))$ at the setpoint $(0, 0)$. Moreover, the optimal value function of (LQR$_\infty$) is given by

$$V_{LQR}(x_0) = \tfrac{1}{2} x_0^\top P x_0. \tag{14}$$

where $P \succ 0$ solves the Algebraic Riccati Equation (ARE)

$$A^\top P + PA - PBR^{-1}B^\top P + Q = 0, \quad P = P^\top \succ 0. \tag{ARE}$$

The solution to the algebraic Riccati equation is readily obtained in MATLAB using the commands `care` (continuous-time) and `dare` (discrete-time).

Finally, recall that the optimal solution to LQR$_\infty$ is given by the feedback $u = Kx = -R^{-1}B^\top P x$. Then, it is straightforward to show that

- $u = Kx$ stabilizes the nonlinear system (Σ) locally, and

- $\frac{1}{2}x^\top Px$ is a local Lyapunov function of the non-linear system (Σ) controlled by $u = Kx$.

The first statement is a standard result in non-linear system theory and can be found in several textbooks and the second statement follows from classical results of Kalman: H.K. Khalil. *Nonlinear Systems*. 3rd. Prentice Hall, New Jersey, 2002; Kalman, "Contributions to the theory of optimal control".

The proof of these two statements is left as an exercise for the reader.

Based on these results, the idea to obtain a terminal region is to consider the infinite-horizon feedback obtained from LQR_∞ as terminal control law and to consider the value function V_{LQR} as terminal penalty V_f. Hence it remains to show on which set $\mathbb{X}_\mathrm{f} \subseteq \mathbb{X}$ this choice is valid and satisfies the conditions of Theorem 2.

We make the following ellipsoidal ansatz for the terminal region

$$\mathbb{X}_\mathrm{f} = \left\{ x \in \mathbb{R}^{n_x} \mid \tfrac{1}{2}x^\top Px \le \rho^2 \right\} \tag{15}$$

with the stage cost from (13) and the terminal penalty $V_\mathrm{f} = V_{LQR}$ from (14). Then the radius ρ of the terminal set can be obtained by solving

$$\min_{\rho} \quad -\rho \tag{16a}$$

subject to

$$\forall x \in \mathbb{X}_\mathrm{f}: \quad Kx \in \mathbb{U} \tag{16b}$$

$$x \in \mathbb{X} \tag{16c}$$

$$\dot{V}_\mathrm{f}(x) + \ell(x, Kx) \le 0. \tag{16d}$$

While the objective is to maximize the size of $\mathbb{X}_\mathrm{f}$, the first constraint captures the input constraints for the terminal feedback law. Observe that the state constraints are handled by the second constraint, which usually will be inactive as in many cases we will have $\mathbb{X}_\mathrm{f} \subset \mathbb{X}$. Finally, the constraint (16d) captures the basic stability condition (3) from Theorem 2. Note that similar arguments as in the proof of Theorem 3 can be used to show that (16d) implies positive invariance of $\mathbb{X}_\mathrm{f}$.

Notice that the optimization problem (16) is subject to an infinite number of constraints as (16d) has to hold for all x in $\mathbb{X}_\mathrm{f}$. Moreover, for generic non-linear system dynamics f the constraint (16d) is non-convex as it reads

$$\dot{V}_\mathrm{f}(x) + \ell(x, Kx) = \nabla V_\mathrm{f}^\top f(x, Kx) + \ell(x, Kx) \le 0.$$

Put differently, unless specific assumptions on the structure of f are invoked, solving (16) can be computationally intense. On the other hand, the computation of the terminal penalty V_f and the terminal control law Kx are straightforward. Finally, observe that any feasible solution $\rho > 0$ to (16d) will specify an admissible choice of $\mathbb{X}_\mathrm{f}$ (15) and V_f. It is also worth to be remarked that, in view of Lemma 3, Assumption **A1**—$(\bar{x}, \bar{u}) = (0, 0) \in \operatorname{int}(\mathbb{X} \times \mathbb{U})$—implies that there exist strictly positive values of ρ feasible in (16).

Before concluding the chapter, we revisit Example 1 and consider a simple remedy.

Example 2 (Pitfall example resolved). *We reconsider the setting form Example 1, i.e. MPC based on*

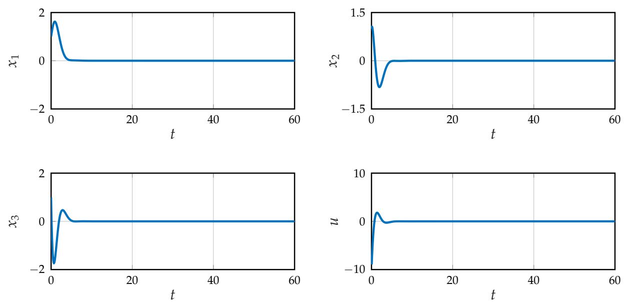
$$\min_{u(\cdot|t_k)} \quad \int_{t_k}^{t_k+T} \|x(\tau|t_k)\|_Q^2 + \|u(\tau|t_k)\|_R^2 \,\mathrm{d}\tau + \|x(t_k+T|t_k)\|_P^2$$

$$\text{subject to } \forall \tau \in [t_k, t_k+T]$$

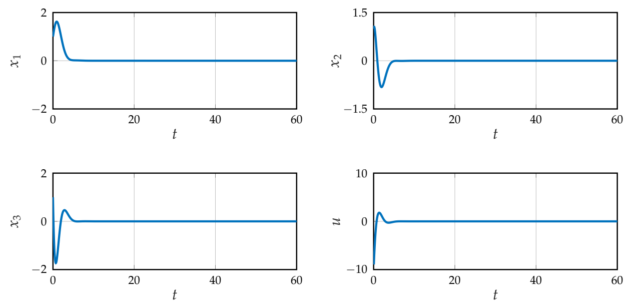
$$\frac{\mathrm{d}x(\tau|t_k)}{\mathrm{d}\tau} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x(\tau|t_k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(\tau|t_k),$$

$$x(t_k|t_k) = x(t_k),$$

where $\|x(t_k+T|t_k)\|_P^2 = x(\tau|t_k)^\top P x(\tau|t_k)$. Note that the end penalty $\|x(t_k+T|t_k)\|_P^2$ uses the solution to the algebraic Riccati equation (ARE). *The numerical setting is as in Example 1, the results are shown in Figure 8. As one can see for $T = 10 \cdot \delta$ and for $T = 8 \cdot \delta$ the solutions are asymptotically stable. Moreover, as the terminal penalty is not only a bound on the cost-to-go, but rather it is the exact cost-to-go, we see obsvere the solutions for both horizon are identical. Indeed one can show that in both cases they match the ones generated by infinite-horizon LQR feedback.*



(a) $T = 10 \cdot \delta$



(b) $T = 8 \cdot \delta$

Figure 8: Closed-loop trajectories for different horizons in Example 2.

Summary

This chapter has recalled classical results that guarantee stability in NMPC via infinite horizons or via terminal regions and penalties. However, as the design of terminal regions is often cumbersome—or in case of zero-terminal constraints their consideration in the numerical solutions can lead to difficulties—we may conclude that there is a clear need for stability conditions which do not require any terminal constraint. As one can already hint from the structure of the proof of Theorem 2, the question of recursive feasibility has to be addressed once one drops the terminal constraint $\mathbb{X}_f$ from the OCP. We will turn towards this issue in the next chapter.

Bibliographic Notes

The results in this chapter have been appeared in the literature in a slightly different order than presented here. The discussion on the infinite-horizon MPC scheme with instantaneous recomputation was given by Jadbabaie et al.[10] it can also be found, e.g. in Findeisen's treatment.[11] The discrete-time pendant to Corollary 1 was given by Keerthi and Gilbert in 1988.[12] The continuous-time variant appeared 1990.[13] Interestingly, the quite intuitive formulation to bound from above the cost-to-go presented in Theorem 3 took another 8-10 years to discover.[14] The generalization which removes the assumptions on the bounds of V is due to Fontes.[15] We also remark that still an interesting to read and highly cited overview on stability in discrete-time MPC with terminal constraints is due to Mayne et al.[16] Finally, the comments on computing terminal regions and terminal control laws follow the original developments of Chen and Allgöwer quite closely. More recent treatments can be found, for example, in the PhD thesis of Böhm.[17]

Finally, we remark that the list of citations is nowhere complete. The recent books[18,19] provide more in-depth overviews.

[10] Jadbabaie, Yu, and Hauser, "Unconstrained receding-horizon control of nonlinear systems".

[11] R. Findeisen. *Nonlinear Model Predictive Control: A Sampled-Data Feedback Perspective*. Fortschr.-Ber. VDI Reihe 8 Nr. 1087. VDI Verlag, Düsseldorf, 2006.

[12] S.S. Keerthi and E.G. Gilbert. "Optimal infinite-horizon feedback laws for a general class of constrained discrete-time systems: Stability and moving-horizon approximations". In: *Journal of Optimization Theory and Applications* 57.2 (1988), pp. 265–293.

[13] D.Q. Mayne and H. Michalska. "Receding horizon control of nonlinear systems". In: *IEEE Trans. Automat. Contr.* 35.7 (1990), pp. 814–824.

[14] H. Chen and F. Allgöwer. "A quasi-infinite horizon nonlinear model predictive control scheme with guaranteed stability". In: *Automatica* 34.10 (1998), pp. 1205–1217.

[15] Fontes, "A General Framework to Design Stabilizing Nonlinear Model Predictive Controllers".

[16] D.Q. Mayne et al. "Constrained model predictive control: Stability and optimality". In: *Automatica* 36.6 (2000), pp. 789–814.

[17] C. Böhm. "Predictive Control using Semi-definite Programming – Efficient Approaches for Periodic Systems and Lur'e Systems". PhD thesis. Universität Stuttgart, 2010.

[18] Rawlings, Mayne, and Diehl, *Model Predictive Control: Theory, Computation, and Design.*

[19] L. Grüne and J. Pannek. *Nonlinear Model Predictive Control: Theory and Algorithms*. 2nd Edition. Communication and Control Engineering. Springer Verlag, 2017.

NMPC without Terminal Constraints

After our discussion of various schemes with terminal constraints in the previous chapter, we now turn towards the question of how to avoid them. Key motivations to avoid terminal constraints are:

- As we have seen in the previous chapter the actual computation of terminal constraints can be quite involved. Moreover, observe that especially $\mathbb{X}_{f,\gamma}$—and to a lesser extend also V_f—are depend on the considered setpoint $\bar{x}$.

Observe that the trick to shift coordinates such that $(\bar{x}, \bar{u}) = (0, 0)$ does not alleviate the need to compute/design a specific terminal region $\mathbb{X}_f$, which is larger than $\{\bar{x}\}$, for each considered setpoint $\bar{x}$. To see this, for example, consider the ansatz that $\mathbb{X}_f$ is a level set of V_f. The underlying optimization problem (16) clearly depends on the distance of the setpoint pair $(\bar{x}, \bar{u})$ to the boundary of the constraints $\mathbb{X}$ and $\mathbb{U}$.

Intentionally, the subsequent treatment will be kept brief as the question of NMPC without terminal constraints will be discussed comprehensively in Part III using a dissipativity framework for NMPC.

Replaced Terminal Constraints

We consider

$$\begin{aligned}
\min_{u(\cdot|t_k)} &\int_{t_k}^{t_k+T} \ell(x(\tau|t_k), u(\tau|t_k))\, \mathrm{d}\tau + \beta V_f(x(t_k+T|t_k)) \\
&\text{subject to } \forall \tau \in [t_k, t_k+T] : \qquad\qquad (OCP_T^{\beta}(x(t_k))) \\
&\frac{\mathrm{d}x(\tau|t_k)}{\mathrm{d}\tau} = f(x(\tau|t_k), u(\tau|t_k)) \\
&x(t|t_k) = x(t_k) \\
&x(\tau|t_k) \in \mathbb{X},\ u(\tau|t_k) \in \mathbb{U},
\end{aligned}$$

wherein in comparison to $OCP_T^{\mathbb{X}_f}(x(t_k))$ no terminal constraint is present and the terminal penalty is multiplied by a positive scalar β. Our standing Assumptions **A1**–**A3** are complemented by the following ones:

Assumption (To replace terminal constraints by the end penalty)**.**

A4' *There exists $\beta_2 \in \mathcal{K}_\infty$, such that $V_T^0(x) \leq \beta_2(\|x\|)$ for all $x \in \mathrm{cl}(\Omega_T^0)$.*

$\mathrm{cl}(\mathbb{X}) \doteq \partial\mathbb{X} \cup \mathbb{X}$ denotes the closure of the set $\mathbb{X}$, and $\partial\mathbb{X}$ denotes the boundary of $\mathbb{X}$.

A5 *For some $\gamma > 0$, let $\mathbb{X}_{f,\gamma} = \{x \in \mathbb{X} \mid V_f(x) \leq \gamma\}$ and V_f satisfy the quasi infinite-horizon NMPC convergence conditions of Theorem 2 for specific values of $\delta > 0, T > 0$.*

A6 *The set Ω_T^0 is bounded and $\bar{x} = 0 \in \mathrm{int}(\Omega_T^0)$.*

Assumption **A4'** is a minor relaxation of **A4** from before; indeed **A4'** is again a weak controllability condition. Moreover, in **A5** observe that the terminal set $\mathbb{X}_f$ is constructed as a level set of the terminal penalty V_f. Hence if in the objective of $OCP_T^{\beta}(x(t_k))$ a lot of weight is put on V_f—i.e. the scalar β is sufficiently large—then one should expect that the state at the end of the prediction horizon $x(t_k + T|t_k)$ will be in $\mathbb{X}_{f,\gamma}$ without the terminal region being explicitly stated in $OCP_T^{\beta}(x(t_k))$.

To the end of verifying this intuitive insight, we draw upon the NMPC scheme based on $OCP_T^0(x(t_k))$ with zero-terminal constraint $\mathbb{X}_f = \{0\}$ and $V_f(x) = 0$. Recall that $V_T^0(x)$ is the associated optimal value function of $OCP_T^0(x(t_k))$, and Ω_T^0 is the set of feasible initial conditions, cf. (12). In this light, the first part of Assumption **A6** can be expected to hold, because Ω_T^0 is the set of states which can be steered to $x = 0$ in time T. The second part of Assumption **A6** will usually hold, as set-points on the boundary of Ω_T^0 would imply that even after convergence to the set-point a small disturbance can render the optimization infeasible.

Theorem 4 (Convergence with Replaced Terminal Constraint)**.**
*Let Assumptions **A1**–**A3**,**A4'**, **A5**, and **A6** hold. Then, there exists $\beta \in (0, \infty)$, such that*

- *the NMPC scheme based on $OCP_T^{\beta}(x(t_k))$ achieves*

$$\lim_{t\to\infty} \|x(t)\| = 0,$$

  for (Σ_{cl}), and

- *the region of attraction contains Ω_T^0.*

Proof. Consider the optimal value function of $OCP_T^{\beta}(x(t_k))$, $V_T^{\beta}(x)$, as a candidate Lyapunov function. From the fact that any optimal solution to $OCP_T^0(x(t_k))$ is feasible in $OCP_T^{\beta}(x(t_k))$ we have that

$$V_T^{\beta}(x) \le V_T^0(x) \overset{\mathbf{A4'}}{\le} \beta_2(\|x\|) \text{for all } x \in \Omega_T^0.$$

From the fact that $V_f(x)$ is positive semi-definite we obtain that for all $k \in \mathbb{N}_0$ we have

$$\beta V_f(x^\star(t_k + T|t_k)) \le \beta_2(\|x^\star(t_k + T|t_k)\|).$$

As $\beta_2(\|x\|)$ is a continuous function it will attain a finite maximum on $\mathrm{cl}(\Omega_T^0)$, which is closed and boudned and hence compact. Thus there exists $\beta \in (0, \infty)$ such that

$$V_f(x^\star(t_k + T|t_k)) \le \frac{\beta_2(\|x^\star(t_k + T|t_k)\|)}{\beta} \le \gamma$$

with γ from Assumption **A5**. In other words, one can choose a finite value for β such that $V_f(x^\star(t_k + T|t_k)) \le \gamma$ holds for all $\in \mathbb{N}_0$. In view of **A5** this implies $x^\star(t_k + T|t_k) \in \mathbb{X}_{f,\gamma}$, i.e. the terminal constraint is satisfied without being explicitly considered in the proof. Finally, invoking Theorem 2 finishes the proof. □

Observe that while the approach taken in the above theorem is elegant, the verification of the conditions underlying still requires to show that there exists a non-vanaishing value γ such that **A5** holds. However, taking the consideration on the computation of ellipsoidal terminal regions in the previous chapter into account, it is easy to check that under mild assumptions the existence of a finite value of β can be deduced without actually computing $\mathbb{X}_{f,\gamma}$, while a quadratic V_f can be obtained via the algebraic Riccati equation (ARE).

Other Approaches

The previous analysis has derived one approach to avoid the potentially tedious computation of terminal regions for NMPC. As we will see below, there also exist other approaches to avoid terminal constraints. We brieftly comment on two of them.

Control Lyapunov Functions as Terminal Penalties

We first recall a standard generalization of Lyapunov functions to systems with controls.

Definition 4. *A local Control-Lyapunov Function (CLF) of the system* (Σ)*, relative to the considered equilibrium $\bar{x}$, is a continuous-function $V : \mathbb{R}^{n_x} \to \mathbb{R}$ for which there is some neighborhood $\mathcal{D}$ of $\bar{x}$ such that the following properties hold:*

(i) V is proper at $\bar{x}$, i.e., the set $\{x \in \mathbb{R}^{n_x} \mid V(x) \leq \gamma\}$ is a compact subset of $\mathcal{D}$ for each γ small enough.

(ii) $V : \mathbb{R}^{n_x} \to \mathbb{R}$ is positive definite on $\mathcal{D}$.

V is positive definite $\Leftrightarrow V(\bar{x}) = 0$ and $V(x) > 0$ for each $x \in \mathcal{D}$, $x \neq \bar{x}$.

(iii) For all $\tilde{x} \in \mathcal{D}$, $x \neq \bar{x}$ there exists $\tau > 0$ and some control $\tilde{u} : [0, \tau) \to \mathbb{U} \subseteq \mathbb{R}^{n_u}$ such that the corresponding state trajectory $x(t) \doteq x(t, \tilde{x}, \tilde{u}(\cdot))$ satisfies

$$V(x(t)) \leq V(\tilde{x}) \text{ for all } t \in [0, \tau) \quad \text{and} \quad V(x(\tau)) < V(\tilde{x}).$$

A function $V : \mathbb{X} \to \mathbb{R}$ is said to be a global CLF for (Σ) *is (i) holds for any $\gamma > 0$, and (ii) and (iii) are satisfied with $\mathcal{D} = \mathbb{R}^{n_x}$.*

The conceptual value of this concept is highlighted in the following theorem.

Definition 5 (Reachablity)**.** *A point $\tilde{x} \in \mathbb{R}^{n_x}$ is said to be globally reachable, if for all $x \in \mathbb{R}^{n_x}$, there exists $T \in \mathbb{R} \cup \infty$ and $\tilde{u} : [0, t) \to \mathbb{U} \subseteq \mathbb{R}^{n_u}$ such that $x(T, x, \tilde{u}) = \tilde{x}$. If instead $\lim_{t \to \infty} x(t, x, \tilde{u}) = \tilde{x}$, the point $\tilde{x} \in \mathbb{R}^{n_x}$ is said to be asymptomatically reachable. Finally, if the above conditions hold only on e neighbourhood of $\tilde{x}$, the point $\tilde{x} \in \mathbb{R}^{n_x}$ is locally (asymptotically) reachable.*

Theorem 5 ($\exists$ CLF $\Rightarrow$ asympt. reachability)**.** *If there exists a local, respectively, global control-Lyapunov function V for* (Σ)*, relative to the considered equilibrium $\bar{x}$, then the equilibrium $\bar{x}$ of* (Σ) *is locally, respectively, globally asymptotically reachable.*

The proof of this result can be found in[20] from where the definition is taken. Observe that in order to check the conditions for control-Lyapunov functions, especially (iii) above, one needs to compute the trajectories $x(t, \tilde{x}, \tilde{u}(\cdot))$, i.e. one needs to solve an ODE.

[20] E.D. Sontag. *Mathematical Control Theory: Deterministic Finite Dimensional Systems*. Vol. 6. Springer, 1998, Chap. 5.7.

Recall that the pivotal advantages of Lyapunov functions for stability analysis is that one does not need to solve any ODE in order to show stability of an equilibrium of a dynamical system. Indeed, as the classical result of Lemma 5 shows, it suffices to verify a differential inequality point-wise in the state space. In case of stability of systems without control input this inequality reads $\nabla V f(x) < 0$ for all $x \in \mathbb{R}^{n_x}$. This reminder lends itself to the question of how one can derive a similar differential inequality for systems with control inputs.

Lemma 4 (Differential characterization of CLFs)**.** *Let V be a continuous function and suppose that on $\mathcal{D} \subseteq \mathbb{R}^{n_x}$, V is continuously differentiable and satisfies item (i) and (ii) of Definition 4. Then, if for all $x \in \mathcal{D}$, $x \neq \bar{x}$, there exists a $u \in \mathbb{U} \subseteq \mathbb{R}^{n_u}$ such that*

$$\nabla V^\top \cdot f(x,u) < 0 \tag{17}$$

then V is a local CLF, i.e. it satisfies item (iii) of Definition 4. If the condition holds with $\mathcal{D} = \mathbb{R}^{n_x}$, then V is a global CLF.

Proof. Choose an arbitrary $\tilde{x} \in \mathcal{D}$, $\tilde{x} \neq \bar{x}$ and any $u \in \mathbb{U}$ such that (17) holds. For small values if $\tau > 0$, the constant control $\tilde{u}(t) \equiv u$ is admissible. As V is of class $\mathcal{C}^1$ we may choose $\tilde{\tau}$ small enough such that $\frac{\mathrm{d}V(x(t,\tilde{x},\tilde{u}(\cdot))}{\mathrm{d}t} < 0$ for all $t \in [0,\tilde{\tau}]$. Hence we have $V(x(t,\tilde{x},\tilde{u}(\cdot)) < V(x)$ for all $t > 0$. $\square$

At this point, one may wonder how the concept of control-Lyapunov functions relates to NMPC without terminal constraints. Notice that the requirements for control-Lyapunov functions from Definition 4 are satisfied by terminal penalties V_f as per Theorem 2. More importantly, any terminal penalty V_f satisfying (5) will also satisfy (17). This highlights two aspects:

- Indeed one may concisely summarize the requirements for terminal penalties V_f as them being local control-Lyapunov functions. Of course one should not forget that the proof of Theorem 2 requires a decay of the terminal penalty which corresponds to the considered stage cost ℓ, cf. (5).
- If one imposes the terminal penalty V_f to be a global control-Lyapunov function (with sufficient decay properties), then one can regard the entire state space $\mathbb{R}^{n_x}$ as a suitable terminal constraint. Hence, under such an assumption, there is no need for any explicit terminal constraint.

Indeed the later point, i.e. imposing the terminal penalty V_f to be a (suitable) global control-Lyapunov function, has been considered in the NMPC literature. The results of Jadbabaie and Hauser[21] formalized this. However, note that state constraints are difficult to handle in such a setting as the geometry of a control-Lyapunov function might not be compatible with the geometry of the constraint set $\mathbb{X}$. Moreover, it is in general not easy to find or to compute control-Lyapunov functions, let alone global ones. We leave

[21] A. Jadbabaie and J. Hauser. "On the stability of receding horizon control with a general terminal cost". In: *IEEE Trans. Automat. Contr.* 50.5 (2005), pp. 674–678.

the formulation of sufficient stability conditions of NMPC based on a global control-Lyapunov function used as terminal penalty as an exercise for the reader.

Stability via Suboptimality Estimates

The two preceding sections have discussed how to avoid explicit terminal constraints in the NMPC design. However, both approaches rely on terminal penalties to ensure convergence and/or stability. Yet, this does not answer the questions of whether terminal penalties are necessary for convergence and stability?

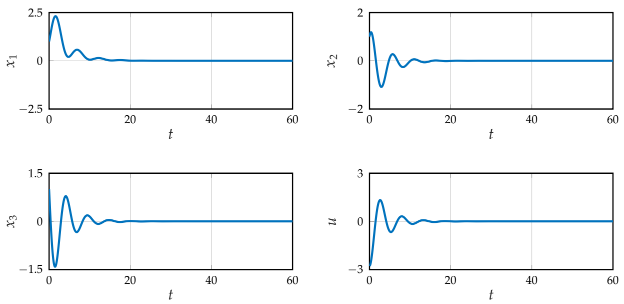
Example 3 (Pitfall becomes motivation). *We re-consider MPC based on the LQ OCP from Examples 1 and 2, i.e.*

$$\min_{u(\cdot|t_k)} \quad \int_{t_k}^{t_k+T} \|x(\tau|t_k)\|_Q^2 + \|u(\tau|t_k)\|_R^2 \, \mathrm{d}\tau$$
$$\text{subject to } \forall \tau \in [t_k, t_k+T]$$
$$\frac{\mathrm{d}x(\tau|t_k)}{\mathrm{d}\tau} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x(\tau|t_k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(\tau|t_k),$$
$$x(t_k|t_k) = x(t_k).$$

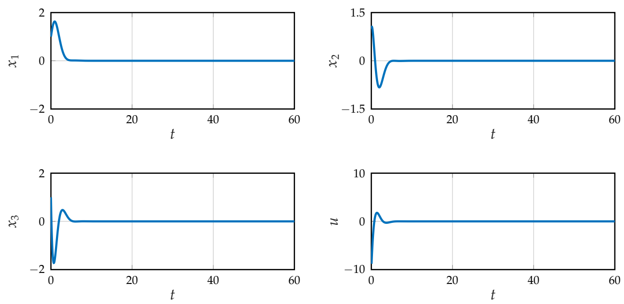
Figure 9 shows the simulation results for $T = N \cdot \delta$ with $N \in \{16, 64\}$. As one can see, the performance of the controller increases with increasing length of the prediction horizon. There is visible difference between the results for $T = 64 \cdot \delta$ and the results of Example 2, which employ the exact cost-to-go as end penalty. This fact is easily understood: using exact cost-to-go as end penalty renders the MPC loop equivalent to an infinite-horizon optimal controller, while long horizons do not exactly reproduce the infinite-horizon feedback but enforce that MPC is a close approximation. This example motivates the analysis of how to choose the prediction horizon in order to guarantee stability and performance.

Finally, observe that the example shows (as did Example 1) that neither terminal constraints nor a terminal penalty are necessary for stability.

Notice that the solution of the algebraic Riccati equation (ARE), which we consider in Example 2 is only the exact cost-to-go if not constraints are active over the entire infinite horizon.



(a) $T = 16 \cdot \delta$



(b) $T = 64 \cdot \delta$

Figure 9: Closed-loop trajectories for different horizons in Example 3.

Subsequently, we shift the focus to MPC schemes without terminal penalties or constraints. That is, we consider

$$\begin{aligned}
&\min_{u(\cdot|t_k)} \int_{t_k}^{t_k+T} \ell(x(\tau|t_k), u(\tau|t_k))\, \mathrm{d}\tau \\
&\text{subject to } \forall \tau \in [t_k, t_k+T]: \qquad\qquad (OCP_T(x(t_k))) \\
&\quad \frac{\mathrm{d}x(\tau|t_k)}{\mathrm{d}\tau} = f(x(\tau|t_k), u(\tau|t_k)) \\
&\quad x(t|t_k) = x(t_k) \\
&\quad x(\tau|t_k) \in \mathbb{X}, u(\tau|t_k) \in \mathbb{U}.
\end{aligned}$$

Notice that in the literature schemes without terminal constraints—but including input and state constraints—are frequently called *unconstrained* MPC schemes.[22] Here, however, we do not use this notion, as it could potentially be misleading.

[22] M. Reble and F. Allgöwer. "Unconstrained Nonlinear Model Predictive Control and Suboptimality Estimates for Continuous-Time Systems". In: *Proc. IFAC World Congress 2011, Milano, Italy, 28.08.-02.09.2011*. 2011, pp. 6733–6738; L. Grüne. "Analysis and design of unconstrained nonlinear MPC schemes for finite and infinite dimensional systems". In: *SIAM Journal on Control and Optimization* 48.2 (2009), pp. 1206–1228; Jadbabaie, Yu, and Hauser, "Unconstrained receding-horizon control of nonlinear systems".

We recall a stability result for such schemes which is based on so-called *suboptimality estimates*. Recall that V_∞ from (2) denotes the infinite-horizon optimal value function corresponding to $OCP_\infty(x(t))$, i.e. with instantaneous recalculation. With slight abuse of notation, let V_∞ now denote the value function for the counterpart to $OCP_\infty(x(t))$, wherein some non-vanashing sampling period $\delta > 0$ is used. Moreover, let $V^{MPC}(x_0)$ denote the closed-loop MPC value function for some horizon T and the sampling period δ, also evaluated on an infinite (application/simulation) horizon, which we defined in (7). Provided optimal solutions exist for all $x \in \mathbb{X}$, we have

$$V_\infty(x) \leq V^{MPC}(x) \quad \forall x \in \mathbb{X}.$$

Indeed, one can show that for the nominal infinite-horizon MPC schemes the feedback obtained with instantaneous feedback is equivalent to the one obtained for non-vanishing sampling periods $\delta > 0$. The reason is that, independently of the considered sampling period, once $OCP_\infty(x(t))$ is solved at $k = 0$, all subsequent optimal solutions can be obtained from the remainder pieces of $(x^\star(\cdot|t_0), u^\star(\cdot|t_0))$, cf. Bellman's Principle of Optimality. We remark that the stability analysis from Theorem 1 needs some adjustments, cf. Corollary 2 below.

Define

$$\alpha \doteq \sup_{x\in\mathbb{X}} \frac{V_\infty(x)}{V^{MPC}(x)}$$

and observe that, as long as $V_\infty(x) < \infty$ on $\mathbb{X}$, by optimality it follows that $\alpha \in (0, 1]$. Note that $\alpha = 1$ implies that the MPC scheme delivers a performance equivalent to the infinite-horizon optimal control, while $\alpha \to 0$ means absolute performance degradation of the MPC scheme. Hence, we may write

$$\alpha V^{MPC}(x) \leq V_\infty(x). \tag{18}$$

Indeed the scalar α is called *suboptimality estimate*, as it compares the infinite-horizon optimal performance to the one of the MPC scheme.

Two questions are immediate:

- Can knowledge about α be exploited for stability analysis?
- How to estimate α for given OCPs?

We will first discuss the former point, before turning towards the latter issue. To this end, we recall a soon to be classical result, which has its roots in dynamic programming.[23]

[23] R. Bellman. "The theory of dynamic programming". In: *Bulletin of the American Mathematical Society* 60.6 (1954), pp. 503–515.

Theorem 6 (Relaxed dynamic programming inequality)**.**
*Consider $OCP_T(x(t_k))$ and let Assumptions **A1**–**A3** hold and let $V_\infty(x_0) < \infty$ on $\mathbb{X}$. If, for all $x_0 \in \mathbb{X}$,*

$$V_T(x_0) \geq V_T(x^\star(\delta|t_k)) + \alpha \int_0^\delta \ell(x^\star(\tau|t_k), u^\star(\tau|t_k))\,\mathrm{d}\tau \qquad (19)$$

with $\alpha \in (0,1]$, then

$$\alpha V_\infty(x_0) \leq \alpha V^{MPC}(x_0) \leq V_T(x_0) \leq V_\infty(x_0). \qquad (20)$$

Proof. As mentioned before, feasibility of $OCP_T(x(t_k))$ and $OCP_\infty(x(t))$ for all $x \in \mathbb{X}$ gives $V_\infty(x) \leq V^{MPC}(x)$, hence we have $\alpha V_\infty(x_0) \leq \alpha V^{MPC}(x_0)$ for any $\alpha \geq 0$.

Moreover, observe that **A2**—lower boundedness $\ell(x,u) \geq \alpha(\|x - \bar{x}\|)$—implies $V_T(x) \geq 0$ and $V_\infty(x) \geq 0$. As the truncation of the infinite-horizon optimal pairs $(x^\star_\infty(\cdot), u^\star_\infty(\cdot))$ to $[0,T]$ is not necessarily optimal in $OCP_T(x(t_k))$, we have that $V_T(x_0) \leq V_\infty(x_0)$.

Now, consider $x(\delta) = x^\star(\delta, x_0; u^\star(\cdot, x(\delta)))$ and let

$$x(k\delta) = x^\star(\delta, x((k-1)\delta); u^\star(\cdot, x((k-1)\delta))).$$

From (19) it follows that

$$V_T(x(k\delta)) - V_T(x((k+1)\delta) \geq \alpha \int_0^\delta \ell(x^\star(\tau|t_k), u^\star(\tau|t_k))\,\mathrm{d}\tau.$$

This gives

$$\sum_{k=0}^{N} V_T(x(k\delta)) - V_T(x((k+1)\delta) \geq \alpha \sum_{k=0}^{N} \int_0^\delta \ell(x^\star(\tau|t_k), u^\star(\tau|t_k))\,\mathrm{d}\tau.$$

Let $x_\infty = \lim_{k\to\infty} x(k\delta)$. Then, in the limit for $N \to \infty$, we obtain

$$V_T(x_0) - V_T(x_\infty) \geq \alpha \sum_{k=0}^{\infty} \int_0^\delta \ell(x^\star(\tau|t_k), u^\star(\tau|t_k))\,\mathrm{d}\tau = \alpha V^{MPC}(x_0).$$

As we assumed $V_\infty(x_0) < \infty$ on $\mathbb{X}$, we have $V_T(x_\infty) \leq V_\infty(x_\infty) < 0$. Moreover, $V_\infty(x_\infty) < 0$ combined with **A2** allows to conclude, via Barbalat's Lemma, that $x_\infty = \lim_{k\to\infty} x(k\delta) = 0$. Hence $V_\infty(x_\infty) = 0$ and we arrive at $V_T(x_0) \geq \alpha V^{MPC}(x_0)$. This concludes the proof. □

The careful reader may have recognized that the inequalities (20) provide an avenue for stability analysis. Recall that for the case of instantaneous recalculation Theorem 1 has established asymptotic stability with V_∞ as Lyapunov function. Hence $\alpha V^{MPC}(x_0) \leq V_\infty(x_0)$ suggests that stability can be shown without any terminal constraint or penalty. This is formalized next. As a preparatory step, we give the following corollary.

Corollary 2 (Asymptotic stability via $OCP_\infty(x(t))$ for $\delta > 0$)**.**
*Consider $OCP_\infty(x(t))$ with non-vanishing sampling period $\delta > 0$ and suppose Assumptions **A1**–**A3** hold. Let **A4** be true for $V_\infty(x)$ on some set $\mathcal{D} \subseteq \mathbb{X}$ and let $OCP_\infty(x(t))$ be feasible at $k = 0$. Then,*

- *$OCP_\infty(x(t))$ is recursively feasible,*
- *the NMPC scheme based on $OCP_\infty(x(t))$ achieves local asymptotic stability of (Σ_{cl}) at $x = 0$, and*
- *$\mathcal{D} \subseteq \mathbb{X}$ is in the region of attraction.*

The proof of this result follows by combination of the main steps from the proof of Theorem 1 (for recursive feasiblity and convergence via Barbalat's Lemma) with the one of Theorem 3 (for the asymptotic stability statement). It is thus omitted.

Theorem 7 (Stability without terminal constraints and penalties)**.** *Consider $OCP_T(x(t_k))$ and let Assumptions **A1–A3** hold. Let **A4** be true for $V_T(x)$ on some set $\mathcal{D} \subseteq \mathbb{X}$ and let $OCP_T(x(t_k))$ be feasible for all $k \in \mathbb{N}$.*

If, for the choice of T and δ, inequality (19) holds with $\alpha \in (0, 1]$, then

- *the NMPC scheme based on $OCP_T(x(t_k))$ achieves local asymptotic stability of (Σ_{cl}) at $x = 0$, and*
- *$\mathcal{D} \subseteq \mathbb{X}$ is in the region of attraction.*

Proof. Observe that (19) implies the decrease of V_T from one sampling instant t_k to t_{k+1}. Moreover, combined with the upper bound on $V_T(x)$ for all $x \in \mathbb{X}$, similar to Step 4 of the proof of Theorem 2, we can conclude convergence of the closed loop (Σ_{cl}) to $x = 0$ via Barbalat's Lemma. Finally, local asymptotic stability can be shown as in the proof of Theorem 3. □

The previous result deserves a discussion. First of all, note that in contrast to the all other stability/convergence results derived so far, the last theorem imposes recursive feasibility as an assumption. But it does not show it. Also notice that in situations with compact input constraints and without state constraints one can usually not expect that the region of attraction is the entire state space $\mathbb{R}^{n_x}$. This is due to the fact that, already for linear unstable systems, at a distance form the origin the unstable eigendynamics have larger effect on the system evolution than magnitude constrained inputs.

The turnpike-based analysis of Part III will provide a handle to overcome recursive feasibility assumptions for problems with input and state constraints.

Moreover, we remark that in view of Theorem 6 suboptimality estimates α will usually depend on the chosen sampling period δ and on the horizon length T. This hints towards the most crucial issue of Theorem 7: how does one compute suboptimality estimates?

Cost Controllability and Suboptimality Estimates

How to compute suboptimality estimate? To this end, we make the following assumption.

Assumption 1 (Exponential cost controllability)**.**

Summary

Bibliographic Notes

- Mention Jadbabaie 2001
- Pin-point earliest results on NMPC without terminal constraints
- Pinpoint the difficulties to consider state constraints
- Point towards relevant references

Trajectory Tracking and Beyond

Summary

Bibliographic Notes

Part III

Economic Model Predictive Control

Towards Generic Stage Costs

EMPC with Terminal Constraints

EMPC without Terminal Constraints

Bibliographic Notes

Part IV

Conclusions and Summary

Comparison of NMPC Formulations

Guidelines for NMPC Design

What wasn't discussed

Research Outlook

Part V

Appendices

Basic Definitions and Well-known Results

Stability

Definition 6 (Class-$\mathcal{K}$ function)**.** • *A scalar function* $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *is said to belong to class* $\mathcal{K}$*, if it is continuous, strictly increasing, and* $\alpha(0) = 0$*.*

- $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *is said to belong to class* $\mathcal{K}_\infty$*, if* $\alpha \in \mathcal{K}$ *and if it is radially unbounded, i.e.* $\alpha(s) \to \infty$ *as* $s \to \infty$*.*

Definition 7 (Stability of equilibria)**.** *The system* $\dot{x} = f(t, x)$ *with* $f(t, 0) = 0$ *is said to be uniformly (locally) stable at* $x = 0$*, if for every* $\epsilon > 0$ *there exists an* $\delta = \delta(\epsilon) > 0$*, which is independent from* t_0*, such that all solutions* $x(\cdot, t_0, x_0)$ *fulfill*

Khalil, Nonlinear Systems
D. Hinrichsen and A.J. Pritchard. *Mathematical Systems Theory I*. Springer, 2005

$$\|x_0\| < \delta \quad \Rightarrow \quad \|x(t, t_0, x_0)\| < \epsilon \quad \text{for all } t \geq t_0 \geq 0.$$

If $x = 0$ *is a uniformly stable equilibrium of* $\dot{x} = f(t, x)$*, and there exists a positive constant* $c = c(t_0)$*, and additionally the solutions fulfill*

i)

$$\lim_{t\to\infty} \|x(t, t_0, x_0)\| = 0 \quad \text{for all } \|x_0\| < c,$$

ii) and for each $\eta > 0$ *there exists* $T = T(\eta) > 0$ *such that*

$$\|x(t, t_0, x_0)\| < \eta, \quad \text{for all } t \geq t_0 + T(\eta), \ \text{for all } \|x_0\| < c,$$

then $x = 0$ *is said to be uniformly (locally) asymptotically stable.*

Lemma 5 (Lyapunov stability)**.** *Let* $\beta_1, \beta_2 \in \mathcal{K}_\infty$*,* $\beta_3 \in \mathcal{K}$*, and the system* $\dot{x} = f(t, x)$ *fulfills* $f(t, 0) = 0$*. Consider some compact domain* $\mathbb{X}$ *containing* $x = 0$ *in its interior, and a function* $V : \mathbb{R}_0^+ \times \mathbb{X} \to \mathbb{R}_0^+$ *such that*

$$\begin{aligned} \beta_1(\|x\|) \leq V(t, x) \leq& \quad \beta_2(\|x\|) \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq& -\beta_3(\|x\|) \end{aligned}$$

holds for all $t \geq 0$ *and all* $x \in \mathbb{X}$*. Then* $x = 0$ *is uniformly asymptotically stable on* $\mathbb{X}$*.*

Lemma 6 (Leibniz)**.** *Let* $\ell : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be differentiable on* $[t_1, t_2] \times [\eta_1, \eta_2]$ *with respect to* η *and continuous with respect to* t *and* η*. Let* $h : \mathbb{R} \to \mathbb{R}$ *be differentiable on* $[\eta_1, \eta_2]$ *with range in* $[t_1, t_2]$*, i.e.,*

$$h : [\eta_1, \eta_2] \to [t_1, t_2].$$

Then it holds that

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\int_0^{h(\eta)} \ell(t,\eta)\,\mathrm{d}t = \int_0^{h(\eta)} \ell_\eta(t,\eta)\,\mathrm{d}t + \ell(h(\eta),\eta)\frac{\mathrm{d}}{\mathrm{d}\eta}h(\eta).$$

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