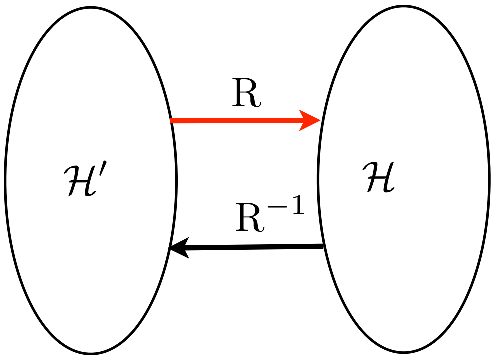
Reproducing kernel Hilbert spaces

- $\mathcal{H} \subseteq \mathcal{S}'(\mathbb{R}^d)$ is a RKHS $\quad \Leftrightarrow \quad \delta(\cdot - \boldsymbol{x}_0) \in \mathcal{H}'$
- Reproducing kernel $h : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ (positive definite)
  - $h(\boldsymbol{x}, \cdot) = h(\cdot, \boldsymbol{x}) \in \mathcal{H}$ for all $\boldsymbol{x} \in \mathbb{R}^d$
  - $f(\boldsymbol{x}) = \langle f, h(\cdot, \boldsymbol{x})\rangle_{\mathcal{H}}$ for all $f \in \mathcal{H}$



- Riesz maps (isometric)
  - $\mathrm{R} : \mathcal{H}' \to \mathcal{H}$
  - $\mathrm{R}^{-1} : \mathcal{H} \to \mathcal{H}'$
- Riesz conjugate: $f^* = \mathrm{R}\{f\} \in \mathcal{H}'$ for all $f \in \mathcal{H}$

$$\langle f, g\rangle_{\mathcal{H}} = \langle f, g^*\rangle = \langle f^*, g\rangle = \langle f^*, g^*\rangle_{\mathcal{H}'}$$

- Explicit inner product
  - $\langle f, g\rangle_{\mathcal{H}} = \langle \mathrm{R}^{-1}\{f\}, g\rangle$
  - $\langle f^*, g^*\rangle_{\mathcal{H}'} = \langle \mathrm{R}\{f^*\}, g^*\rangle \quad = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f^*(\boldsymbol{x}) h(\boldsymbol{x}, \boldsymbol{y}) g^*(\boldsymbol{y}) \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \quad = \langle f^* \otimes g^*, h\rangle$

Form GSPs to classical Gaussian processes

- Gaussian process on $\mathbb{R}^d$: $\{G(\boldsymbol{x})\}_{\boldsymbol{x}\in\mathbb{R}^d}$ with $G(\boldsymbol{x}) = \langle G, \delta(\cdot - \boldsymbol{x})\rangle$
  where $G \sim \mathcal{N}(\mu_G, \mathrm{R})$ (GSP)

  Necessary and sufficient conditions:
  - covariance operator = Riesz map $\mathrm{R} : \mathcal{H}' \to \mathcal{H}$ where $\mathcal{H}$=RKHS
  - $\mu_G \in \mathcal{H}$
- Characteristic form $\widehat{\mathscr{P}}_G : \mathcal{H}' \to \mathbb{R}$ with $\mathcal{H}' \supseteq \mathcal{S}(\mathbb{R}^d)$

$$\widehat{\mathscr{P}}_G(\varphi) = \mathbb{E}\{\mathrm{e}^{\mathrm{j}\langle G, \varphi\rangle}\} = \exp\left(-\tfrac{1}{2}\|\varphi\|^2_{\mathcal{H}'} + \mathrm{j}\langle \mu_G, \varphi\rangle\right)$$

- Covariance function = reproducing kernel

$$\mathbb{E}\left\{G(\boldsymbol{x})G(\boldsymbol{y})\right\} = \mathrm{R}\{\delta(\cdot - \boldsymbol{y})\}(\boldsymbol{x}) = h(\boldsymbol{x}, \boldsymbol{y})$$

- Specific case
  - $\langle f, g\rangle_{\mathcal{H}_{\mathrm{L}}} = \langle (\mathrm{L}^*\mathrm{L} + \mathrm{R}_{\boldsymbol{\phi}})\{f\}, g\rangle$
  - $\langle f^*, g^*\rangle_{\mathcal{H}'} = \langle \underbrace{(\mathrm{A}_{\boldsymbol{\phi}} + \mathrm{R}_{\boldsymbol{p}})}_{\mathrm{R}}\{f^*\}, g^*\rangle$