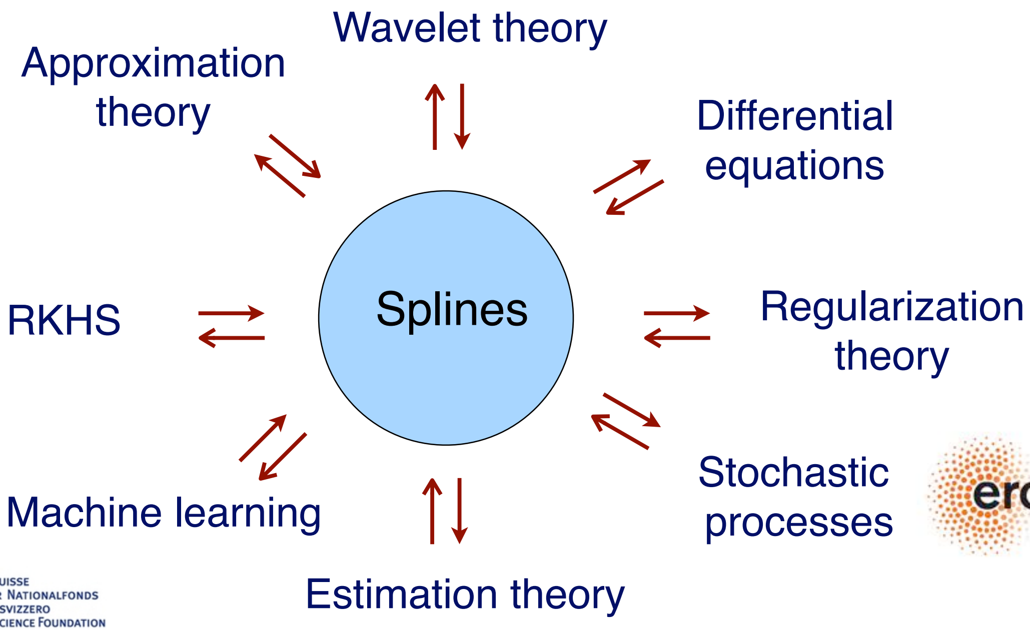
Splines as a unifying mathematical concept

Functional analysis



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Signal processing

Numerical analysis

The problems to be solved

Operator L: spline-admissible with null space $\mathcal{N}_{\mathrm{L}} = \mathrm{span}\{p_n\}_{n=1}^{N_0}$

- Characterization of a L-spline s

$$\mathrm{L}s = w_\delta = \sum_{k=1}^{K} a_k \delta(\cdot - \boldsymbol{x}_k)$$

s.t. $\boldsymbol{\phi}(s) = \mathbf{0}$ (boundary conditions)

$$\Rightarrow \quad s = \mathrm{L}_{\boldsymbol{\phi}}^{-1}\{w_\delta\} = \sum_{k=1}^{K} a_k \rho_{\mathrm{L}}(\cdot - \boldsymbol{x}_k) + \sum_{n=1}^{N_0} b_n p_n$$

- Generation of sparse stochastic processes

Solution of stochastic differential equation: $\mathrm{L}s = w \quad \Rightarrow \quad s = \mathrm{L}_{\boldsymbol{\phi}}^{-1}\{w\}$

- Variational formulation of inverse problems

Tikhonov flavor:

$$\arg \min_{f \in \mathcal{H}_{\mathrm{L}}} F(\boldsymbol{\nu}(f), \mathbf{y}) + \lambda \|\mathrm{L}f\|_{L_2(\mathbb{R}^d)}^2$$

= $\mathrm{L}^*\mathrm{L}$-spline with knots at $\boldsymbol{x}_m$ when $\nu_m = \delta(\cdot - \boldsymbol{x}_m)$

$\Rightarrow$ General form of solution: $s = \sum_{m=1}^{M} a_m \mathrm{L}_{\boldsymbol{\phi}}^{-1} \mathrm{L}_{\boldsymbol{\phi}}^{-1*}\{\nu_m\} + \sum_{n=1}^{N_0} b_n p_n$

Generalized TV/sparse flavor:

$$\arg \min_{f \in \mathcal{M}_{\mathrm{L}}} F(\boldsymbol{\nu}(f), \mathbf{y}) + \lambda \|\mathrm{L}f\|_{\mathcal{M}(\mathbb{R}^d)}$$

= L-spline with adaptive knots $(\boldsymbol{x}_k)_{k=1}^{K}$ and $K < M$

Mathematical challenges

- Inversion of operator L

  $\exists$ unique L_ϕ^{-1} such that

  $(i) : \mathrm{L}\mathrm{L}_\phi^{-1}\{w\} = w$ and $(ii) : \phi(\mathrm{L}_\phi^{-1}\{w\}) = \mathbf{0}$ (boundary conditions)

- Definition of native space = RKHS

  $\mathcal{H}_{\mathrm{L}} = \{f \in \mathcal{S}'(\mathbb{R}^d) : \|\mathrm{L}f\|_{L_2(\mathbb{R}^d)} < \infty\}$???

  $\mathcal{M}_{\mathrm{L}} = \{f \in \mathcal{S}'(\mathbb{R}^d) : \|\mathrm{L}f\|_{\mathcal{M}(\mathbb{R}^d)} < \infty\}$???

  Question: what is the relation between those space and stochastic processes ???

  Solution: Show that $\mathcal{H}_{\mathrm{L}}$ is a RKHS and provide its **reproducing kernel**

- How does this concern you

  You will need to do the work (=apply the theory) for your own family of operators L

RKHS

Desirata: $\mathcal{H}_L \subseteq C_{\mathrm{b},\alpha}(\mathbb{R}^d)$ (continuous functions of slow growth)

Inner product: $\langle f, g\rangle_{\mathcal{H}_{\mathrm{L}}} = \langle \mathrm{L}f, \mathrm{L}g\rangle + ????$ (to solve unicity problem)

- Reproducing kernel property

  (i) $h(\cdot, \boldsymbol{x}_0) \in \mathcal{H}_{\mathrm{L}}$ for all $\boldsymbol{x}_0 \in \mathbb{R}^d$

  (ii) $f(\boldsymbol{x}_0) = \langle h(\cdot, \boldsymbol{x}_0), f\rangle_{\mathcal{H}_{\mathrm{L}}}$ for all $f \in \mathcal{H}_{\mathrm{L}}$ and $\boldsymbol{x}_0 \in \mathbb{R}^d$.

  $$\Rightarrow \quad \boxed{\langle h(\cdot, \boldsymbol{x}_0), h(\cdot, \boldsymbol{y}_0)\rangle_{\mathcal{H}_{\mathrm{L}}} = h(\boldsymbol{x}_0, \boldsymbol{y}_0)}$$

  "lazy" formula for computing Gram matrices

- Why is this useful

  Any $f \in \mathcal{H}_{\mathrm{L}}$ can be approached **as closely as desired** by an expansion of the form

  $$f_K = \sum_{k=1}^{K} a_k h(\cdot, \boldsymbol{x}_k)$$ with **a finite number** of terms K