Proper regularization operator

Definition
The operator $\mathrm{L} : \mathcal{H}_{\mathrm{L}} \to L_2(\mathbb{R}^d)$ with null space $\mathcal{N}_{\mathrm{L}} = \mathrm{span}\{p_n\}_{n=1}^{N_0} \subseteq \mathcal{H}_{\mathrm{L}}$ is a proper regularization operator for the measurement operator $\boldsymbol{\nu} : \ f \mapsto \boldsymbol{\nu}(f) = (\langle \nu_1, f\rangle, \dots, \langle \nu_M, f\rangle)$ if the following technical conditions are met:

1. L is spline-admissible
2. $\nu_1, \dots, \nu_M \in \mathcal{H}'_{\mathrm{L}}$
3. For all $q \in \mathcal{N}_{\mathrm{L}}$, $\|\boldsymbol{\nu}(q)\|_2 \geq 0$ with equality if and only if $q = 0$.

- Criterion

  The singular values of $\mathbf{P} = [\boldsymbol{\nu}(p_1) \ \cdots \ \boldsymbol{\nu}(p_{N_0})]$ are strictly positive and bounded

- Relevant Green's function

  $G_{\mathrm{L}^*\mathrm{L}}(\boldsymbol{x}, \boldsymbol{y})$ such that $(\mathrm{L}^*\mathrm{L})\{G_{\mathrm{L}^*\mathrm{L}}(\cdot, \boldsymbol{y})\} = \delta(\cdot - \boldsymbol{y})$

Representer theorem for linear inverse problems

- $\boldsymbol{\nu} : f \mapsto \boldsymbol{\nu}(f) = (\langle \nu_1, f\rangle, \dots, \langle \nu_M, f\rangle)$ is a continuous linear operator $\mathcal{H}_{\mathrm{L}} \to \mathbb{R}^M$ that extracts M measurements from the signal f;
- $\mathrm{L} : \mathcal{H}_{\mathrm{L}} \to L_2(\mathbb{R}^d)$ is a proper regularization operator;
- $\{p_n\}_{n=1}^{N_0}$ is a basis of the null space of the regularization operator;
- $F : \mathbb{R}^M \times \mathbb{R}^M \to \mathbb{R}$ is a strictly convex and coercive loss function;
- $\mathbf{y} \in \mathbb{R}^M$ is a given data vector and $\lambda \in \mathbb{R}^+$ an adjustable regularization parameter.

Theorem
The solution of the general minimization problem

$$\arg\min_{f \in \mathcal{H}_{\mathrm{L}}} F(\boldsymbol{\nu}(f), \mathbf{y}) + \lambda \|\mathrm{L}f\|^2_{L_2(\mathbb{R}^d)}$$

is **unique** and has the generic **linear parametric form**

$$f_{(\lambda)} = \sum_{m=1}^{M} a_m \varphi_m + \sum_{n=1}^{N_0} b_n p_n$$

with $\varphi_m = \mathrm{A}\{\nu_m\} = \int_{\mathbb{R}^d} G_{\mathrm{L}^*\mathrm{L}}(\cdot, \boldsymbol{y}) \nu_m(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}, \quad \mathbf{a} = (a_m) \in \mathbb{R}^M, \quad \mathbf{b} = (b_n) \in \mathbb{R}^{N_0}$,
subject to the "**orthogonality**" constraint: $\langle \mathbf{a}, \boldsymbol{\nu}(p_n)\rangle = 0$ for $n = 1, \dots, N_0$.

Your work

- Interpolation problem

  Input: set of (potentially noisy) data points: $f(\boldsymbol{x}_m) \approx y_m, \quad m = 1, \dots, M$

$$f_{\mathrm{opt}} = \arg \min_{f \in \mathcal{H}_{\mathrm{L}}} \left(\sum_{m=1}^{M} \big(y_m - f(\boldsymbol{x}_m)\big)^2 + \lambda \|\mathrm{L}f\|_{L_2}^2 \right)$$

- List of tasks
  - Use “representer theorem” to deduce the parametric form of the solution of the above interpolation problem when L is LSI with non-trivial null space.
    Hints: Identify the underlying ν_m. $F(\mathbf{z}, \mathbf{y}) = \|\mathbf{z} - \mathbf{y}\|^2$ is strictly convex and coercive.
  - Show that f_{opt} is a $\mathrm{L}^*\mathrm{L}$-splines with knots at $\{\boldsymbol{x}_m\}$.
  - Derive an (numerical) algorithm for finding the optimal coefficients $\mathbf{a} = (a_1, \dots, a_M)$ and $\mathbf{b} = (b_1, \dots, b_{N_0})$.
  - Illustrate the scheme for $\mathrm{L} = \mathrm{D}$ as well as for other differential operator(s).
  - Implement the algorithm and present results of your data fitting (plots) illustrating the effect of the regularisation parameter λ.