RKHS associated with invertible operator

- Native space

$$\mathcal{H}_{\mathrm{L}} = \left\{f : \mathbb{R}^d \to \mathbb{R} \text{ s.t. } \|f\|_{\mathcal{H}_{\mathrm{L}}} \triangleq \|\mathrm{L}f\|_{L_2(\mathbb{R}^d)} < \infty\right\} \subseteq L_2(\mathbb{R}^d)$$

Coercive regularisation operator: $\mathrm{L} : \mathcal{H}_{\mathrm{L}} \to L_2(\mathbb{R}^d)$

$$c\|f\|_{L_2(\mathbb{R}^d)} \le \|\mathrm{L}f\|_{L_2(\mathbb{R}^d)} \quad = \|f\|_{\mathcal{H}_{\mathrm{L}}}$$

continuity $\Leftrightarrow$ specifies a valid norm

coercivity

$\Leftrightarrow$ Existence of (stable) inverse operator: $\mathrm{L}^{-1} : L_2(\mathbb{R}^d) \to \mathcal{H}_{\mathrm{L}}$

- Inner product: $\langle f, g\rangle_{\mathcal{H}_{\mathrm{L}}} = \langle \mathrm{L}f, \mathrm{L}g\rangle \quad = \langle \underbrace{(\mathrm{L}^*\mathrm{L})f}_{f^* \in \mathcal{H}'_{\mathrm{L}}}, g\rangle$

Determination of reproducing kernel

$$h(\boldsymbol{x}, \boldsymbol{y}) = \mathrm{R}\{\delta(\cdot - \boldsymbol{y})\}(\boldsymbol{x}) \quad \text{with} \quad \mathrm{R} = (\mathrm{L}^*\mathrm{L})^{-1} : \mathcal{H}'_{\mathrm{L}} \to \mathcal{H}_{\mathrm{L}}$$

- Reproducing kernel property

(i) $\quad h(\cdot, \boldsymbol{x}_0) \in \mathcal{H}_{\mathrm{L}}$ for all $\boldsymbol{x}_0 \in \mathbb{R}^d$

(ii) $\quad f(\boldsymbol{x}_0) = \langle h(\cdot, \boldsymbol{x}_0), f\rangle_{\mathcal{H}_{\mathrm{L}}}$ for all $f \in \mathcal{H}_{\mathrm{L}}$ and $\boldsymbol{x}_0 \in \mathbb{R}^d$.

$$\Rightarrow \quad \boxed{\langle h(\cdot, \boldsymbol{x}_0), h(\cdot, \boldsymbol{y}_0)\rangle_{\mathcal{H}_{\mathrm{L}}} = h(\boldsymbol{x}_0, \boldsymbol{y}_0)}$$

- Explicit determination for LSI (=convolution) operator

$\hat{L}(\boldsymbol{\omega}) = \mathcal{F}\{\mathrm{L}\delta\}(\boldsymbol{\omega})$: frequency response of operator

$$h(\boldsymbol{x}, \boldsymbol{y}) = \rho_{\mathrm{L}^*\mathrm{L}}(\boldsymbol{x} - \boldsymbol{y}) \quad \text{where} \quad \rho_{\mathrm{L}^*\mathrm{L}}(\boldsymbol{x}) = \mathcal{F}^{-1}\left\{\frac{1}{|\widehat{L}(\boldsymbol{\omega})|^2}\right\}(\boldsymbol{x})$$

Regularized least-squares fit

- Interpolation problem

  Input: set of (potentially noisy) data points: $f(\boldsymbol{x}_m) \approx y_m, \quad m = 1, \dots, M$

$$f_{\text{opt}} = \arg \min_{f \in \mathcal{H}_L} \left(\sum_{m=1}^{M} \big(y_m - f(\boldsymbol{x}_m)\big)^2 + \lambda \|\mathrm{L}f\|_{L_2}^2 \right)$$

- Outcome of representer theorem

  Unique solution $f_{\text{opt}} \in \operatorname{span}\{h(\cdot, \boldsymbol{x}_m)\}_{m=1}^{M}$

$$\Leftrightarrow \quad \exists \mathbf{a} = (a_m) \in \mathbb{R}^M \quad \text{s.t.} \quad f_{\text{opt}}(\boldsymbol{x}) = \sum_{m=1}^{M} a_m h(\boldsymbol{x}, \boldsymbol{x}_m)$$

Your task(s)

- Show that f_{opt} is a $\mathrm{L}^*\mathrm{L}$-splines with knots at $\{\boldsymbol{x}_m\}$
- Derive an (numerical) algorithm for finding the optimal coefficients $\mathbf{a} = (a_1, \dots, a_M)$
- Derive $h(\cdot, \boldsymbol{x})$ for your favourite operator L
- Implement the algorithm and present results of your data fitting (plots) illustrating the effect of the regularisation parameter λ
- **Extra**: Derive a B-spline based algorithm for $\{x_m\}_{k \in \mathbb{Z}}$ (cardinal setting)

- Controlled generation of data
  - Generate a random set of M points $x_m \in [0, T]$ (uniform poisson distribution)
  - Generate the samples $z_m = f(x_m)$ of a zero-mean Gaussian process with correlation function $\mathbb{E}\{f(x)f(y)\} = h(x, y)$

    Hint: $\mathbf{z} \sim \mathcal{N}(\mathbf{0}, \mathbf{C})$ with $[\mathbf{C}]_{m,n} = h(x_m, x_n)$
  - Generate noisy data by adding i.i.d. Gaussian noise: $y_m = f(x_m) + \epsilon$.