Linear System Theory

Lecture 4

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Outline

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2. Algorithm: Smith Form.
3. 3.1 input decoupling zeros
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   3.3 input-output decoupling zeros
   3.4 zeros
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1. Recap

system description $\}$ ξ pseudo-state

Polynomial matrix $P(s)$

$$P(s) = \begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix} \qquad P \begin{bmatrix} \xi \\ -u \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}$$

in state-space description $T(s) = sI - A$ with $s = \lambda$

and considering only $T(s)$ we can "compute" the invariant polynomials of $T(s)$ meaning that we compute the invariant poles of A.

Strict system equivalence

Equivalence throug unimodular matrices

. unimodular matrices are square matrices nonsingular whose inverse are polynomial matrices.

$$|U(s)| \in \mathbb{R}$$ (it is not a polynomial)

$$\begin{bmatrix} T_1 & U_1 \\ -V_1 & W_1 \end{bmatrix} = \begin{bmatrix} & & \\ & U_1 & \\ & & \end{bmatrix} \begin{bmatrix} T & U \\ -V & W \end{bmatrix} \begin{bmatrix} & & \\ & U_2 & \\ & & \end{bmatrix}$$

$$\begin{bmatrix} M & 0 \\ X & I \end{bmatrix} \qquad \begin{bmatrix} N & Y \\ 0 & I \end{bmatrix}$$

unimodular matrices.

2. Smith Form

By elementary row operations (resp. column) an $n \times n$ polynomial matrix $A(s)$ can be brought to upper (or lower) triangular form.

Definition: Elementary unimodular matrices

$$T_1 = \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & O & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & c & & \\ & & & & & 1 & \\ & O & & & & & 1 \\ & & & & & & \ddots \\ & & & & & & & 1 \end{bmatrix}$$

$c \in F$

$c \in \mathbb{R}$

$$T_2 = \begin{bmatrix} 1 & & & & & \\ & 1 & & & 1 & \\ & \cancel{1}\,0 & \cdots & & \cdots\ \cancel{0} & \\ & & 1 & & & \\ & & & 1 \ \ddots & 1 & \\ & 1\,\cancel{0} & & & \cancel{1} & 1 \end{bmatrix} = \begin{array}{c} \\ \\ i \\ \\ \\ j \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccc} & & & i & & j & \end{array} \\ \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 0 & \cdots & 1 & \\ & & & 1 & & & \\ & & & & \ddots & & \\ & & & & 1 & & \\ \cdots & & & 1 & \cdots & 0 & \\ & & & & & & \ddots \end{bmatrix} \end{array}$$

$$T_3 = \begin{array}{c} \\ \\ i \rightarrow \\ \\ \end{array} \begin{bmatrix} 1 & & & \\ & \ddots & & O \\ O & & \ddots & \\ \cdots\, c(s) & & & \\ O & & O & 1 \end{bmatrix}$$

$$\begin{array}{c} \uparrow \\ j \end{array}$$

single entry where a polynomial appears otherwise identity matrix

$c(s) \in \mathbb{R}(s)$

$i > j$ in the lower triangular part. (below the diagonal).

$S_1 = T_1$, $S_2 = T_2$ $S_3 = T_3^T$

Smith decomposition:

$$P(s) = \begin{bmatrix} p_{11}(s) & \cdots & \\ \vdots & & \\ & & p_{nn}(s) \end{bmatrix} \xrightarrow[\text{Form}]{\text{Smith}} \begin{bmatrix} d_1(s) & & & 0 \\ & d_2(s) & & \\ & & \ddots & \\ 0 & & & d_n(s) \end{bmatrix}$$

Algorithm:

STEP 1: Find the lowest degree polynomial in s in the 1^{st} column

$$d^{\circ} p_{j1} \leq d^{\circ} p_{k1} \quad \forall k$$

STEP 2: Bring the j^{th} line) (row) (resp. column) in the first row.

STEP 3: Use polynomial division on the elements of the 1^{st} column by dividing them with the lowest degree entry found in STEP 1.

$$p_{k1}(s) = \alpha_{k1}(s) p_{j1}(s) + r_{k1}(s)$$

$$d^{\circ} r_{k1}(s) \leq d^{\circ} p_{j1}(s)$$

Use $T_3 = \begin{array}{l} \\ k \rightarrow \end{array} \begin{bmatrix} 1 & & 0 \\ -\alpha_k & \ddots & \\ & 0 & 1 \end{bmatrix}$

Repeat STEPS 1-2-3 until the first column looks like

$$\begin{bmatrix} c_{11} & c_{12} & \cdots \\ 0 & c_{22} & \\ 0 & \vdots & \\ 0 & \vdots & \\ \vdots & & \\ 0 & & \end{bmatrix}$$

1st possibility: → iterate to have upper triangular canonical form.

or

<u>STEP 4</u> do STEPS 1-2-3 <u>on the first row</u> using matrices S_1, S_2, S_3 and <u>first column using T_1, T_2, T_3</u>

until

$$\begin{bmatrix} c_{11} & 0 & 0 & 0 & 0 \\ 0 & c_{22} & c_{23} & \cdots & \\ 0 & c_{32} & \ddots & & \\ 0 & & & \ddots & \\ 0 & & \cdots & & \end{bmatrix}$$

(Choose least order polynomial among all elements of the matrix) and bring to first position.)

<u>STEP 5</u> do steps 1 to 4 with the submatrix

→ until $\begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots & \\ & & & d_n \end{bmatrix}$ is obtained.

Definition the resulting (equivalent strict systems)

diagonal matrix is called the Smith form of the original polynomial matrix.

$$P = \begin{bmatrix} T & U \\ -V & W \end{bmatrix}$$

3. decoupling zeros, zeros, poles, i-d. z., o-d. z., i-o-d. z.

3.1 consider the matrix

3.1. i-d. zeros

input decoupling ≡ i-d.

$$\begin{pmatrix} T & U \end{pmatrix}$$

and compute its Smith Form using the algorithm.

collect all the zeros of the diagonal entries in the Smith Form.

$$\begin{bmatrix} d_1 & & \\ & \ddots & 0 \\ & & d_r \end{bmatrix} \qquad \begin{array}{c} d_i = 0 \\ i=1,\dots,r \end{array} \longrightarrow \{\beta_i\}$$

βs are the input-decoupling zeros

3.2 output-decoupling zeros (o-d. z.)

Consider $\begin{pmatrix} T \\ -V \end{pmatrix}$ in transposed form

Perform on $\begin{pmatrix} T^T & -V^T \end{pmatrix}$ a Smith decomposition using the algorithm to achieve

$$\left(\begin{array}{ccc|c} \tilde{d}_1 & & 0 & 0 \\ & \ddots & & \\ 0 & & \tilde{d}_r & \end{array}\right)$$

and compute the zeros of every $\tilde{d}_i$ to get the set of output-decoupling zeros : $\{\gamma_i\}$

3.3 input-output decoupling zeros (i-o-d. z.)

Remove the $\{\beta_i\}$ from $\underline{(T \quad U)}$ to get $\underline{(T_1 \quad U_1)}$

i.e. apply the unimodular matrices obtained to bring the $(T \quad U)$ to Smith form by inverting them using the identity in the diagonal instead of the d's

extend the unimodular matrices so that they can apply on $P(s)$

$\begin{pmatrix} T_1 \\ -V_1 \end{pmatrix}$ is therefore obtained

$\Longmapsto$ compute the Smith Form of $\begin{pmatrix} T_1^T & -V_1^T \end{pmatrix}$ and compute the zeros of the diagonal entries of the resulting Smith-Form.

$$\longrightarrow \quad \{\theta_i\}$$

Remove the $\{\theta_i\}$ from the $\{\gamma_i\}$ to find the input-output decoupling zeros. (i-o-d z.).

$$\{\delta_i\} = \{\gamma_i\} - \{\theta_i\}$$

3.4 Zeros

Simply compute the Smith-Form of $T(s)$. Let the zeros of the diagonal entries of the Smith form be $\{\eta_i\}$

3.5 Poles

$$\{\alpha_i\} = \{\eta_i\} - \{\{\beta_i, \gamma_i\} - \{\delta_i\}\}$$

$\{\beta_i\}$ i.d.

$\{\gamma_i\}$ o.d.

$\{\delta_i\}$ i.o.d

$\{\eta_i\}$ zeros

$\{\alpha_i\}$ poles of $G(s)$.

$$P(s) = \left[\begin{array}{ccc|c} I_2 & 0 & 0 & 0 \\ 0 & s^2(s+1) & s(s+2) & -s \\ 0 & 0 & s+2 & 1 \\ \hline 0 & 0 & -1 & 0 \end{array}\right]$$

$\{\eta_i\} = \{0, 0, -1, -2\}$

$\{\beta_i\} = \{0\}$

$\{\gamma_i\} = \{0, 0, -1\}$

$\{\theta_i\} = \{0, -1\}$

$\{\delta_i\} = \{0\}$

$\{\alpha_i\} = \{-2\}$.