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On strict system equivalence and similarity†

PAUL A. FUHRMANN‡

It is shown that if a natural state space model is associated with each polynomial system matrix then the similarity of the models is equivalent to the strict system equivalence (s.s.e.) of the system matrices. A minor departure from Rosenbrock's definition of s.s.e. is needed.

1. Introduction

Recently the author (1976) described a new approach to the exposition of the theory of finite dimensional linear systems over arbitrary fields. This approach has the advantage of synthesizing the abstract module approach, state-space theory and the theory of polynomial system matrices as expounded by Rosenbrock (1970). The object of this note is to show how the same methods enable us to clarify the question of strict system equivalence of two polynomial system matrices, a problem left open (Rosenbrock 1970). We do this by associating with each factorization of a proper rational transfer function, a natural state-space model as well as the polynomial system matrix. The strict system equivalence of two such polynomial matrices is equivalent in turn to the similarity of the two corresponding models. In order to achieve this we need however to modify slightly the definition of strict system equivalence. It is our hope that this note is enough to convince the reader with the naturalness of the modified definition. We restrict ourselves to discrete-time systems which has the advantage of allowing us to develop the results over arbitrary fields. We will restrict ourselves however to proper transfer functions for the maintenance of causality.

2. Preliminaries

Let F be an arbitrary field, $F[\lambda]$ the ring of polynomials over F , F^n the vector space over F , of all n -tuples of elements in F and $F^n[\lambda]$ the free module over $F[\lambda]$, of all n -tuples with polynomial entries. Thus $F^n[\lambda]$ can be identified also with the set of all vector polynomials with coefficients in F^n . Similarly $F^{m \times n}$ and $F^{m \times n}[\lambda]$ denote the set of all $m \times n$ matrices with elements in F and $F[\lambda]$ respectively. The ring $F[\lambda]$ is entire and hence its field of quotients can be constructed. This field, denoted by $F(\lambda)$, is the field of rational functions. An element f of $F(\lambda)$ is called a proper rational function if in a representation $f = p/q$ as the quotient of two polynomials we have $\deg p \leq \deg q$; f is called strictly proper if $\deg p < \deg q$. These notions are well defined, as they are clearly independent of the representation used.

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Important in the constructions to follow are the submodules and quotient modules of $F^n[\lambda]$. For the proofs of the results summarized here the reader is referred to the paper by Fuhrmann (1976).

A subset M of $F^n[\lambda]$ is a submodule of $F^n[\lambda]$ if and only if $M = DF^n[\lambda]$ for some polynomial matrix D in $F^{n \times n}[\lambda]$. A submodule is called a full submodule if in such a representation D is non-singular. If M is a full submodule and M has two representations $M = DF^n[\lambda] = EF^n(\lambda)$, then D and E differ by at most a right unimodular factor. M is a full submodule if and only if the quotient module $F^n[\lambda]/M$ is a torsion module over $F[\lambda]$. In that case $F^n[\lambda]/M$ is also a finite dimensional vector space over F . In order to have a concrete representation for the abstract quotient module $F^n[\lambda]/M = F^n[\lambda]/DF^n[\lambda]$, which consists of a set of equivalence classes, we introduce a projection operator Π in $F^n(\lambda)$. Each rational function f decomposes uniquely in the form $f = p + g$, where p is a polynomial and g a strictly proper rational function. Now we let $\Pi f = g$. We extend Π to a map $\Pi : F^n(\lambda) \rightarrow F^n(\lambda)$ by letting

$$\Pi \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} \Pi f_1 \\ \vdots \\ \Pi f_n \end{pmatrix} \quad (2.1)$$

Given a non-singular matrix D in $F^{n \times n}[\lambda]$ we define the map $\pi_D : F^n[\lambda] \rightarrow F^n[\lambda]$ by

$$\pi_D f = D \Pi(D^{-1}f) \quad \text{for } f \in F^n[\lambda] \quad (2.2)$$

Clearly π_D is a projection map, that is $\pi_D^2 = \pi_D$, $\text{Ker } \pi_D = DF^n[\lambda]$ and $K_D = \text{Im } \pi_D$ is a module over $F[\lambda]$ which is isomorphic to $F^n[\lambda]/DF^n[\lambda]$. In K_D we define a linear map $S(D)$ by

$$S(D)f = \pi_D \chi f \quad \text{for } f \in K_D \quad (2.3)$$

Here χ denotes the identity polynomial, i.e. $\chi(\lambda) = \lambda$. We refer to the class of operators defined by (2.3) as canonical models. In fact every linear transformation in a finite dimensional vector space over F is similar to a canonical model and two canonical models $S(D)$ and $S(D_1)$ are similar if and only if the polynomial matrices D and D_1 are equivalent, i.e. have the same Smith canonical form.

3. Representations and realizations

We consider now a proper rational matrix function G in $F^{m \times n}(\lambda)$. Assume G has a representation of the form

$$G(\lambda) = V(\lambda)D(\lambda)^{-1}U(\lambda) + W \quad (3.1)$$

Here we assume that W is a constant matrix, $G(\lambda) - W = V(\lambda)D(\lambda)^{-1}U(\lambda)$ is a strictly proper rational matrix and V, D, U are polynomial matrices in $F^{m \times r}[\lambda]$, $F^{r \times n}[\lambda]$ and $F^{r \times n}[\lambda]$ respectively. No assumptions concerning the left co-primeness of D and U or the right co-primeness of D and V are made.

Since G is proper rational we have a formal expansion for G in the form

$$G(\lambda) = \sum_{n=0}^{\infty} G_{-n} \lambda^{-n} \quad (3.2)$$

and of course we must have $G_0 = W$.

Consider now a discrete-time system $\{A, B, C, D\}$ in state-space representation. Thus the dynamic equations are

$$\left. \begin{aligned} x_{n+1} &= Ax_n + Bu_n \\ y_n &= Cx_n + Du_n \end{aligned} \right\} \quad (3.3)$$

and we say that $\{A, B, C, D\}$ is a realization of G if

$$G_0 = D$$

and

$$G_{-n} = CA^{n-1}B, \quad n \geq 1 \quad (3.4)$$

Two systems $\{A, B, C, D\}$ and $\{A_1, B_1, C_1, D_1\}$ are similar if there exists an invertible linear transformation R such that

$$A_1 = RAR^{-1}, \quad B_1 = RB, \quad C_1R = C \quad \text{and} \quad D_1 = D \quad (3.5)$$

Now we use the representation (3.1) as the basis for a realization of G . We choose K_D as our state space and take $S(D)$, defined by (2.3), as the generator in our realization. Define two linear transformations $B : F^n \rightarrow K_D$ and $C : K_D \rightarrow F^m$ in the following way :

$$B\xi = \pi_D(U\xi) \quad \text{for} \quad \xi \in F^n \quad (3.6)$$

and

$$Cp = (VD^{-1}p)_{-1} \quad \text{for} \quad p \in K_D \quad (3.7)$$

Here $(VD^{-1}p)_{-1}$ refers to the coefficient of λ^{-1} in a formal expansion

$$(VD^{-1}p)(\lambda) = \sum_{n=-\infty}^{\infty} (VD^{-1}p)_n \lambda^n$$

Of course, as $(VD^{-1}p)(\lambda)$ is rational, only a finite number at most of the positive indexed coefficients are non-zero.

Now we claim that the system $\{S(D), B, C, W\}$ is a realization of G . It suffices to show that

$$G_{-(n+1)} = CS(D)^n B \quad \text{for} \quad n \geq 0 \quad (3.8)$$

That (3.8) holds, follows from the next computation :

$$\begin{aligned} CS(D)^n B\xi &= CS(D)^n \pi_D U\xi \\ &= C\pi_D \chi^n \pi_D U\xi = (VD^{-1}\pi_D \chi^n \pi_D U\xi)_{-1} \\ &= (VD^{-1}\pi_D \chi^n U\xi)_{-1} = (VD^{-1}D\pi D^{-1}\chi^n U\xi)_{-1} \\ &= (V\pi D^{-1}U\chi^n \xi)_{-1} = (VD^{-1}U\chi^n \xi)_{-1} \\ &= (\chi^n (G - W)\xi)_{-1} = (\chi^n G\xi)_{-1} = G_{-(n+1)}\xi \end{aligned}$$

It follows from Theorem 6.1 of Fuhrmann (1976) and the discussion that follows that the realization $\{S(D), B, C, W\}$ constructed here is controllable if and only if D and U are left co-prime and observable if and only if D and V are right co-prime.

4. Similarity and strict system equivalence

We consider now two different representations of a proper transfer function G :

$$G = W + VD^{-1}U = W_1 + V_1D_1^{-1}U_1 \quad (4.1)$$

Here all functions are polynomial matrices, W and W_1 are such that $G - W$ and $G - W_1$ are strictly proper rational matrix functions. Thus necessarily $W = W_1$. The dimensions of D and D_1 may differ. To the two representations of G correspond two canonical state-space models $\{S(D), B, C, W\}$ and $\{S(D_1), B_1, C_1, W\}$ respectively, where B_1 and C_1 are defined by formulae analogous to (3.6) and (3.7) respectively.

Assume now that the two state-space models are similar and the similarity is given by the linear transformation $Z: K_D \rightarrow K_{D_1}$. Thus Z is a one-to-one map of K_D onto K_{D_1} , which makes the following diagram commutative.

$$\begin{array}{ccc}
 & F^m & \\
 B \swarrow & & \searrow B_1 \\
 K_D & \xrightarrow{Z} & K_{D_1} \\
 S(D) \downarrow & & \downarrow S(D_1) \\
 K_D & \xrightarrow{Z} & K_{D_1} \\
 C \swarrow & & \searrow C_1 \\
 & F^n &
 \end{array} \quad (4.2)$$

In particular $ZS(D) = S(D_1)Z$, that is Z intertwines the transformations $S(D)$ and $S(D_1)$. The structure of transformations that intertwine canonical models is known and given by Theorem 4.5 of Fuhrmann (1976). Thus there exist polynomial matrices M and M_1 such that

$$MD = D_1M_1 \quad (4.3)$$

and Z is given by

$$Zp = \pi_{D_1}Mp \quad \text{for } p \in K_D \quad (4.4)$$

Since Z is assumed invertible we must have the left co-primeness of M and D_1 and the right co-primeness of D and M_1 .

We consider next the relations between the respective input and output maps. We have $ZB = B_1$, hence for each $\xi \in F^m$

$$\begin{aligned}
 \pi_{D_1}(U_1\xi) &= \pi_{D_1}M\pi_D(U\xi) \\
 &= \pi_{D_1}(MU\xi)
 \end{aligned} \quad (4.5)$$

The last equality follows from (4.3) which is equivalent to

$$ZDF^r[\lambda] \subset D_1F^r[\lambda] \quad (4.6)$$

From (4.5) it follows that

$$\pi_{D_1}(u_1 - MU)\xi = 0 \quad \text{for } \xi \in F^m \quad (4.7)$$

and this implies the existence of a polynomial matrix Y_1 such that

$$U_1 - MU = D_1 Y_1 \quad (4.8)$$

or

$$U_1 = MU + D_1 Y_1 \quad (4.9)$$

Similarly we have $C_1 Z = C$ and more generally

$$CS(D)^n = C_1 ZS(D)^n = C_1 S(D_1)^n Z \quad (4.10)$$

Therefore we get for $p \in K_D$

$$(VD^{-1}\pi_D \chi^n p)_{-1} = (V_1 D_1^{-1} \pi_{D_1} \chi^n \pi_{D_1} M p)_{-1}$$

or

$$(V \Pi D^{-1} \chi^n p)_{-1} = (V_1 \Pi D_1^{-1} M \chi^n p)_{-1}$$

which implies in turn

$$((V - V_1 M_1) D^{-1} \chi^n p)_{-1} = 0 \quad \text{for all } n \geq 0$$

Hence $(V - V_1 M_1) D^{-1}$ is necessarily equal to some polynomial matrix X . For this X we have

$$V - V_1 M_1 = XD \quad (4.11)$$

Now, following Rosenbrock (1970), we associate with each representation of the form (3.1) of a proper rational transfer function G a polynomial system matrix

$$\begin{pmatrix} D & U \\ -V & W \end{pmatrix} \quad (4.12)$$

Assume now that for two such representations the corresponding canonical state-space representations are similar. It is clear that eqns. (4.3), (4.9) and (4.11) imply the equality

$$\begin{pmatrix} M & U \\ X & I \end{pmatrix} \begin{pmatrix} D & U \\ -V & W \end{pmatrix} = \begin{pmatrix} D_1 & U_1 \\ -V_1 & W_1 \end{pmatrix} \begin{pmatrix} M_1 & -Y_1 \\ 0 & I \end{pmatrix} \quad (4.13)$$

That $XU + W = V_1 Y_1 + W_1$ holds, follows from $W = W_1$ and the fact that

$$\begin{aligned} XU - V_1 Y_1 &= (V - V_1 M_1) D^{-1} U - V_1 D_1^{-1} (U_1 - MU) \\ &= VD^{-1} U - V_1 M_1 D^{-1} U - V_1 D_1^{-1} U_1 + V_1 D_1^{-1} MU \\ &= 0 \end{aligned}$$

as $VD^{-1} U = V_1 D_1^{-1} U_1$, both being equal to the strictly proper rational part of G , whereas the equality

$$V_1 M_1 D^{-1} U = V_1 D_1^{-1} MU$$

is a straight consequence of (4.3).

The converse result holds also. Let the two polynomial system matrices

$$\begin{pmatrix} D & U \\ -V & W \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} D_1 & U_1 \\ -V_1 & W_1 \end{pmatrix}$$

be connected through eqn. (4.13) where moreover W and W_1 are constant, M and D_1 are left co-prime and M_1 and D are right co-prime, then it follows that the canonical state-space models are similar and in particular the corresponding transfer functions $W + VD^{-1}U$ and $W_1 + V_1D_1^{-1}U_1$ have to be equal. To summarize we have proved the following theorem.

Theorem 4.1

Let $G = VD^{-1}U + W = V_1D_1^{-1}U_1 + W_1$ be two representations of the proper rational function G such that W, W_1 are constant matrices, V, D, U, V_1, D_1, U_1 are polynomial matrices for $VD^{-1}U$ and $V_1D_1^{-1}U_1$ are strictly proper rational functions. Then the two respective canonical state-space models $\{S(D), B, C, W\}$ and $\{S(D_1), B_1, C_1, W_1\}$ are similar if and only if there exist polynomial matrices, M, M_1, X and Y_1 for which M and D_1 are left co-prime, M_1 and D are right co-prime and for which (4.13) holds.

Theorem 4.1. suggests strongly that Rosenbrock's original definition of strict system equivalence should be modified. We suggest the following.

Definition 4.2

Let

$$P(\lambda) = \begin{pmatrix} D(\lambda) & U(\lambda) \\ -V(\lambda) & W \end{pmatrix} \quad \text{and} \quad P_1(\lambda) = \begin{pmatrix} D_1(\lambda) & U_1(\lambda) \\ -V_1(\lambda) & W_1 \end{pmatrix}$$

be two polynomial system matrices. We say that P and P_1 are strictly system equivalent if there exist polynomial matrices M, M_1, X and Y_1 for which M and D_1 are left co-prime, M_1 and D are right co-prime and (4.13) holds.

As a consequence of this definition of strict system equivalence Theorem 4.1 can be restated shortly in the following way.

Theorem 4.3

Two polynomial system matrices are strictly system equivalent if and only if their corresponding canonical state-space models are similar.

We want to remark that Definition 4.2 has the advantage of not restricting the matrices D and D_1 to be of the same size. Moreover, it associates strict system equivalence in a very close way with the similarity of state-space models.

Of course, we are left now with two differing notions of strict system equivalence. To avoid ambiguity Rosenbrock (1970, p. 52) suggested the use of the term unimodular strict system equivalence for the equivalence relation.

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