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INVARIANTS AND CANONICAL FORMS UNDER DYNAMIC COMPENSATION*

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Abstract. This paper is concerned with the development of a complete abstract invariant as well as a set of canonical forms under dynamic compensation for linear systems characterized by proper, rational transfer matrices. More specifically, it is shown that one can always associate with any proper rational transfer matrix, $T(s)$, a special lower left triangular matrix, $\xi_T(s)$, called the interactor. This matrix is then shown to represent an abstract invariant under dynamic compensation which, together with the rank of $T(s)$, represents a complete abstract invariant. A set of canonical forms under dynamic compensation is also developed along with appropriate dynamic compensation.

1. Introduction. The primary purpose of this paper will be to exhibit invariants and a set of canonical forms for linear dynamical systems which are equivalent under dynamic compensation. Critical to this purpose will be the introduction of a special lower triangular matrix $\xi_T(s)$ associated with any $T(s)$ in S and called the "interactor" of T. The role of the interactor in resolving the closely allied questions of exact model matching and inverse systems is also displayed.

Section 2 contains a precise description of equivalence under dynamic compensation as well as some elementary properties of this notion. The interactor $\xi_T(s)$ is introduced in § 3 and is shown to be an abstract invariant under dynamic compensation in § 4. The main results on invariants and canonical forms are also established in § 4 and some final observations are made in § 5.

2. Dynamic compensation. We begin with some definitions.

DEFINITION 2.1. Let S be the set of all proper $(p\times m)$ transfer matrices of full rank $r(=\min\{p, m\})$ with first r rows, $T_r(s)$, of rank r. Let S_+, S_- be the subsets of S given by

$$\begin{aligned} S_+ &= \{T(s)\in S \mid T(s) \quad \text{is } p\times m \quad \text{with } m-p\geqq 0\},\\ S_- &= \{T(s)\in S \mid T(s) \quad \text{is } p\times m \quad \text{with } m-p<0 \text{ and } T_m(s) \text{ of rank } m\},\end{aligned}$$

respectively.

We observe that $S_+\cap S_-=\varnothing$ and that $S_+\cup S_-=S$.

DEFINITION 2.2. Let $T(s)$ be a given $p\times m$ element of S. Then, any $m\times k$ transfer matrix $T_c(s)$ in S is called a *dynamic compensator of* $T(s)$.

The operation of $T_c(s)$ can be represented in "open loop" form by the block diagram of Fig. 1, where $y(s)=T(s)\,u(s)$ represents the (Laplace transform of the) zero state dynamical behavior of the given system and $u(s)=T_c(s)\,v(s)$ that of the compensator.

Since $T(s)T_c(s)$ is again a proper transfer matrix, it is readily shown that $T_c(s)$ can be represented in terms of input dynamics and state feedback [1], [2]. More

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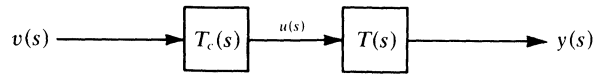

FIG. 1

precisely, if $T(s)=R(s)P^{-1}(s)$ with $R(s)$ and $P(s)$ relatively right prime polynomial matrices [3], then there are polynomial matrices $F(s)$, $G(s)$ and $L(s)$ such that $T_c(s)=P(s)P_c^{-1}(s)L(s)$ is proper with $P_c(s)=G(s)P(s)-F(s)$. Thus, the operation of $T_c(s)$ can be represented in "closed loop" form by the block diagram of Fig. 2, with $z(s)$ the (Laplace transform of the) partial state of the given system, $F(s)z(s)$ the state feedback "part", and $G(s)^{-1}L(s)$ the input dynamic "part" of

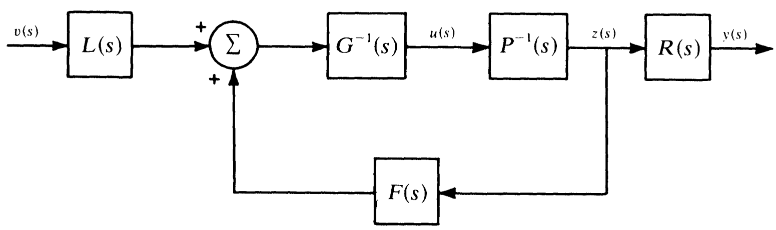

FIG. 2

$T_c(s)$ (see [1], [2]). Knowledge of the portions of a dynamical system which remain unaltered (or invariant) under a particular form of compensation is ultimately tied to a number of important questions of control system analysis and synthesis such as model matching and decoupling.

DEFINITION 2.3. If $T_1(s)$ and $T_2(s)$ are elements of S, then $T_1(s)$ and $T_2(s)$ are equivalent under dynamic compensation if

$$\begin{aligned} T_1(s)T_{1c}(s) &= T_2(s), \\ T_2(s)T_{2c}(s) &= T_1(s) \end{aligned} \tag{2.4}$$

for some $T_{1c}(s)$ and $T_{2c}(s)$ in S.

If $T_1(s)$ and $T_2(s)$ are equivalent under dynamic compensation, we write $T_1(s)E_dT_2(s)$. It is clear that E_d defines an equivalence relation on S. The main purpose of this paper is the characterization of the orbits of this equivalence relation by the determination of invariants and a set of canonical forms.

The following elementary observations are required because we deal with proper transfer matrices of different dimensions.

OBSERVATION 2.5. *If* $T_1(s)E_dT_2(s)$, *then* $T_1(s)$ *and* $T_2(s)$ *have the same rank.*

OBSERVATION 2.6. S_+ *and* S_- *are stable under dynamic compensation* (*i.e., if* $T_1(s)\in S_+$, *say, and* $T_1(s)E_dT_2(s)$, *then* $T_2(s)\in S_+$).

OBSERVATION 2.7. *If $T_1(s)$ is a $p \times m$ element of S_- and $T_1(s)E_dT_2(s)$, then $T_2(s)$ is also a $p \times m$ element of S_-, and both $T_{1c}(s)$ and $T_{2c}(s)$ are nonsingular.*

3. The interactor $\xi_T(s)$. Let $T(s)$ be an element of S. We shall, in this section, determine a unique nonsingular, lower left triangular polynomial matrix $\xi_T(s)$ associated with $T(s)$ and called the interactor of $T(s)$. The constructive procedure which will be outlined is similar to that given by Silverman [4] in the case of state space representations, although unlike Silverman's algorithm, it does yield a unique $\xi_T(s)$ for transfer matrix representations.

LEMMA 3.1. *Let $T(s)$ be an $m \times m$ element of S. Then there is a unique, nonsingular $(m \times m)$, lower left triangular polynomial matrix $\xi_T(s)$ of the form*

$$\xi_T(s) = H_T(s)\,\mathrm{diag}\,[s^{f_1}, \cdots, s^{f_m}], \tag{3.2}$$

where

$$H_T(s) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ h_{21}(s) & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ & & \cdots & 0 \\ h_{m1}(s) & h_{m2}(s) & \dots & 1 \end{bmatrix} \tag{3.3}$$

and $h_{ij}(s)$ is divisible by s (or is zero) such that

$$\lim_{s\to\infty} \xi_T(s)T(s) = K_T \tag{3.4}$$

with K_T nonsingular.

Proof. We first prove the existence of such a $\xi_T(s)$. It is well-known [2] that $T(s)$ can always be factored as the product $R(s)P^{-1}(s)$ with $R(s)$, $P(s)$ relatively right prime polynomial matrices and $P(s)$ column proper. Let $\partial_i(P) = d_i$, $i = 1, \cdots, m$, denote the column degrees of $P(s)$ and let $\sum_{i=1}^m d_i = n$. Now, $\det R(s)$ is a nonzero polynomial of degree q (since $T(s)$ is nonsingular) with $q \leqq n$ (since $T(s)$ is proper). There are unique integers μ_i, $i = 1, \cdots, m$, such that

$$\lim_{s\to\infty} s^{\mu_i}T_i(s) = \tau_i, \qquad i = 1, \cdots, m, \tag{3.5}$$

where $T_i(s)$ is the ith row of $T(s)$ and τ_i is both finite and nonzero. We define the first row $\xi_T(s)_1$ of $\xi_T(s)$ by

$$\xi_T(s)_1 = (s^{\mu_1}, 0, \cdots, 0) \tag{3.6}$$

so that

$$\lim_{s\to\infty} \xi_T(s)_1 T(s) = \xi_1 = \tau_1. \tag{3.7}$$

If τ_2 is linearly independent of ξ_1, then we set

$$\xi_T(s)_2 = (0, s^{\mu_2}, 0, \cdots, 0) \tag{3.8}$$

so that

$$\lim_{s\to\infty} \xi_T(s)_2 T(s) = \xi_2 = \tau_2. \tag{3.9}$$

On the other hand, if τ_2 and ξ_1 are linearly dependent so that $\tau_2=\alpha_1^1\xi_1$ with $\alpha_1^1\neq 0$, then we let

(3.10) $$\tilde{\xi}_T^1(s)_2=s^{\mu_2^1}[(0,s^{\mu_2},0,\cdots,0)-\alpha_1^1\xi_T(s)_1],$$

where μ_2^1 is the unique integer for which $\lim_{s\to\infty}\tilde{\xi}_T^1(s)_2T(s)=\tilde{\xi}_2^1$ is both finite and nonzero. If $\tilde{\xi}_2^1$ is linearly independent of ξ_1, then we set

(3.11) $$\xi_T(s)_2=\tilde{\xi}_T^1(s)_2$$

and note that

(3.12) $$\lim_{s\to\infty}\xi_T(s)_2T(s)=\tilde{\xi}_2^1$$

is linearly independent of ξ_1. If not, then $\tilde{\xi}_2^1=\alpha_1^2\xi_1$ and we let

(3.13) $$\tilde{\xi}_T^2(s)_2=s^{\mu_2^2}[\tilde{\xi}_T^1(s)_2-\alpha_1^2\xi_T(s)_1],$$

where μ_2^2 is the unique integer for which $\lim_{s\to\infty}\tilde{\xi}_T^2(s)_2T(s)=\tilde{\xi}_2^2$ is both finite and nonzero. If $\tilde{\xi}_2^2$ and ξ_1 are linearly independent, then we set $\xi_T(s)_2=\tilde{\xi}_T^2(s)_2$ and if not, we repeat the procedure until either linear independence is obtained or $\mu_1+\mu_2^k=n-q$.[1] In case $\mu_1+\mu_2^k=n-q$, set $f_3=0,\cdots,f_m=0$ and the corresponding $h_{ij}=0$. The remaining rows of $\xi_T(s)$ are defined recursively in an entirely analogous manner. In other words, we obtain either (i) a matrix $\xi_T(s)$ of the form (3.2) such that (3.4) is satisfied or (ii) $\xi_T(s)_1,\cdots\xi_T(s)_r$, $r\leqq m$, such that $\lim_{s\to\infty}\cdot\xi_T(s)_jT(s)=\xi_j$ with $\xi_1,\cdots,\xi_r$ linearly independent and $\sum_1^r f_i=n-q$. In case (ii), we set $f_{r+1}=0,\cdots,f_m=0$ and the corresponding $h_{ij}=0$ to obtain $\xi_T(s)$. *If* $r=m$, then

(3.14) $$\lim_{s\to\infty}\xi_T(s)T(s)=\begin{bmatrix}\xi_1\\ \xi_2\\ \vdots\\ \xi_m\end{bmatrix}=K_T$$

is nonsingular since the ξ_i are linearly independent. If $r<m$, then $\xi_T(s)T(s)=\xi_T(s)R(s)P^{-1}(s)$ is a proper transfer matrix as each step produces a proper transfer matrix. But then

(3.15) $$\partial_i(\xi_TR)=\bar{d}_i\leqq d_i=\partial_i(P)$$

so that

$$\sum_{i=1}^m\partial_i(\xi_TR)=\sum_{i=1}^m\bar{d}_i\leqq\sum d_i=n.$$

However,

(3.16) $$\begin{aligned}\text{degree }(\det\xi_TR)&=\text{degree }(\det\xi_T)=\text{degree }(\det R)\\&=n-q+q,\end{aligned}$$

[1] Note $\mu_1+\mu_2^k$ cannot exceed $n-q$ as ξ_1, $\tilde{\xi}_2^k$ are finite and nonzero.

which implies $\sum_{i=1}^{m} \partial_i(\xi_T R) = \sum_{i=1}^{m} \bar{d}_i \geqq n$. It follows that $\bar{d}_i = d_i$ and hence, that $\xi_T(s)R(s)$ is column proper with the same column degrees as $P(s)$. Thus,

$$\lim_{s\to\infty} \xi_T(s)R(s)P^{-1}(s) = K_T \tag{3.17}$$

is nonsingular [2].

We now prove uniqueness. Let $\xi_T(s) = H_T(s)\,\mathrm{diag}\,[s^{f_1}, \cdots, s^{f_m}]$ and $\hat{\xi}_T(s) = \hat{H}_T(s)\,\mathrm{diag}\,[s^{\hat{f}_1}, \cdots, s^{\hat{f}_m}]$ satisfy (3.4). Then, $\xi_T R$ and $\hat{\xi}_T R$ are column proper with $\partial_i(\xi_T R) = \partial_i(P) = \partial_i(\hat{\xi}_T R)$. It follows [2] that

$$[\xi_T R][\hat{\xi}_T R]^{-1} = H_T(s)\,\mathrm{diag}\,[s^{f_1-\hat{f}_1}, \cdots, s^{f_m-\hat{f}_m}]\hat{H}_T^{-1}(s),$$
$$[\hat{\xi}_T R][\xi_T R]^{-1} = \hat{H}_T(s)\,\mathrm{diag}\,[s^{\hat{f}_1-f_1}, \cdots, s^{\hat{f}_m-f_m}]H_T^{-1}(s)$$

are both proper. Since $H_T(s)$ and $\hat{H}_T(s)$ are of the form (3.3), $f_i = \hat{f}_i$ for $i = 1, \cdots, m$. Now, both $H_T(s)$ and $\hat{H}_T(s)$ are unimodular, lower left triangular matrices with diagonal entries 1. Moreover, $H_T(s)\hat{H}_T^{-1}(s) = U(s)$ is unimodular, proper and satisfies $\lim_{s\to\infty} U(s) = L$ with L nonsingular. Since each $h_{ij}(s)$ is divisible by s (or is zero) and each $\hat{h}_{ij}(s)$ is divisible by s (or is zero), $U(s) = I$ and $H_T(s) = \hat{H}_T(s)$. The proof of the lemma is now complete.

LEMMA 3.18. *Let $T(s)$ be a $p \times m$ element of S_+ with $p < m$. Then there is a unique $p \times p$ matrix $\xi_T(s)$ of the form*

$$\xi_T(s) = H_T(s)\,\mathrm{diag}\,[s^{f_1}, \cdots, s^{f_p}], \tag{3.19}$$

where

$$H_T(s) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ h_{21}(s) & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ h_{p1}(s) & h_{p2}(s) & \cdots & 1 \end{bmatrix} \tag{3.20}$$

and $h_{ij}(s)$ is divisible by s (or is zero) such that

$$\lim_{s\to\infty} \xi_T(s)T(s) = K_T \tag{3.21}$$

with K_T of rank p.

Proof. Let $T(s) = R(s)P^{-1}(s)$ with $R(s)$, $P(s)$ relatively right prime and $P(s)$ column proper. Then the $p \times m$ matrix $R(s)$ is of rank p and there are row vectors $r_{p+1}, \cdots, r_m$ (with polynomial entries) such that

$$R_e(s) = \begin{bmatrix} R \\ r_{p+1} \\ \vdots \\ r_m \end{bmatrix} \tag{3.22}$$

is nonsingular and $T_e(s) = R_e(s)P^{-1}(s)$ is proper (i.e., is an element of S). By virtue of Lemma 3.1, there is a $\xi_{T_e}(s)$ of the form (3.2) such that $\lim_{s\to\infty} \xi_{T_e}(s)T_e(s) = K_e$

is nonsingular. Let

$$\xi_{T_e}(s)=\left[\begin{array}{c|c}\xi_T(s) & O_{p,m-p} \\ \hline X_{m-p,p} & X_{m-p,m-p}\end{array}\right] \tag{3.23}$$

where $\xi_T(s)$ is a $p\times p$ matrix and $X_{i,j}$ is an $i\times j$ matrix. Then $\xi_T(s)$ is necessarily of the form (3.19) and

$$\lim_{s\to\infty}\xi_T(s)R(s)P^{-1}(s)=K_T \tag{3.24}$$

with K_T of rank p. The uniqueness of $\xi_{T_e}(s)$ implies that $\xi_T(s)$ is unique.

LEMMA 3.25. *Let $T(s)$ be a $p\times m$ element of S, and let $T_m(s)$ denote the nonsingular matrix consisting of the first m rows of $T(s)$. Then there is a unique $p\times p$ matrix $\xi_T(s)$ of the form*

$$\xi_T(s)=\begin{bmatrix}\xi_{T_m}(s) & 0 \\ -\gamma_1(s) & \gamma_2(s)\end{bmatrix} \tag{3.26}$$

where $\gamma_1(s)$, $\gamma_2(s)$ are relatively left prime and $\gamma_2(s)$ is a nonsingular lower left triangular matrix in Hermite normal form [3] *with monic diagonal entries such that*

$$\lim_{s\to\infty}\xi_T(s)T(s)=K_T \tag{3.27}$$

with K_T a constant matrix of rank m whose final $p-m$ rows are zero.

Proof. Let $T(s)=R(s)P^{-1}(s)$ with $R(s)$, $P(s)$ relatively right prime. Then

$$R(s)=\begin{bmatrix}R_m(s) \\ R_{p-m}(s)\end{bmatrix} \tag{3.28}$$

so that $T_m(s)=R_m(s)P^{-1}(s)$. Since $T_m(s)$ is nonsingular, $R_m(s)$ is nonsingular. It follows that there is a (unique) pair of polynomial matrices $\gamma_1(s)$, $\gamma_2(s)$ such that

$$R_{p-m}(s)R_m^{-1}(s)=\gamma_2^{-1}(s)\gamma_1(s), \tag{3.29}$$

where $\gamma_1(s)$, $\gamma_2(s)$ are relatively left prime polynomial matrices and $\gamma_2(s)$ is a nonsingular lower left triangular matrix in Hermite normal form with monic diagonal entries. However, (3.29) implies that

$$\gamma_2(s)R_{p-m}(s)-\gamma_1(s)R_m(s)=0. \tag{3.30}$$

Since $\xi_{T_m}(s)$ is unique by Lemma 3.1 and $\gamma_2^{-1}(s)\gamma_1(s)$ represents a unique factorization (since the Hermite normal form of $\gamma_2(s)$ is unique) of $R_{p-m}(s)R_m^{-1}(s)$, the matrix $\xi_T(s)$ exists and is unique.

DEFINITION 3.31. If $T(s)$ is an element of S, then $\xi_T(s)$ is called the *interactor* of $T(s)$.

We note that the interactor is defined for all proper transfer matrices in S.

We illustrate the construction of $\xi_T(s)$ in the following two examples.

Example 3.32. Let

$$T(s)=\begin{bmatrix}\dfrac{1}{s+1} & \dfrac{1}{s+2} & \dfrac{1}{s+3} \\ 0 & \dfrac{1}{s+3} & 1\end{bmatrix}.$$

Then $f_1=1$, $f_2=0$ and $\xi_T(s)_1=(s \quad 0)$ so that $\lim_{s\to\infty} \xi_T(s)_1T(s)=\xi_1=(1 \quad 1 \quad 1)=\tau_1$. Now, $\tau_2=(0 \quad 0 \quad 1)$ is linearly independent of ξ_1 and so, $\xi_T(s)_2=(0 \quad 1)$. Thus,

$$\xi_T(s)=\begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \lim_{s\to\infty} \xi_T(s)T(s)=\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}=K_T$$

with K_T a constant matrix of rank 2.

Example 3.33. Let

$$T(s)=\begin{bmatrix} \dfrac{1}{s+1} & \dfrac{1}{s+2} \\ \dfrac{1}{s+3} & \dfrac{1}{s+4} \end{bmatrix}.$$

Then $f_1=1, f_2=1$ and $\tau_1=(1 \quad 1), \tau_2=(1 \quad 1)$. So $\xi_T(s)_1=(s \quad 0)$ and τ_2 is linearly dependent on ξ_1 with $\tau_2=1\cdot\xi_1$. Thus, $\tilde{\xi}_T^1(s)_2=s[(0 \quad s)-(s \quad 0)]=(-s^2 \quad s^2)$ and $\lim_{s\to\infty} \tilde{\xi}_T^1(s)_2T(s)=\tilde{\xi}_2^1=(-2 \quad -2)=-2\cdot\xi_1$. Since $\tilde{\xi}_2^1$ depends linearly on ξ_1, we continue by setting $\tilde{\xi}_T^2(s)_2=s[(-s^2 \quad s^2)+2(s \quad 0)]=(-s^3 \quad +2s^2 \quad s^3)$. Then $\tilde{\xi}_2^2=(6 \quad 8)$ is not linearly dependent on ξ_1 and so,

$$\xi_T(s)=\begin{bmatrix} s & 0 \\ -s^3+2s^2 & s^3 \end{bmatrix},$$

$$\lim_{s\to\infty} \xi_T(s)T(s)=\begin{bmatrix} 1 & 1 \\ 6 & 8 \end{bmatrix}=K_T$$

with K_T a constant matrix of rank 2.

4. Invariants and canonical forms. We begin with some lemmas.

LEMMA 4.1. *Let $T(s)$ be a $p\times m$ element of S_+. Then there is a* (*not necessarily proper*) *$m\times p$ transfer matrix $\theta_T(s)$ such that* (i) $T(s)\theta_T(s)=I_p$ *and* (ii) $\theta_T(s)\xi_T^{-1}(s)$ *is proper.*

Proof. If $p=m$, then let $\theta_T(s)=T^{-1}(s)$. Since $\lim_{s\to\infty} \xi_T(s)T(s)=K_T$ is nonsingular, $\lim_{s\to\infty} T^{-1}(s)\xi_T^{-1}(s)=K_T^{-1}$ is nonsingular. It therefore follows that $T^{-1}(s)\xi_T^{-1}(s)$ is proper.

If $p<m$, we append $m-p$ row vectors from the standard basis to $T(s)$ to obtain a proper nonsingular $m\times m$ transfer matrix

$$T_e(s)=\begin{bmatrix} T(s) \\ E \end{bmatrix}. \tag{4.2}$$

Then

$$\xi_{T_e}(s)=\begin{bmatrix} \xi_T(s) & 0_{p,m-p} \\ 0_{m-p,p} & I_{m-p} \end{bmatrix}. \tag{4.3}$$

Let $\theta_T(s)$ be the $m\times p$ transfer matrix consisting of the first p columns of $T_e^{-1}(s)$.

Then $T(s)\theta_T(s)=I_p$. Since $T_e^{-1}(s)\xi_{T_e}^{-1}(s)$ is proper and

$$T_e^{-1}(s)=[\theta_T(s) \quad *_{-m,m-p}],$$
$$\xi_{T_e}^{-1}(s)=\begin{bmatrix}\xi_T^{-1}(s) & 0_{p,m-p}\\ 0_{m-p,p} & I_{m-p}\end{bmatrix}, \tag{4.4}$$

it follows that $\theta_T(s)\xi_T^{-1}(s)$ is proper.

THEOREM 4.5. *Let $T_1(s)$ be a $p\times m$ element of S_+ and let $T_2(s)$ be a $p\times q$ element of S_+. Then there is an element $T(s)$ of S such that*

$$T_1(s)T(s)=T_2(s) \tag{4.6}$$

if and only if $\xi_{T_1}(s)\xi_{T_2}^{-1}(s)$ is proper.

Proof. If $\xi_{T_1}(s)\xi_{T_2}^{-1}(s)$ is proper, then $\xi_{T_1}(s)T_2(s)=[\xi_{T_1}(s)\xi_{T_2}^{-1}(s)][\xi_{T_2}(s)T_2(s)]$ is proper. Hence, $T(s)=\theta_{T_1}(s)T_2(s)=[\theta_{T_1}(s)\xi_{T_1}^{-1}(s)][\xi_{T_1}(s)T_2(s)]$ is proper. But $T_1(s)T(s)=[T_1(s)\theta_{T_1}(s)]T_2(s)=I_pT_2(s)=T_2(s)$.

On the other hand, if there is an element $T(s)$ of S such that $T_1(s)T(s)=T_2(s)$, then $\xi_{T_1}(s)T_2(s)=[\xi_{T_1}(s)T_1(s)]T(s)$ is proper. But $\xi_{T_1}(s)\xi_{T_2}^{-1}(s)=\xi_{T_1}(s)I_p\xi_{T_2}^{-1}(s)=\xi_{T_1}(s)[T_2(s)\theta_{T_2}(s)]\xi_{T_2}^{-1}(s)=[\xi_{T_1}(s)T_2(s)][\theta_{T_2}(s)\xi_{T_2}^{-1}(s)]$ is then a proper transfer matrix.

It is of interest to note that Theorem 4.5 represents a direct resolution to the question of existence of solutions to the well-known model matching problem, which has recently been expanded somewhat and termed the "minimal design problem".

COROLLARY 4.7. *$\xi_T(s)$ is an abstract invariant for E_d on S_+.*

Proof. Suppose that $T_1(s)E_dT_2(s)$ with $T_1(s)\in S_+$ (hence, by Observation 2.6, $T_2(s)\in S_+$). Then Theorem 4.5 implies that both $\xi_{T_1}(s)\xi_{T_2}^{-1}(s)$ and $\xi_{T_2}(s)\xi_{T_1}^{-1}(s)$ are proper $p\times p$ transfer matrices. In view of the uniqueness part of the proof of Lemma 3.1, it follows that $\xi_{T_1}(s)=\xi_{T_2}(s)$.

LEMMA 4.8. *The invariant $\xi_T(s)$ is complete on S_+.*

Proof. Let $T_1(s)$ and $T_2(s)$ be elements of S_+ such that $\xi_{T_1}(s)=\xi_{T_2}(s)$. If G is a constant $m\times p$ matrix such that $\xi_{T_1}G$ is nonsingular, then $\xi_{T_1}(s)T_1(s)G$ is a $p\times p$ element of S with $\lim_{s\to\infty}\xi_{T_1}(s)T_1(s)G=\xi_{T_1}G$ nonsingular. Thus, $[\xi_{T_1}(s)T_1(s)G]^{-1}$ is in S and so is

$$T_{1c}(s)=G[\xi_{T_1}(s)T_1(s)G]^{-1}\xi_{T_2}(s)T_2(s). \tag{4.9}$$

Since $\xi_{T_1}(s)=\xi_{T_2}(s)$, $T_{1c}(s)=G[T_1(s)G]^{-1}T_2(s)$ and

$$T_1(s)T_{1c}(s)=[T_1(s)G][T_1(s)G]^{-1}T_2(s)=T_2(s). \tag{4.10}$$

Similarly, there is a $T_{2c}(s)$ is S with $T_2(s)T_{2c}(s)=T_1(s)$ and so, $T_1(s)E_dT_2(s)$.

We note that Theorem 4.5 has a number of interesting consequences, such as Corollary 4.7, as well as the following corollaries.

COROLLARY 4.11. *Let $T(s)$ be an element of S_+. Then $T(s)$ has a proper right inverse $T_{ri}(s)$ if and only if $\xi_T(s)=I$.*

Proof. $T(s)T_{ri}(s)=I$ if and only if $\xi_T(s)\xi_I^{-1}(s)$ is proper. But $\xi_I(s)=I_p$ and $\xi_T(s)$ is proper if and only if $\xi_T(s)=I$.

COROLLARY 4.12. *Let $T(s)$ be an element of S_-. Then $T(s)$ has a proper left inverse if and only if $\xi_{T'_m}(s)=I$ (where $T'(s)$ is the transpose of $T(s)$).*

We return now to our study of invariants and canonical forms.

DEFINITION 4.13. *If $T(s)\in S$, let ρ_T denote the rank of $T(s)$, and let $S_-, q=\{T(s)\in S_- | \rho_T=q\}$.*

LEMMA 4.14. *$\xi_T(s)$ is an abstract invariant for E_d on S_-.*

Proof. Suppose that $T_1(s)E_dT_2(s)$ with $T_1(s)$, $T_2(s)$ in S_-. Then, clearly, $T_{1m}(s)E_dT_{2m}(s)$ and so, by Corollary 4.7, $\xi_{T_{1m}}(s)=\xi_{T_{2m}}(s)$.

By virtue of Observation 2.7, since $[-\gamma_{21}(s), \gamma_{22}(s)]T_1(s)T_{1c}(s)=[-\gamma_{21}(s), \gamma_{22}(s)]T_2(s)=0$, we have

$$-\gamma_{21}(s)R_{1m}(s)+\gamma_{22}(s)R_{1p-m}(s)=0. \tag{4.15}$$

But then $\gamma_{22}^{-1}(s)\gamma_{21}(s)=\gamma_{12}^{-1}(s)\gamma_{11}(s)$ are both relatively left prime factorizations of the *same* transfer matrix with both $\gamma_{22}(s)$ and $\gamma_{12}(s)$ lower left triangular matrices in (unique) Hermite normal form with monic diagonal entries. Thus, $\gamma_{22}(s)=\gamma_{12}(s)$ and $\xi_{T_1}(s)=\xi_{T_2}(s)$.

LEMMA 4.16. *$\xi_T(s)$ is complete on S_-, q.*

Proof. Let $T_1(s)$ and $T_2(s)$ be elements of S_-, q such that $\xi_{T_1}(s)=\xi_{T_2}(s)$. Then $\xi_{T_{1q}}(s)=\xi_{T_{2q}}(s)$ and

$$\begin{aligned} T_{1c}(s)&=[\xi_{T_{1q}}(s)T_{1q}(s)]^{-1}[\xi_{T_{2q}}(s)T_{2q}(s)]\\ &=T_{1q}^{-1}(s)T_{2q}(s)\end{aligned} \tag{4.17}$$

is an element of S. But

$$\begin{aligned} \xi_{T_1}(s)T_1(s)T_{1c}(s)&=\begin{bmatrix}\xi_{T_{1q}}(s)T_{1q}(s)\\ 0\end{bmatrix}T_{1q}^{-1}(s)T_{2q}(s)\\ &=\begin{bmatrix}\xi_{T_{2q}}(s)\\ 0\end{bmatrix}T_{2q}(s)\\ &=\xi_{T_2}(s)T_2(s)=\xi_{T_1}(s)T_2(s)\end{aligned} \tag{4.18}$$

and so, $T_1(s)T_{1c}(s)=T_2(s)$. Similarly, there is a $T_{2c}(s)$ in S with $T_2(s)T_{2c}(s)=T_1(s)$ and so, $T_1(s)E_dT_2(s)$. We now state the main result of this paper.

THEOREM 4.19. *Let ψ be the function on S given by*

$$\psi(T(s))=(\rho_T, \xi_T(S)). \tag{4.20}$$

Then ψ is a complete abstract invariant for equivalence under dynamic compensation.

Proof. By virtue of Observation 2.5, Corollary 4.7 and Lemma 4.14, ψ is an invariant.

As for the completeness of ψ, it will be sufficient in view of Lemmas 4.8 and 4.16 to show that if $\psi(T_1(s))=\psi(T_2(s))$, then either $T_1(s)$ and $T_2(s)$ are in S_+ or $T_1(s)$ and $T_2(s)$ are in S_-. Suppose that $\psi(T_1(s))=\psi(T_2(s))$ and that (say) $T_1(s)$ is a $p_1\times m_1$ element of S_+ and $T_2(s)$ is a $p_2\times m_2$ element of S_-. Then $\rho_{T_1}=p_1\leqq m_1$ and $\rho_{T_2}=m_2<p_2$. But $p_1=m_2<p_2$ and so the $p_1\times p_1$ matrix $\xi_{T_1}(s)$ could not equal the $p_2\times p_2$ matrix $\xi_{T_2}(s)$. Thus, the theorem is proved.

It is important to note that this theorem establishes the fact that any two dynamical systems (with transfer matrices in S) are equivalent under dynamic compensation if and only if their transfer matrices have equal rank and their interactors are equal.

DEFINITION 4.21. A subset C of S is called *a set of canonical forms for S under E_d* if, for each $T(s)$ in S, there is a unique $C_T(s)$ in C with $T(s)E_dC_T(s)$.

Let $C_+ = \{\xi_T^{-1}(s) | T(s) \in S_+\}$ and

$$C_- = \left\{ \begin{bmatrix} \xi_{T_m}^{-1}(s) \\ T_{p-m}(s)T_m^{-1}(s)\xi_{T_m}^{-1}(s) \end{bmatrix} \middle| T(s) \in S_- \right\}.$$

We then have the following theorem.

THEOREM 4.22. *If $C = C_+ \cup C_-$, then C is a set of canonical forms for S under E_d.*

Proof. If $T(s) \in S_+$, we set

$$T_c(s) = G[\xi_T(s)T(s)G]^{-1},$$

where G is any $m \times p$ constant matrix such that $\xi_T G$ is nonsingular. Then $T(s)T_c(s) = \xi_T^{-1}(s)$ and $T(s) = \xi_T^{-1}(s)T_c(s)^{-1}$ so that $T(s)E_d\xi_T(s)^{-1}$.

If $T(s) \in S_-$, we set

$$T_c(s) = T_m^{-1}(s)\xi_{T_m}^{-1}(s)$$

so that $T(s)T_c(s) = C_T(s)$ and $T(s) = C_T(s)T_c(s)^{-1}$ (i.e. $T(s)E_dC_T(s)$).

Example 4.23. Let $T_1(s)$ be the element of S given by

$$T_1(s) = \begin{bmatrix} \dfrac{1}{s+2} & 0 \\ \dfrac{2}{s+3} & \dfrac{s+1}{s+3} \\ \dfrac{1}{(s+2)(s+3)} & \dfrac{1}{s+3} \\ \dfrac{2s+7}{(s+2)(s+3)} & \dfrac{1}{s+3} \end{bmatrix},$$

and let $T_2(s)$ be the element of S given by

$$T_2(s) = \begin{bmatrix} \dfrac{1}{s+1} & 0 \\ \dfrac{s^2+6s+7}{(s+1)(s+3)} & 1 \\ \dfrac{s+4}{(s+1)(s+3)} & \dfrac{1}{s+1} \\ \dfrac{3s+10}{(s+1)(s+3)} & \dfrac{1}{s+1} \end{bmatrix}.$$

Are $T_1(s)$ and $T_2(s)$ equivalent under dynamic compensation? Since both $T_1(s)$ and $T_2(s)$ are of rank 2, we need only examine $\xi_{T_1}(s)$ and $\xi_{T_2}(s)$. Since

$$\lim_{s\to\infty}\begin{bmatrix} s & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix} \frac{1}{s+2} & 0 \\ \frac{2}{s+3} & \frac{s+1}{s+3}\end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}$$

and

$$\lim_{s\to\infty}\begin{bmatrix} s & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{s^2+6s+7}{(s+1)(s-3)} & 1\end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1\end{bmatrix}$$

we have $\xi_{T_1,2}(s) = \xi_{T_2,2}(s)$. To determine the remaining rows of $\xi_{T_1}(s)$ and $\xi_{T_2}(s)$, we note that

$$T_1(s) = \begin{bmatrix} R_{11}(s) \\ R_{12}(s)\end{bmatrix} P_1^{-1}(s),$$

$$T_2(s) = \begin{bmatrix} R_{21}(s) \\ R_{22}(s)\end{bmatrix} P_2^{-1}(s),$$

where

$$R_{11}(s) = \begin{bmatrix} 1 & 0 \\ 1 & s+1\end{bmatrix},\quad R_{12}(s) = \begin{bmatrix} 0 & 1 \\ 2 & 1\end{bmatrix},\quad P_1(s) = \begin{bmatrix} s+2 & 0 \\ -1 & s+3\end{bmatrix}$$

and

$$R_{21}(s) = \begin{bmatrix} s+3 & 0 \\ s^2+6s+7 & s+1\end{bmatrix},\quad R_{22}(s) = \begin{bmatrix} s+4 & 1 \\ 3s+10 & 1\end{bmatrix},$$

$$P_2(s) = \begin{bmatrix} (s+1)(s+3) & 0 \\ 0 & s+1\end{bmatrix},$$

respectively. Thus,

$$R_{12}(s)R_{11}^{-1}(s) = \begin{bmatrix} -\frac{1}{s+1} & \frac{1}{s+1} \\ \frac{2s+1}{s+1} & \frac{1}{s+1}\end{bmatrix} = \gamma_{12}^{-1}(s)\gamma_{11}(s),$$

$$R_{22}(s)R_{21}^{-1}(s) = \begin{bmatrix} -\frac{1}{s+1} & \frac{1}{s+1} \\ \frac{2s+1}{s+1} & \frac{1}{s+1}\end{bmatrix} = \gamma_{22}^{-1}(s)\gamma_{21}(s),$$

where

$$\gamma_{12}(s)=\gamma_{22}(s)=\begin{bmatrix} s+1 & 0 \\ -1 & 1 \end{bmatrix},$$
$$\gamma_{11}(s)=\gamma_{21}(s)=\begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}$$

and

$$\xi_{T_1}(s)=\xi_{T_2}(s)=\begin{bmatrix} s & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & s+1 & 0 \\ -2 & 0 & -1 & 1 \end{bmatrix}.$$

In other words, $T_1(s)E_dT_2(s)$.

The unique[2] dynamic compensator, $T_{1c}(s)$, which equates the two systems is now given by (4.17), i.e.,

$$T_{1c}(s)=T_{1q}^{-1}(s)T_{2q}(s)$$
$$=\begin{bmatrix} \dfrac{s+2}{s+1} & 0 \\ \dfrac{s+3}{s+1} & \dfrac{s+3}{s+1} \end{bmatrix},$$

and $C_{T_1}(s)=C_{T_2}(s)$, the canonical form for both $T_1(s)$ and $T_2(s)$ is given by

$$C_{T_1}(s)=C_{T_2}(s)=\begin{bmatrix} \dfrac{1}{s} & 0 \\ 0 & 1 \\ \dfrac{-1}{s^2+s} & \dfrac{1}{s+1} \\ \dfrac{2s+1}{s^2+3s} & \dfrac{1}{s+1} \end{bmatrix}.$$

5. Concluding remarks. We have now exhibited a complete abstract invariant, $\psi(T(s))=(\rho_T, \xi_T(s))$ for transfer matrix equivalence under dynamic compensation; i.e. we have shown that for systems characterized by full rank, proper transfer matrices, the rank and the interactor determine equivalence under dynamic compensation.

The relevance of this observation with respect to the question of exact model matching and proper right inverses was also shown.

In establishing completeness, explicit expressions are obtained for the requisite dynamic compensators. We further determined a set of canonical forms for the

[2] One can readily establish that the dynamic compensators which equate two equivalent systems in S_- are unique.

class of systems considered and developed the explicit compensators which produce the canonical forms.

Subsequent investigations will build on the results presented here, and will employ the interactor to resolve numerous related questions; e.g. the development of complete abstract invariants for system equivalence under state feedback compensation, the derivation of new and direct procedures for (dynamically and triangularly) decoupling systems via both dynamic and state feedback compensation, and more efficient resolutions to model matching via both stable and minimal order compensation (the minimal design problem).

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