INT. J. CONTROL, 1982, VOL. 35, NO. 4, 701–725

Structure and Smith–MacMillan form of a rational matrix at infinity

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The use of special row and column operations in the reduction of a rational matrix to its Smith–MacMillan form at infinity is investigated. The connections between this reduction procedure and the valuation approach are established. A graphical method for finding the Smith–MacMillan form of a rational matrix at infinity from its Bode magnitude array, and some new results on realization theory for polynomial matrices are presented.

1. Introduction

For the past decade it has been accepted that the most satisfactory definition of the finite poles and the finite zeros of a rational matrix $T(s)$ is via its Smith–MacMillan form. In order that the finite pole–zero structure of $T(s)$ be preserved during the reduction process to the Smith–MacMillan form, row and column operations represented by polynomial matrices with no finite poles or finite zeros (i.e. unimodular matrices) are used. It has been noted (Kailath 1980) that these (unimodular) operations will in general, destroy the pole–zero structure of $T(s)$ at infinity since they may have poles and zeros there.

If the pole–zero structure at infinity is required, it is usual to apply a bilinear transform

$$s=\frac{\alpha w+\beta}{\gamma w+\delta} \quad (\gamma\neq 0 \text{ and } \alpha\delta-\beta\gamma\neq 0)$$

which has the effect of sending the point $s=\alpha/\gamma$ to $w=\infty$, and bringing the point $s=\infty$ to $w=-\delta/\gamma$ where it may be studied in terms of the Smith–MacMillan form of $\tilde{T}(w)$

$$\tilde{T}(w)=T\left(\frac{\alpha w+\beta}{\gamma w+\delta}\right)$$

Usually the constants α, β, γ, δ, which may be chosen arbitrarily, are selected so that the point $s=\alpha/\gamma$ is a *regular* point, i.e. there are no poles or zeros of $T(s)$ there.

This type of approach to studying the point at infinity becomes difficult to use if $T(s)$ has dimension larger than say 3×3, or if the individual entries of $T(s)$ contain high powers of s. Since the pole–zero structure at infinity is important for a number of reasons such as studying the high gain and high frequency behaviour of systems, an improved calculation procedure for finding the Smith–MacMillan form of a rational matrix at infinity is sought.

Received 6 November 1981.

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If a rational matrix $T(s)$ is related to another rational matrix $S(s)$ by

$$T(s)=P(s)S(s)Q(s),\ (P(s),\ Q(s) \text{ square non-singular rational matrices})$$

then $T(s)$ and $S(s)$ have the same Smith–MacMillan form at $s=s_0$ (including $s_0=\infty$) if $P(s)$ and $Q(s)$ have no poles or zeros there. (See Van Dooren *et al.* 1979 for a proof.) This result means that one could also find the Smith–MacMillan form at infinity using elementary operations if their matrix representations have no poles or zeros at infinity. Such a set of elementary operations are represented by a special kind of *square proper rational* matrices having precisely this desired property.

It will be demonstrated in the sequel how these elementary operations may be used to reduce a rational matrix to its Smith–MacMillan form at infinity. The link between these operations and the valuation theory approach (Kailath 1980, Verghese 1978, Verghese and Kailath 1981) is made clear. The authors believe that the above approach to finding the Smith–MacMillan form at infinity is easier to use than the valuation approach when the matrix is sparse and of large dimension. This is largely due to the rapid increase in the number of minors of different orders which have to be considered in the latter approach.

Section 2 of the paper deals with notation and a number of definitions which will be used during the ensuing analysis and discussion.

Section 3 introduces the aforementioned elementary (bi-proper) operations and then shows how these may be used to find the Smith–MacMillan form of a rational matrix at infinity. It is shown that the Smith–MacMillan form at infinity is a canonical form for a suitably defined equivalence relation. A graphical method of finding this canonical form from the Bode magnitude array of $T(s)$ is also discussed. Whenever possible, arguments are illustrated with numerical examples.

Section 4 presents some new results on the structure of quadruples (J, I, B, C) which are strongly irreducible (Verghese *et al.* 1979, 1981) realizations of polynomial matrices. It is shown that if a polynomial matrix has more rows than columns (or is square) and it is *column proper*, or it has more columns than rows (or is square) and it is *row proper* then a strongly irreducible realization may be written down.

2. Background mathematics and notation

Let $\mathbb{R}$ be the field of reals, $\mathbb{R}[s]$ the ring of polynomials with coefficients in $\mathbb{R}$, and $\mathbb{R}(s)$ the field of rational functions $t(s)=\alpha(s)/\beta(s)$, $\alpha(s)$, $\beta(s)\in\mathbb{R}[s]$, $\beta(s)\neq 0$.

A *discrete valuation* (Atiyah and Macdonald 1969) on $\mathbb{R}(s)$ is a mapping $\delta:\ \mathbb{R}(s)\to\mathbb{Z}\cup\{+\infty\}$ ($\mathbb{Z}$: the ring of integers) such that

$$\delta(t_1(s)t_2(s))=\delta(t_1(s))+\delta(t_2(s)) \tag{1}$$

$$\delta(t_1(s)+t_2(s))\geqslant \min\{\delta(t_1(s)),\ \delta(t_2(s))\} \quad \text{for all } t_1(s),\ t_2(s)\in\mathbb{R}(s) \tag{2}$$

$$\delta(0):\ =+\infty \tag{3}$$

Now if $t(s)=\alpha(s)/\beta(s)\in\mathbb{R}(s)$ and we take

$$\delta(t(s)):=\deg\beta(s)-\deg\alpha(s)=q\in\mathbb{Z}\cup\{+\infty\} \tag{4}$$

it is clear that the function $\delta(\cdot)$ defined in (4) satisfies (1), (2) and (3) (where $\deg(0):=-\infty$), and it therefore serves as a discrete valuation on $\mathbb{R}(s)$.

A $t(s) \in \mathbb{R}(s)$ is now defined to be a *proper* rational function if $\delta(t(s)) \geq 0$, i.e. if $\deg \beta(s) \geq \deg \alpha(s)$. It is clear that the set of proper rational functions $\mathbb{R}_{pr}(s)$ endowed with the operations of addition and multiplication is a commutative ring with unity element (the real number 1) and no zero divisors. $\mathbb{R}_{pr}(s)$ is therefore an integral domain.

The *units* in $\mathbb{R}_{pr}(s)$ are those proper rational functions $t(s) \in \mathbb{R}_{pr}(s)$ for which there exists a $t'(s) \in \mathbb{R}_{pr}(s)$ such that $t(s)t'(s) = 1$. This implies that $t(s) = \alpha(s)/\beta(s) \in \mathbb{R}_{pr}(s)$ is a unit if and only if $\deg \alpha(s) = \deg \beta(s)$ and therefore $\delta(t(s)) = 0$. The units in $\mathbb{R}_{pr}(s)$ we call *bi-proper* rational functions.

Let $t(s) = \alpha(s)/\beta(s) \in \mathbb{R}(s)$ and write

$$t(s) = \frac{\alpha(s)}{\beta(s)} = \frac{\alpha(s)}{\beta(s)} s^q s^{-q} = u(s)s^{-q}, \quad (q = \deg \beta(s) - \deg \alpha(s)) \tag{5}$$

from which it is seen that every $t(s) \in \mathbb{R}(s)$ can be written as a product of a unit $u(s)$ in $\mathbb{R}_{pr}(s)$ and a factor s^{-q} where $q = \delta(t(s))$.

Example 1

(*a*) Let $t(s) = (s+1)/s(s+2)(s+3)$, then $\delta(t(s)) = 3 - 1 = 2 = q$ and $t(s)$ can be written as : $t(s) = [s(s+1)/(s+2)(s+3)](1/s^2)$.

(*b*) If $t(s) = (s+2)(s+3)/(s+1)$, then $\delta(t(s)) = 1 - 2 = -1 = q$ and $t(s)$ can be written as : $t(s) = [(s+2)(s+3)/s(s+1)]s$.

Now for every pair $t_1(s), t_2(s) \not\equiv 0$ of rational functions there exists a proper rational function $p(s) \in \mathbb{R}_{pr}(s)$ and a rational function $r(s) \in \mathbb{R}(s)$ such that

$$t_1(s) = t_2(s)p(s) + r(s) \tag{6}$$

and either $r(s) \equiv 0$ or else $\delta(r(s)) < \delta(t_2(s))$. In particular we have

(i) if $q_1 = \delta(t_1(s)) \geq \delta(t_2(s)) = q_2$ and $t_i(s) = u_i(s)s^{-q_i}$, $i = 1, 2$, then

$$p(s) = u_2^{-1}(s)u_1(s)s^{-(q_1-q_2)}$$

and $r(s) \equiv 0$, and

(ii) if $q_1 < q_2$ then $p(s) = 0$ and $r(s) = t_1(s)$.

Definition 1

Given two rational functions $t_1(s), t_2(s) \not\equiv 0$ then we say that $t_1(s)$ *is divisible by* $t_2(s)$ if $\delta(t_1(s)) \geq \delta(t_2(s))$. (Note that if $t_1(s)$ is divisible by $t_2(s)$ then the 'quotient' $p(s)$ is proper and the division is 'exact', i.e. $r(s) = 0$.)

Example 2

(*a*) Let $t_1(s) = s^2$, $\delta(t_1(s)) = -2 = q_1$, $t_2(s) = s^3$, $\delta(t_2(s)) = -3 = q_2$. Then $t_1(s) = s^2 = t_2(s)p(s) = s^3(1/s)$ where $\delta(p(s)) = \delta(1/s) = 1$, i.e. $p(s)$ is proper.

(*b*) Let $t_1(s) = 1/s+1$, $\delta(t_1(s)) = 1 = q_1$ and $t_2(s) = s^3$, $\delta(t_2(s)) = -3 = q_2$ so that $q_1 > q_2$. Writing $t_1(s) = [s/(s+1)](1/s) = u_1(s)s^{-q_1}$ and $t_2(s) = 1\ s^3 = u_2(s)s^{-q_2}$ then $p(s) = u_2^{-1}(s)u_1(s)s^{-(q_1-q_2)} = [s/(s+1)](1/s^4) = 1/(s+1)s^3$.

We now note that for the integral domain of proper rational functions $\mathbb{R}_{pr}(s)$ the discrete valuation defined in (4) serves as a 'stathm' (MacDuffee 1933) (or 'degree') and so (6) describes as Euclidean division algorithm. Thus $\mathbb{R}_{pr}(s)$ is a Euclidean ring and therefore a principal ideal ring.

Consider now the set $\mathbb{R}_{pr}{}^{p\times m}(s)$ of $p\times m$ matrices with elements in $\mathbb{R}_{pr}(s)$. A matrix $T(s)\in\mathbb{R}_{pr}{}^{p\times m}(s)$ is called a *proper rational matrix*. For $p=m$, $\mathbb{R}_{pr}{}^{p\times p}(s)$ is a ring. The units in $\mathbb{R}_{pr}{}^{p\times p}(s)$ are those $p\times p$ proper rational matrices $U(s)$ for which there exists a $p\times p$ proper rational matrix $U'(s)\in\mathbb{R}_{pr}{}^{p\times p}(s)$ such that $U(s)U'(s)=I_p$. The units $U(s)\in\mathbb{R}_{pr}{}^{p\times p}(s)$ we call $\mathbb{R}_{pr}(s)$-*unimodular* or *bi-proper* rational matrices.

It can easily be proved (MacDuffee 1933, Ch. 3, Thm. 20.1) that $U(s)\in\mathbb{R}_{pr}{}^{p\times p}(s)$ is a biproper matrix if and only if det $U(s)$ is a unit in $\mathbb{R}_{pr}(s)$ (i.e. a bi-proper rational function). It can also be verified that $U(s)\in\mathbb{R}_{pr}{}^{p\times p}(s)$ is a bi-proper rational matrix if and only if $\lim_{s\to\infty} U(s)=E_U\in\mathbb{R}^{p\times p}$ where det $E_U\neq 0$.

In system theoretic terms the above equivalent conditions imply that if $A_U\in\mathbb{R}^{n\times n}$, $B_U\in\mathbb{R}^{n\times p}$, $C_U\in\mathbb{R}^{p\times n}$, $E_U\in\mathbb{R}^{p\times p}$ is a canonical (minimal) realization of a bi-proper rational matrix $U(s)\in\mathbb{R}_{pr}{}^{p\times p}(s)$ then $A_U-B_UE_U{}^{-1}C_U$, B_UE_U, $-C_UE_U{}^{-1}$, $E_U{}^{-1}$ represents a minimal realization of the *unique* inverse $U'(s)=U(s)^{-1}\in\mathbb{R}_{pr}{}^{p\times p}(s)$ of $U(s)$.

3. Smith–MacMillan form of a rational matrix at infinity

As in standard discussions on the classical Smith form (Gantmacher 1959), we begin by introducing three elementary column or row operations on a rational matrix $T(s)\in\mathbb{R}^{p\times m}(s)$. These are :

1(a). Interchanging any two columns of $T(s)$. This is achieved by multiplying $T(s)$ on the right by the $m\times m$ bi-proper rational matrix

$$T_{R1}=\begin{array}{cl} \left[\begin{array}{ccccccccccc} 1 &&&&&&&&&& \\ &\ddots&&&&&&&&& \\ &&1&&&&&&&& \\ &&&0&\cdots&\cdots&1&&&& \\ &&&\vdots&1&&\vdots&&&& \\ &&&\vdots&&\ddots&\vdots&&&& \\ &&&\vdots&&1&\vdots&&&& \\ &&&1&\cdots&\cdots&0&&&& \\ &&&&&&&1&&& \\ &&&&&&&&\ddots&& \\ &&&&&&&&&&1 \end{array}\right] & \begin{array}{l} \\ \\ \\ \leftarrow \text{row } i \\ \\ \\ \\ \leftarrow \text{row } j \\ \\ \\ \\ \end{array} \\ \begin{array}{ccccccccccc} &&&\uparrow&&&\uparrow&&&& \\ &&&\text{col } i&&&\text{col } j&&&& \end{array} & \end{array} \qquad (7)$$

1(b). Interchanging only two rows of $T(s)$ by multiplying on the left by the $p\times p$ bi-proper rational matrix T_{L1} which has exactly the same structure as T_{R1}.

2(a). Multiplying column i by a bi-proper rational function (unit) $u(s)\in\mathbb{R}_{pr}(s)$. This is achieved by multiplying $T(s)$ on the right by the $m\times m$ bi-proper rational matrix

$$T_{R2}(s)=\begin{array}{cc}\left[\begin{array}{ccccccc}1&&&&&&\\&\ddots&&&&&\\&&1&&&&\\&&&u(s)&&&\\&&&&1&&\\&&&&&\ddots&\\&&&&&&1\end{array}\right] & \begin{array}{l}\\ \\ \\ \leftarrow \text{row } i\\ \\ \\ \\ \end{array}\\ \begin{array}{ccccccc}&&&\uparrow\ \text{col } i&&&\end{array} & \end{array}\tag{8}$$

2(*b*). Multiplying row i by a unit ; by multiplying $T(s)$ on the left by the $p \times p$ bi-proper rational matrix $T_{L2}(s)$ which has exactly the same structure as $T_{R2}(s)$.

3(*a*). Adding to column i a multiple by an element $t(s) \in \mathbb{R}_{pr}(s)$ of column j. This is achieved by multiplying $T(s)$ on the right by the $m \times m$ bi-proper rational matrix $T_{R3}(s)$ given by

$$T_{R3}(s)=\begin{array}{cc}\left[\begin{array}{ccccccccc}1&&&&&&&&\\&\ddots&&&&&&&\\&&1&&&&&&\\&&\vdots&1&&&&&\\&&\vdots&&\ddots&&&&\\&&\vdots&&&1&&&\\&&t(s)&\cdots&\cdots&\cdots&1&&\\&&&&&&&\ddots&\\&&&&&&&&1\end{array}\right] & \begin{array}{l}\\ \\ \leftarrow \text{row } i\\ \\ \\ \\ \leftarrow \text{row } j\\ \\ \\ \end{array}\end{array}\tag{9}$$

3(*b*). Adding to row i a multiple by an element $t(s)$ of row j : by multiplying $T(s)$ on the left by the $p \times p$ bi-proper rational matrix

$$T_{L3}(s)=\begin{array}{c}\left[\begin{array}{ccccccccc}1&&&&&&&&\\&\ddots&&&&&&&\\&&1&\cdots&\cdots&\cdots&t(s)&&\\&&&1&&&\vdots&&\\&&&&\ddots&&\vdots&&\\&&&&&1&\vdots&&\\&&&&&&1&&\\&&&&&&&\ddots&\\&&&&&&&&1\end{array}\right]\\ \begin{array}{ccccccccc}&&\uparrow\ \text{col } i&&&&\uparrow\ \text{col } j&&\end{array}\end{array}\tag{9'}$$

These row and column operations we call *elementary* operations, and their matrix representations given by (7), (8), (9) or (9)′ are called *elementary matrices*.

We can now state

Theorem 1

Let $T(s) \in \mathbb{R}^{p\times m}(s)$, $\text{rank}_{\mathbb{R}(s)} T(s) = \min(p, m)$. Then by means of left elementary operations, $T(s)$ can be brought to one of the following upper right triangular forms

$$\begin{bmatrix} t_{11}(s) & t_{12}(s) & \dots & t_{1p}(s) & \dots & t_{1m}(s) \\ 0 & t_{22}(s) & \dots & t_{2p}(s) & \dots & t_{2m}(s) \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & t_{pp}(s) & \dots & t_{pm}(s) \end{bmatrix} \quad (p \leqslant m) \tag{10}$$

$$\begin{bmatrix} t_{11}(s) & t_{12}(s) & \dots & t_{1m}(s) \\ 0 & t_{22}(s) & \dots & t_{2m}(s) \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & t_{mm}(s) \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \quad (p \geqslant m) \tag{10$'$}$$

where if $[t_{1k}(s), t_{2k}(s), \dots, t_{k-1,k}(s), t_{kk}(s), 0 \dots 0]^{\mathrm{T}}$ is the kth column of (10) or (10)$'$, $k = 1, 2, \dots, \min(p, m)$ then either

$$t_{ik}(s) = 0 \quad \text{for all } i = 1, 2, \dots, k-1$$

or if

$$t_{ik}(s) \neq 0 \quad \text{for some } i = 1, 2, \dots, k-1$$

then

$$\delta(t_{kk}(s)) > \delta(t_{ik}(s)) \quad \text{for each such } i$$

Similarly, by means of right elementary operations $T(s)$ can be brought to one of the following lower left triangular forms

$$\begin{bmatrix} t_{11}(s) & 0 & \dots & 0 & \dots & 0 \\ t_{21}(s) & t_{22}(s) & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ t_{p1}(s) & t_{p2}(s) & \dots & t_{pp}(s) & \dots & 0 \end{bmatrix} \quad (p \leqslant m) \tag{11}$$

$$\begin{bmatrix} t_{11}(s) & 0 & \dots & 0 \\ t_{21}(s) & t_{22}(s) & \dots & 0 \\ \vdots & \vdots & & \vdots \\ t_{m1}(s) & t_{m2}(s) & \dots & t_{mm}(s) \\ \vdots & \vdots & & \vdots \\ t_{p1}(s) & t_{p2}(s) & \dots & t_{pm}(s) \end{bmatrix} \quad (p \geqslant m) \tag{11$'$}$$

where if $[t_{k1}(s), t_{k2}(s), \dots, t_{k,k-1}(s), t_{kk}(s), 0 \dots 0]$ is the kth row of (11) or (11)$'$, $k = 1, 2, \dots, \min(p, m)$ then either

$$t_{kj}(s) = 0 \quad \text{for all } j = 1, 2, \dots, k-1$$

or if

$$t_{kj}(s) \neq 0 \quad \text{for some } j = 1, 2, \dots, k-1$$

then

$$\delta(t_{kk}(s)) > \delta(t_{kj}(s)) \quad \text{for each such } j$$

Proof

We only indicate how to obtain (10) or (10)′ by row operations. The triangular forms (11) or (11)′ can be obtained using similar reasoning and column operations.

Assume that the first column of $T(s)$ contains at least one non-zero element. Among these elements choose one of least valuation $\delta(\cdot)$, and then bring it to position (1, 1) by a row permutation. This means that for the elements $t_{11}(s), t_{21}(s), \ldots, t_{p1}(s)$ of the first column of the resulting matrix we will have

$$\delta(t_{11}(s)) \leqslant \delta(t_{i1}(s)) \quad i=2, 3, \ldots, p$$

so we can write

$$t_{i1}(s) = t_{11}(s)p_{i1}(s) \quad i=2, 3, \ldots, p$$

with $p_{i1}(s) \in \mathbb{R}_{pr}(s)$. We now subtract from row i a multiple by $p_{i1}(s)$ of row 1 for all $i=2, \ldots, p$. After this step the first column of the resulting matrix will be : $[t_{11}(s) \ 0 \ldots 0]^T$.

Now consider the remaining $(p-1)\times(m-1)$ matrix (ignore the first row and column) ; and employ similar arguments to those above to reduce it to a matrix having the form

$$\begin{bmatrix} t_{11}(s) & t_{12}(s) & t_{13}(s) & \cdots & t_{1m}(s) \\ 0 & t_{22}(s) & t_{23}(s) & \cdots & t_{2m}(s) \\ 0 & 0 & t_{33}(s) & \cdots & t_{2m}(s) \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & t_{p3}(s) & \cdots & t_{pm}(s) \end{bmatrix} \tag{12}$$

In the second column two situations may arise. These are :

(i) $\delta(t_{22}(s)) \leqslant \delta(t_{12}(s))$, in which case $t_{12}(s)$ can be written as $t_{12}(s) = t_{22}(s)p_{22}(s)$, $p_{22}(s) \in \mathbb{R}_{pr}(s)$ and so by multiplying row 2 of (12) by $p_{22}(s) \in \mathbb{R}_{pr}(s)$ and subtracting it from row 1 we arrive at a matrix having the form

$$\begin{bmatrix} t_{11}(s) & 0 & t_{13}(s) & \cdots & t_{1m}(s) \\ 0 & t_{22}(s) & t_{23}(s) & \cdots & t_{2m}(s) \\ 0 & 0 & t_{33}(s) & \cdots & t_{3m}(s) \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & t_{p3}(s) & \cdots & t_{pm}(s) \end{bmatrix} \tag{13}$$

(ii) $\delta(t_{22}(s)) > \delta(t_{12}(s))$, in which case we cannot reduce this column any further.

Continuing in this way we arrive at a rational matrix having the desired form. □

From Theorem 1 we obtain

Corollary 1

If $T(s) \in \mathbb{R}_{pr}{}^{p\times p}(s)$ is a bi-proper rational matrix then it may be represented as the product of a finite number of elementary matrices.

Proof

By Theorem 1 $T(s)$ may be brought to the form

$$\hat{T}(s)=\begin{bmatrix} \hat{t}_{11}(s) & \hat{t}_{12}(s) & \dots & \hat{t}_{1p}(s) \\ 0 & \hat{t}_{22}(s) & \dots & \hat{t}_{2p}(s) \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \hat{t}_{pp}(s) \end{bmatrix} \tag{14}$$

by left elementary operations. Since $T(s)$ is a bi-proper rational matrix and the elementary matrices used to reduce $T(s)$ to $\hat{T}(s)$ are bi-proper rational matrices, $\hat{T}(s)$ is also bi-proper rational, i.e. det $\hat{T}(s)=\hat{t}_{11}(s)\hat{t}_{22}(s)\dots\hat{t}_{pp}(s)$ is a bi-proper rational function. Since the elements $\hat{t}_{ij}(s)$ of $\hat{T}(s)$ are proper rational functions, the above implies that $\hat{t}_{ii}(s)$, $i\in\mathbf{p}$ are bi-proper rational functions and so $\delta(\hat{t}_{ii}(s))=0$, for all $i=1, 2, \dots, p$.

Now from Theorem 1 we have that either (i) $\hat{t}_{ik}(s)=0$ for all $i=1, 2, \dots, k-1$, or (ii) $\delta(\hat{t}_{kk}(s))>\delta(\hat{t}_{ik}(s))$ for some $i=1, 2, \dots, k-1$. But since $\delta(\hat{t}_{ii}(s))=0$ for all $i=1, 2, \dots, p$, (ii) implies $\delta(\hat{t}_{ik}(s))<0$, i.e. $\hat{t}_{ik}(s)$ is non-proper, which is a contradiction since $\hat{T}(s)$ is proper, in fact bi-proper. Hence case (ii) cannot arise.

Thus $\hat{T}(s)$ has the form: $\hat{T}(s)=\text{diag}\,(\hat{t}_{11}(s), \dots, \hat{t}_{pp}(s))$, i.e. it is equal to a product of elementary matrices of type 2, and hence $\hat{T}(s)$ can be reduced to the unit matrix I_p by means of left elementary operations of type 2. Then, conversely, the unit matrix I_p can be transformed to $T(s)$ by means of elementary operations whose matrices are $T_{L1}^{1}, \dots, T_{L1}^{p}$, which proves the result.

3.1. *Equivalence at infinity*

Definition 2

Let $(T_1(s), T_2(s))\in\mathbb{R}^{p\times m}(s)\times\mathbb{R}^{p\times m}(s)$. Then $T_1(s)$ and $T_2(s)$ are called *equivalent* (*at infinity*) if there exist bi-proper rational matrices $T_L(s)\in\mathbb{R}_{pr}^{p\times p}(s)$ and $T_R(s)\in\mathbb{R}_{pr}^{m\times m}(s)$ such that

$$T_L(s)T_1(s)T_R(s)=T_2(s) \tag{15}$$

Clearly (15) defines an equivalence relation on $\mathbb{R}^{p\times m}(s)$, (i.e. a subset $\mathcal{E}^\infty\subset\mathbb{R}^{p\times m}(s)\times\mathbb{R}^{p\times m}(s)$ which has the properties of reflexivity, symmetry and transitivity) and if $T_1(s)$ and $T_2(s)$ are equivalent (at infinity) we denote this by writing:

$$(T_1(s), T_2(s))\in\mathcal{E}^\infty \quad \text{or by } T_1(s)\sim T_2(s)\ (\mathcal{E}^\infty)$$

The $\mathcal{E}^\infty$-equivalence class: $[T(s)]_{\mathcal{E}^\infty}$ or the *orbit* of a fixed $T(s)\in\mathbb{R}^{p\times m}(s)$ is the set

$$\begin{aligned}[T(s)]_{\mathcal{E}^\infty} &= \{T_i(s)\in\mathbb{R}^{p\times m}(s)\,|\,T_i(s)=T_L(s)T(s)T_R(s) \text{ where } T_L(s)\in\mathbb{R}_{pr}^{p\times p}(s), \\ &\qquad T_R(s)\in\mathbb{R}_{pr}^{m\times m}(s) \text{ are bi-proper rational matrices}\} \\ &= \{T_i(s)\in\mathbb{R}^{p\times m}(s)\,|\,(T_i(s), T(s))\in\mathcal{E}^\infty\}\subset\mathbb{R}^{p\times m}(s)\end{aligned}$$

We can now state

Theorem 2 (Smith–MacMillan form of a rational matrix at infinity)

Let $T(s) \in \mathbb{R}^{p\times m}(s)$ with $\operatorname{rank}_{\mathbb{R}(s)} T(s) = r$. Then $T(s)$ is equivalent (at infinity) to a diagonal matrix having the form

$$S_{T(s)}{}^{\infty} = \begin{bmatrix} s^{q_1} & & & & & & & & \\ & s^{q_2} & & & & & & & \\ & & \ddots & & & & & & \\ & & & s^{q_k} & & & & & \\ & & & & 1/s^{\bar{q}_{k+1}} & & & & \\ & & & & & \ddots & & & \\ & & & & & & 1/s^{\bar{q}_r} & & \\ & & & & & & & 0 & \\ & & & & & & & & \ddots \\ & & & & & & & & & 0 \end{bmatrix} \tag{16}$$

where

$$q_1 \geqslant q_2 \geqslant \dots \geqslant q_k \geqslant 0 \tag{17}$$

$$\bar{q}_r \geqslant \bar{q}_{r-1} \geqslant \dots \geqslant \bar{q}_{k+1} > 0 \tag{18}$$

Proof

Among all the elements $t_{ij}(s)$ of $T(s)$ that are not equal to zero we choose one which has least valuation $\delta(\cdot)$ and by suitable permutations of rows and columns we bring this element to position (1, 1). We now write: $t_{11}(s) = u_{11}(s)s^{-q_{11}}$ where $u_{11}(s)$ is bi-proper and $q_{11} = \delta(t_{11}(s))$. Multiplying the first row or column of $T(s)$ by $u_{11}(s)^{-1}$ (operation of type 2) brings the first row or column of the resulting matrix $\tilde{T}(s) = [\tilde{t}_{ij}(s)]$ to

$$[s^{-q_{11}}, u_{12}(s)s^{-q_{12}}, \dots, u_{1m}(s)s^{-q_{1m}}] \tag{19}$$

or

$$[s^{-q_{11}}, u_{21}(s)s^{-q_{21}}, \dots, u_{p1}(s)s^{-q_{p1}}]^{\mathrm{T}} \tag{20}$$

respectively, where $u_{ij}(s)$ are bi-proper and $q_{1j} = \delta(\tilde{t}_{1j}(s)) \geqslant q_{11}$, $j = 2, \dots, m$ or $q_{i1} = \delta(t_{i1}(s)) \geqslant q_{11}$, $i = 2, \dots, p$.

Now writing: $\tilde{t}_{1j}(s) = u_{1j}(s)s^{-q_{1j}} = s^{-q_{11}}p_{1j}(s)$, $j = 2, \dots, m$, where $p_{1j}(s) = u_{1j}(s)s^{-(q_{1j}-q_{11})} \in \mathbb{R}_{\mathrm{pr}}(s)$, we see that by column operations of type 3, i.e. by subtracting from the jth column ($j = 2, \dots, m$) the first multiplied by $p_{1j}(s)$, we can reduce the first row of $\tilde{T}(s)$ to the form $[s^{-q_{11}}, 0, \dots, 0]$. Similarly, with row operations of type 3 and similar arguments the first column of the resulting matrix can be reduced to the form $[s^{-q_{11}}, 0, \dots, 0]^{\mathrm{T}}$.

The resulting matrix $\hat{T}(s)$ will have the form

$$\hat{T}(s) = \begin{bmatrix} s^{-q_{11}} & 0 & \dots & 0 \\ 0 & \hat{t}_{22}(s) & \dots & \hat{t}_{2m}(s) \\ \vdots & \vdots & & \vdots \\ 0 & \hat{t}_{p2}(s) & \dots & \hat{t}_{pm}(s) \end{bmatrix}$$

As $t(s)$ in (9) and (9)' has a non-negative valuation (by definition), and since the valuation of a product and the valuation of a sum obey (1) and (2) respectively, we observe that $\delta(s^{-q_{11}}) \leqslant \delta(\hat{t}_{ij}(s))$ for $i=2, \ldots, p$ and $j=2, \ldots, m$. This means that we never have to consider the possibility of generating a term with a lower valution than that of $s^{-q_{11}}$.

This is illustrated by the following example

$$\begin{bmatrix} 1 & 0 \\ t(s) & 1 \end{bmatrix} \begin{bmatrix} s^{-q_{11}} & u_{12}s^{-q_{12}} \\ u_{21}s^{-q_{21}} & u_{22}s^{-q_{22}} \end{bmatrix} = \begin{bmatrix} s^{-q_{11}} & u_{12}s^{-q_{12}} \\ u_{21}s^{-q_{21}} + t(s)s^{-q_{11}} & u_{22}s^{-q_{22}} + t(s)u_{12}s^{-q_{12}} \end{bmatrix}$$

We know that

(a) $\delta(t(s)) \geqslant 0$

(b) $\delta(s^{-q_{11}}) \leqslant \min \{\delta(u_{12}s^{-q_{12}}),\ \delta(u_{21}s^{-q_{21}}),\ \delta(u_{22}s^{-q_{22}})\}$

By applying the rules governing the properties of the valuations of sums and products we have that

$$\delta(s^{-q_{11}}) \leqslant \min \{\delta(u_{12}s^{-q_{21}} + t(s)s^{-q_{11}}),\ \delta(u_{22}s^{-q_{22}} + t(s)u_{12}s^{-q_{12}})\}$$

which shows that we can never generate a term with valuation less than $\delta(s^{-q_{11}})$.

It is now easy to see that the reduction process may be continued to the second row and column and so on until (16) is obtained. The above in conjunction with Corollary 1 proves Theorem 2. □

In view of Theorem 2 and the arguments in the introduction regarding the infinite poles and the infinite zeros of a rational matrix, we can now state :

Definition 3

If p_∞ is the number of q_i's in (17) with $q_i > 0$, $i \in \mathbf{k}$ then we say that $T(s)$ has p_∞ *poles at infinity*, each one of order $q_i > 0$. Also if z_∞ is the number of $\bar{q}_i$'s in (18) with $\bar{q}_i > 0$ then we say that $T(s)$ has z_∞ *zeros at infinity*, each one of order $\bar{q}_i > 0$.

The above reduction process using elementary operations will now be illustrated with an example.

Example 3

Let

$$T(s) = \begin{bmatrix} 1 & 0 \\ s+1 & 1 \end{bmatrix}$$

O1. Bring the least valuation element to position (1, 1). This is achieved as follows

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ s+1 & 1 \end{bmatrix} = \begin{bmatrix} s+1 & 1 \\ 1 & 0 \end{bmatrix}$$

O2. Multiply the first column by the unit $s/(s+1)$

$$\begin{bmatrix} s+1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} s/(s+1) & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} s & 1 \\ s/(s+1) & 0 \end{bmatrix}$$

O3. Add $-1/s$ times column one to column two

$$\begin{bmatrix} s & 1 \\ s/(s+1) & 0 \end{bmatrix}\begin{bmatrix} 1 & -1/s \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} s & 0 \\ s/(s+1) & -1/(s+1) \end{bmatrix}$$

O4. Add $-1/(s+1)$ times row one to row two

$$\begin{bmatrix} 1 & 0 \\ -1/(s+1) & 1 \end{bmatrix}\begin{bmatrix} s & 0 \\ s/(s+1) & -1/(s+1) \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & -1/(s+1) \end{bmatrix}$$

O5. Multiply column two by the unit $-(s+1)/s$

$$\begin{bmatrix} s & 0 \\ 0 & -1/(s+1) \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -(s+1)/s \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & 1/s \end{bmatrix}$$

Hence $T(s)$ has one infinite pole of order 1 and one infinite zero of order 1.

Theorems 1 and 2 give rise to the following :

Proposition 1

Let $A(s) \in \mathbb{R}_{\mathrm{pr}}^{p\times m}(s)$, $B(s) \in \mathbb{R}_{\mathrm{pr}}^{q\times m}(s)$, with $p+q \geqslant m$. Then the following statements are equivalent :

S1. The proper rational matrix

$$T(s) := \begin{bmatrix} A(s) \\ B(s) \end{bmatrix} \in \mathbb{R}_{\mathrm{pr}}^{(p+q)\times m}(s)$$

has no zeros at infinity.

S2. The Smith–MacMillan form of $T(s)$ at infinity is

$$S_{T(s)}^{\infty} = \begin{bmatrix} I_m \\ 0 \end{bmatrix}$$

S3. There exists a bi-proper rational matrix $U_{\mathrm{L}}(s) \in \mathbb{R}_{\mathrm{pr}}^{(p+q)\times(p+q)}(s)$ such that

$$U_{\mathrm{L}}(s)T(s) = \begin{bmatrix} I_m \\ 0 \end{bmatrix}$$

S4. There exist proper rational matrices $X(s) \in \mathbb{R}_{\mathrm{pr}}^{m\times p}(s)$ and $Y(s) \in \mathbb{R}_{\mathrm{pr}}^{m\times q}(s)$ such that

$$[X(s),\ Y(s)]\begin{bmatrix} A(s) \\ B(s) \end{bmatrix} = I_m$$

S5. There exist proper rational matrices

$$C(s)\in\mathbb{R}_{\mathrm{pr}}{}^{p\times[(p+q)-m]}(s)$$

and

$$D(s)\in\mathbb{R}_{\mathrm{pr}}{}^{q\times[(p+q)-m]}(s)$$

such that the proper rational matrix

$$\begin{bmatrix} A(s) & C(s) \\ B(s) & D(s) \end{bmatrix}\in\mathbb{R}_{\mathrm{pr}}{}^{(p+q)\times(p+q)}(s)$$

is bi-proper.

S6. $\lim_{s\to\infty} T(s)=E\in\mathbb{R}^{(p+q)\times m}$ and rank $E=m$.

Proof

S1 $\Rightarrow$ S2. This follows directly from Theorem 1 and Definition 2.

S2 $\Rightarrow$ S3. S2 implies the existence of bi-proper rational matrices

$$T_{\mathrm{L}}(s)\in\mathbb{R}_{\mathrm{pr}}{}^{(p+q)\times(p+q)}(s)$$

and

$$T_{\mathrm{R}}(s)\in\mathbb{R}_{\mathrm{pr}}{}^{m\times m}(s)$$

such that

$$T(s)=T_{\mathrm{L}}(s)S_{T(s)}{}^{\infty}\,T_{\mathrm{R}}(s)=\begin{bmatrix} T_{\mathrm{L}1}(s) & T_{\mathrm{L}2}(s) \\ T_{\mathrm{L}3}(s) & T_{\mathrm{L}4}(s) \end{bmatrix}\begin{bmatrix} I_m \\ 0 \end{bmatrix}T_{\mathrm{R}}(s)$$

$$=\begin{bmatrix} T_{\mathrm{L}1}(s) & T_{\mathrm{L}2}(s) \\ T_{\mathrm{L}3}(s) & T_{\mathrm{L}4}(s) \end{bmatrix}\begin{bmatrix} T_{\mathrm{R}}(s) & 0 \\ 0 & I_{(p+q)-m} \end{bmatrix}\begin{bmatrix} I_m \\ 0 \end{bmatrix}=U_{\mathrm{L}}{}^{-1}(s)\begin{bmatrix} I_m \\ 0 \end{bmatrix}$$

where $U_{\mathrm{L}}{}^{-1}(s)\in\mathbb{R}_{\mathrm{pr}}{}^{(p+q)\times(p+q)}(s)$ bi-proper. Hence S3 follows.

S3 $\Rightarrow$ S4. By appropriately partitioning $U_{\mathrm{L}}(s)$ in S3 we have

$$\begin{bmatrix} U_{\mathrm{L}1}(s) & U_{\mathrm{L}2}(s) \\ U_{\mathrm{L}3}(s) & U_{\mathrm{L}4}(s) \end{bmatrix}\begin{bmatrix} A(s) \\ B(s) \end{bmatrix}=\begin{bmatrix} I_m \\ 0 \end{bmatrix} \tag{21}$$

whence $[X(s),\ Y(s)]=[U_{\mathrm{L}1}(s),\ U_{\mathrm{L}2}(s)]$.

S4 $\Rightarrow$ S5. Notice that

$$\begin{bmatrix} \hat{U}_{\mathrm{L}1}(s) & \hat{U}_{\mathrm{L}2}(s) \\ \hat{U}_{\mathrm{L}3}(s) & \hat{U}_{\mathrm{L}4}(s) \end{bmatrix}:=\begin{bmatrix} U_{\mathrm{L}1}(s) & U_{\mathrm{L}2}(s) \\ U_{\mathrm{L}3}(s) & U_{\mathrm{L}4}(s) \end{bmatrix}^{-1}$$

(where the sizes of the matrices $\hat{U}_{\mathrm{L}i}(s)$ are equal to those of the matrices $U_{\mathrm{L}i}(s)$) is bi-proper and from (21) it follows that

$$\begin{bmatrix} A(s) \\ B(s) \end{bmatrix}=\begin{bmatrix} \hat{U}_{\mathrm{L}1}(s) \\ \hat{U}_{\mathrm{L}3}(s) \end{bmatrix}$$

Therefore $C(s) := \hat{U}_{L2}(s)$ and $D(s) := \hat{U}_{L4}(s)$.

S5 $\Rightarrow$ S6. $\begin{bmatrix} A(s) & C(s) \\ B(s) & D(s) \end{bmatrix}$ being bi-proper implies (see § 2) that

$$\lim_{s\to\infty} \begin{bmatrix} A(s) & C(s) \\ B(s) & D(s) \end{bmatrix} = \begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix} = K \in \mathbb{R}^{(p+q)\times(p\times q)}$$

with rank $K = p+q$. Therefore

$$\lim_{s\to\infty} \begin{bmatrix} A(s) \\ B(s) \end{bmatrix} = \begin{bmatrix} K_1 \\ K_3 \end{bmatrix} = E \in \mathbb{R}^{(p+q)\times m}$$

which has rank m.

S6 $\Rightarrow$ S1. This is obvious. □

Remark

From statement S4, it is clear that Proposition 1 can also be interpreted as expressing various equivalent necessary and sufficient conditions for the existence of left (or right) proper inverses: $\hat{T}(s) = [X(s), Y(s)]$ of a proper rational matrix $T(s) = \begin{bmatrix} A(s) \\ B(s) \end{bmatrix}$.

Corollary 2

Let $T(s) \in \mathbb{R}^{p\times p}(s)$. Then $T(s)$ is a bi-proper rational matrix if and only if $S_{T(s)}{}^{\infty} = I_p$.

Corollary 3

Let $T_1(s) \in \mathbb{R}^{p\times m}(s)$, $K_L \in \mathbb{R}^{p\times p}$, $K_R \in \mathbb{R}^{m\times m}$, K_L, K_R non-singular and $T_2(s) := K_L T_1(s) K_R$ then $T_1(s)$ and $T_2(s)$ have the same finite and the same infinite Smith–MacMillan forms.

Proof

K_L and K_R are both unimodular and biproper. □

3.2. *Canonical forms for $\mathcal{E}^{\infty}$ and complete sets of invariants at infinity*

Let $T(s) \in \mathbb{R}^{p\times m}(s)$, with $\text{rank}_{\mathbb{R}(s)} T(s) = r \leqslant \min(p, m)$ and consider the quotient of $\mathbb{R}^{p\times m}(s)$ by $\mathcal{E}^{\infty}$, i.e. consider the set (denoted by) $\mathbb{R}^{p\times m}(s)/\mathcal{E}^{\infty}$ of $\mathcal{E}^{\infty}$-equivalence classes: $[T(s)]_{\mathcal{E}^{\infty}}$ when $T(s)$ runs through the elements of $\mathbb{R}^{p\times m}(s)$. We can characterize these equivalence classes by determining complete sets of invariants and canonical forms. We have:

Theorem 3

Let $T(s) \sim S_{T(s)}(\mathcal{E}^{\infty})$ as in Theorem 2. Then the map

$$g: \mathbb{R}^{p\times m}(s) \to \mathbb{Z} \times \mathbb{Z} \times \dots \times \mathbb{Z};\ T(s) \mapsto (q_1, q_2, \dots, q_k;\ \bar{q}_{k+1}, \dots, \bar{q}_r)$$

is a complete invariant for $\mathcal{E}^{\infty}$, i.e. the indices q_i, $i \in \mathbf{k}$, $\bar{q}_j$, $j = k+1, \dots, r$ (as an ordered set) constitute a complete set of invariants for $\mathcal{E}^{\infty}$. Also the map

$f: \mathbb{R}^{p\times m}(s)\to\mathbb{R}^{p\times m}(s)$; $T(s)\mapsto S_{T(s)}{}^{\infty}$ is a canonical map for $\mathcal{E}^{\infty}$ on $\mathbb{R}^{p\times m}(s)$ and $S_{T(s)}{}^{\infty}$ is a canonical form for $\mathcal{E}^{\infty}$ on $\mathbb{R}^{p\times m}(s)$, which we also call the *canonical form of $T(s)$ at infinity*.

Proof

Let us denote by $\xi_j(T)\in\mathbb{Z}$ the least valuation among the valuations of all minors of $T(s)$ of order j, $j\in\mathbf{r}$, and define the rational functions $i_j(s)$, $j\in\mathbf{r}$

$$\left.\begin{aligned} i_1(s) &:= s^{\xi_0(T)-\xi_1(T)}, \quad (\xi_0(T):=0)\\ i_2(s) &:= s^{\xi_1(T)-\xi_2(T)}\\ &\vdots\\ i_r(s) &:= s^{\xi_{r-1}(T)-\xi_r(T)} \end{aligned}\right\} \tag{22}$$

Let $\bar{T}(s)\in\mathbb{R}^{p\times m}(s)$ be any other rational matrix within the $\mathcal{E}^{\infty}$-equivalence class $[T(s)]_{\mathcal{E}^{\infty}}$ of $T(s)$. Then by definition there exist bi-proper rational matrices $T_{\mathrm{L}}(s)\in\mathbb{R}_{\mathrm{pr}}{}^{p\times p}(s)$ and $T_{\mathrm{R}}(s)\in\mathbb{R}_{\mathrm{pr}}{}^{m\times m}(s)$ such that $\bar{T}(s)=T_{\mathrm{L}}(s)T(s)T_{\mathrm{R}}(s)$. If by $T^{i_1,i_2,\ldots,i_j}_{k_1,k_2,\ldots,k_j}$ we denote the jth order minor of $T(s)$ which is composed from rows $i_1,\ldots,i_j$ and columns $k_1,\ldots,k_j$; $j\in\mathbf{r}$ then by the Binet–Cauchy formula we have

$$\bar{T}^{i_1,\ldots,i_j}_{k_1,\ldots,k_j}=\sum T_{\mathrm{L}}{}^{i_1,\ldots,i_j}_{\alpha_1,\ldots,\alpha_j}\, T^{\alpha_1,\ldots,\alpha_j}_{\beta_1,\ldots,\beta_j}\, T_{\mathrm{R}}{}^{\beta_1,\ldots,\beta_j}_{k_1,\ldots,k_j} \tag{23}$$

$$1\leqslant\alpha_1\leqslant\ldots\leqslant\alpha_j\leqslant p,\quad 1\leqslant\beta_1\leqslant\ldots\leqslant\beta_j\leqslant m,\quad j\in\mathbf{r}$$

Making use of the properties of a discrete valuation, § 2 (Atiyah and Macdonald 1969), and observing that the valuations of the elements of $T_{\mathrm{L}}(s)$ and $T_{\mathrm{R}}(s)$ are non-negative, it follows that

$$\xi_j(\bar{T})\geqslant\xi_j(T),\quad j\in\mathbf{r} \tag{24}$$

By the symmetry property of the equivalence relation $\mathcal{E}^{\infty}$, we can also write:

$$\xi_j(T)\geqslant\xi_j(\bar{T}),\quad j\in\mathbf{r} \tag{25}$$

and hence that

$$\xi_j(\bar{T})=\xi_j(T),\quad j\in\mathbf{r} \tag{26}$$

Therefore we have that

$$\bar{T}(s)\sim T(s)(\mathcal{E}^{\infty})\Rightarrow\xi_j(\bar{T})=\xi_j(T),\quad j\in\mathbf{r} \tag{27}$$

Hence $\bar{T}(s)$ and $T(s)$ have the same set of rational functions $i_j(s)$, $j\in\mathbf{r}$ defined in (22). Now by hypothesis: $T(s)\sim S_{T(s)}{}^{\infty}(\mathcal{E}^{\infty})$ and therefore from (27) $\xi_j(T)=\xi_j(\bar{T})=\xi_j(S_{T(s)}{}^{\infty})$, $j\in\mathbf{r}$ and it simply follows that

$$\left.\begin{aligned} \xi_1(T)&=q_1\\ \xi_2(T)&=q_1+q_2\\ &\vdots\\ \xi_k(T)&=\sum_{i=1}^{k}q_i\\ \xi_{k+1}(T)&=\left(\sum_{i=1}^{k}q_1\right)-\bar{q}_{k+1}\\ &\vdots\\ \xi_r(T)&=\sum_{i=1}^{k}q_i-\sum_{j=k+1}^{r}\bar{q}_j \end{aligned}\right\} \tag{28}$$

i.e.

$$\left.\begin{aligned} &i_1(s)=s^{q_1}, \quad i_2(s)=s^{q_2}, \ \ldots, \quad i_k(s)=s^{q_k} \\ &i_{k+1}(s)=\frac{1}{s^{\bar{q}_{k+1}}}, \quad i_{k+2}(s)=\frac{1}{s^{\bar{q}_{k+2}}}, \ \ldots, \quad i_r(s)=\frac{1}{s^{\bar{q}_r}} \end{aligned}\right\} \tag{29}$$

Hence the map $g: \ T(s)\mapsto(q_1, \ldots, q_k; \ \bar{q}_{k+1}, \ldots, \bar{q}_r)$ is an invariant for $\mathscr{E}^{\infty}$. We show now that g is a complete invariant, i.e. that $g(T(s))=g(\bar{T}(s))$ implies that $T(s)\sim\bar{T}(s)(\mathscr{E}^{\infty})$.

Let $g(T(s))=g(\bar{T}(s))=(q_1, \ldots, q_k; \ \bar{q}_{k+1}, \ldots, \bar{q}_r)$ as an ordered set in $\mathbb{Z}\times \ldots \times \mathbb{Z}$. Then $T(s)\sim S_{T(s)}{}^{\infty}(\mathscr{E}^{\infty})$ and $\bar{T}\sim S_{\bar{T}(s)}{}^{\infty}(\mathscr{E}^{\infty})$ where

$$S_{T(s)}{}^{\infty}\equiv S_{\bar{T}(s)}{}^{\infty}=\operatorname{diag}\,(s^{q_1}, \ldots, s^{q_k}, 1/s^{\bar{q}_{k+1}}, \ldots, 1/s^{\bar{q}_r}, 0 \ldots 0)$$

and therefore $T(s)\sim\bar{T}(s)(\mathscr{E}^{\infty})$. The above arguments imply also that $f: \ T(s)\mapsto S_{T(s)}{}^{\infty}$ is a canonical map and $S_{T(s)}{}^{\infty}$ is a canonical form for $\mathscr{E}^{\infty}$ on $\mathbb{R}^{p\times m}(s)$. □

Definition 4

The rational functions $i_i(s)$, $i\in\mathbf{r}$ in (29) are defined as the *invariant rational functions* of $T(s)$ at infinity.

Remark

Note that formulae (22) indicate a simple algorithm for the computation of the Smith–MacMillan form of $T(s)$ at infinity $S_{T(s)}{}^{\infty}$.

Example 4

Let

$$T(s)=\begin{bmatrix} 1/(s+1)^2 & s^3 & s^2/(s+1) \\ (s+2)/(s^2+0{\cdot}2s+1) & 1/s^3 & 1/(s+2)^2 \end{bmatrix}$$

then the least valuation $\xi_1(T)$ among the valuations of all first order minors (i.e. elements) of $T(s)$ is :

$$\xi_1(T)=\min\,\{2, \ -3, \ -1, \ +1, \ 3, \ 2\}=-3$$

The least valuation $\xi_2(T)$ among all second order minors is :

$$\xi_2(T)=\min\,\{-2, \ 0, \ -1\}=-2 \quad \text{and} \quad \xi_0(T)=0$$

Therefore $i_1(s)=s^{\xi_0-\xi_1}=s^3$, $i_2(s)=s^{\xi_1-\xi_2}=s^{-3-(-2)}=1/s$ and so

$$S_{T(s)}{}^{\infty}=\begin{bmatrix} s^3 & 1 & 0 \\ 0 & 1/s & 0 \end{bmatrix}$$

i.e. $T(s)$ has one pole at infinity of order three and one zero at infinity of order one.

3.3. *Graphical determination of the Smith–MacMillan form of a rational matrix at infinity*

It is well known that in the case of scalar rational functions the difference between the denominator and the numerator polynomial degrees (i.e. the valuation) is given by the asymptotic roll-off rate of the Bode magnitude plot. Hence the roll-off rate determines the Smith–MacMillan form at infinity in the scalar case. Under certain genericity assumptions, this idea may be generalized to the matrix case. This provides the practical designer with a tool which may be used to evaluate the Smith–MacMillan form of a rational matrix $T(s)$ at infinity (and hence also in the square case the asymptotic root-locus behaviour) from an inspection of the open-loop Bode magnitude array. To show this let $T(s) \in \mathbb{R}^{p \times m}(s)$ and denote by $T^{i_1, i_2, \ldots, i_k}_{j_1, j_2, \ldots, j_k}$ the kth order minor composed of rows $i_1, i_2, \ldots, i_k$ and columns $j_1, j_2, \ldots, j_k$ of $T(s)$. Then

$$T^{i_1, i_2, \ldots, i_k}_{j_1, j_2, \ldots, j_k} = \sum^{k!} (-1)^{\mu} (t_{i_1 p_1}, t_{i_2 p_2}, \ldots, t_{i_k p_k}) \tag{30}$$

where $p_1, p_2, \ldots, p_k$ is a permutation of $j_1, j_2, \ldots, j_k$ and the sum is taken over all possible permutations (the exponent μ is the number of transpositions required to go from the natural order $j_1, j_2, \ldots, j_k$ to $p_1, p_2, \ldots, p_k$). Taking valuations of both sides of (30) yields

$$\delta\{T^{i_1, i_2, \ldots, i_k}_{j_1, j_2, \ldots, j_k}\} \geqslant \min \{\delta(t_{i_1 p_1}, t_{i_2 p_2}, \ldots, t_{i_k p_k})\} = \min \left(\sum_{l=1}^{k} \delta(t_{i_l p_l}) \right) \tag{31}$$

Since

$$\xi_k(T) := \min \{\delta(T^{i_1, i_2, \ldots, i_k}_{j_1, j_2, \ldots, j_k})\} \tag{32}$$

we have

$$\xi_k(T) \geqslant \min \left(\sum_{l=1}^{k} \delta(t_{i_l p_l}) \right) \tag{33}$$

where the minima are taken over all indices, $1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant p$, $1 \leqslant j_1 < j_2 < \ldots < j_k \leqslant m$.

Note that $\delta(t_{ij})$ are the asymptotic roll-off rates of the individual entries of the rational matrix $T(s)$ (a valuation of 1 corresponds to a 20 db/dec roll-off rate and so on). Before the $\delta(t_{ij})$'s can be used to find $\xi_k(T)$ via (33), we have to establish the conditions under which the inequality in this equation may be removed. These conditions are given by the following lemma :

Lemma 1

Given two rational functions $t_1(s)$ and $t_2(s)$, then

$$\delta(t_1(s) + t_2(s)) = \min \{\delta(t_1(s)), \delta(t_2(s))\}$$

if

$$\delta(t_1(s)) \neq \delta(t_2(s))$$

Proof

Let $t_i(s) = n_i(s)/d_i(s)$, $i = 1, 2$ where $n_i(s)$ and $d_i(s)$ are polynomials, then

$$t_1(s) + t_2(s) = \frac{n_1(s)d_2(s) + n_2(s)d_1(s)}{d_1(s)d_2(s)}$$

since $\delta(t_1(s)) \neq \delta(t_2(s))$ we have that

$$\deg(n_1(s)) + \deg(d_2(s)) \neq \deg(n_2(s)) + \deg(d_1(s))$$

therefore

$$\begin{aligned}\delta(t_1(s)+t_2(s)) &= \deg(d_1(s)) + \deg(d_2(s)) - \max[\{\deg(n_1(s)) \\ &\qquad + \deg(d_2(s))\}, \{\deg(n_2(s)) - \deg(d_1(s))\}] \\ &= \min\{\delta(t_1(s)), \delta(t_2(s))\} \qquad \square\end{aligned}$$

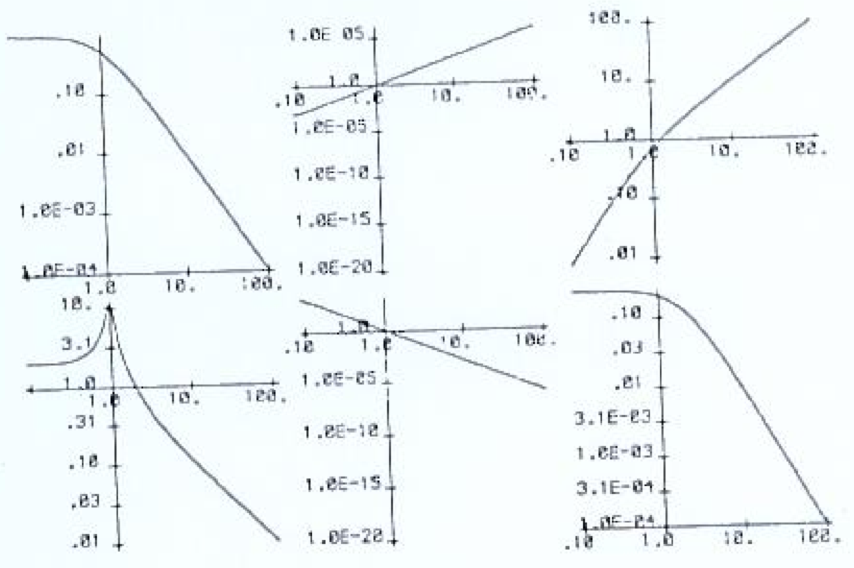
This lemma shows that if there is only one least valuation product term $\sum_{i=1}^{k} t_{ip_i}$ in (33) then equality holds in (33). If there is more than one least valuation product term, however, cancellations *may* occur during the formation of the least valuation minor of order k. Almost any small perturbation to the transfer matrix will remove these valuation-increasing cancellations and decrease the order of some of the infinite zeros and generate finite ones. In the generic case, therefore, equality holds in (33) even if there is more than one least valuation product term in this expression. These ideas will now be clarified with the aid of examples.

Bode magnitude diagrams corresponding to Example 4 are shown in the Figure and the valuation matrix $V = \{\delta(t_{ij}(s))\}$ is given by

$$V = \begin{bmatrix} 2 & -3 & -1 \\ 1 & 3 & 2 \end{bmatrix}$$

From V it follows that

$$\xi_1(T) = -3 \quad \text{and} \quad \xi_2(T) = \min\{2+3,\ 1-3,\ 2+2,\ 1-1,\ 2-3,\ 3-1\} = -2$$



Bode magnitude array for Example 4.

Hence

$$S_{T(s)}^{\infty}(s)=\begin{bmatrix} s^3 & 0 & 0 \\ 0 & 1/s & 0 \end{bmatrix}$$

which agrees with the earlier calculation. Note that no cancellation difficulties arose in this case. As has already been mentioned, the graphical method fails if cancellations occur between numerator coefficients of maximum degree when forming least valuation minors of different orders. For example, given a

$$T(s)=\begin{bmatrix} 1/(s+1) & 1/(s+3) \\ 1/(s+2) & 1/(s+4) \end{bmatrix}$$

the Smith–MacMillan form at infinity may be calculated as

$$S_{T(s)}^{\infty}=\begin{bmatrix} 1/s & 0 \\ 0 & 1/s^3 \end{bmatrix}$$

If we were to use a valuation matrix obtained from the Bode magnitude array of $T(s)$, which is

$$V=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

we would calculate the Smith–MacMillan form at infinity as

$$S_{T(s)}^{\infty}=\begin{bmatrix} 1/s & 0 \\ 0 & 1/s \end{bmatrix}$$

This discrepancy is due to numerator cancellations which occurred during the formation of the second order minor. If $T(s)$ is perturbed to

$$T'(s)=\begin{bmatrix} 1/(s+1) & 1/(s+3) \\ 1/(s+2) & 1{\cdot}00001/(s+4) \end{bmatrix}$$

the Smith–MacMillan form at infinity is

$$S_{T'(s)}^{\infty}=\begin{bmatrix} 1/s & 0 \\ 0 & 1/s \end{bmatrix}$$

which agrees with the graphical result. The interested reader may verify that the perturbation of $T(s)$ resulted in the generation of two finite zeros at $(-2{\cdot}5 \pm j447{\cdot}2)$.

4. Realization theory

Let $T(s)\in\mathbb{R}^{p\times m}(s)$ with MacMillan degree (Rosenbrock 1970) $\delta_m(T(s))=n$. It has been shown (Verghese *et al.* 1979, 1981, Rosenbrock 1974) that associated with $T(s)$ there is a system of first order differential equations and an output expression having the form

$$\left.\begin{aligned} J_T\dot{x}_T(t) &= A_T x_T(t) + B_T u(t) \\ y_T(t) &= C_T x_T(t) \end{aligned}\right\} \tag{34}$$

where, if $T(s)$ is non-proper, then $J_T \in \mathbb{R}^{n\times n}$ and is singular, $A_T \in \mathbb{R}^{n\times n}$, $B_T \in \mathbb{R}^{n\times m}$, $C_T \in \mathbb{R}^{p\times n}$ and such that :

(i) $T(s) = C_T(sJ_T - A_T)^{-1}B_T$ (35)

(ii) the rational (polynomial) matrices

$$[sJ_T - A_T, \ -B_T], \begin{bmatrix} sJ_T - A_T \\ C_T \end{bmatrix} \tag{36}$$

have no finite or infinite zeros.

Such a quadruple (J_T, A_T, B_T, C_T) has been defined as a '*strongly irreducible*' realization of $T(s)$ (Verghese *et al.* 1979, 1981).

Let us now write

$$T(s) = T_{\mathrm{sp}}(s) + E(s) \tag{37}$$

where $T_{\mathrm{sp}}(s)$ is strictly proper and $E(s)$ is polynomial. If $T(s)$ is an entirely polynomial matrix, i.e. $T_{\mathrm{sp}}(s) \equiv 0$, then it turns out that a *strongly irreducible* realization of $T(s) \equiv E(s)$ can be obtained from a minimal realization of the strictly proper rational matrix $(1/w)E(1/w)$ (Verghese *et al.* 1979). We state this result formally by the following :

Proposition 2

Let $E(s) \in \mathbb{R}^{p\times m}[s]$ and let $J_E \in \mathbb{R}^{n_1\times n_1}$, $B_E \in \mathbb{R}^{n_1\times m}$, $-C_E \in \mathbb{R}^{p\times n_1}$ be a *minimal realization* of $(1/w)E(1/w)$ where $n_1 = \delta_m[(1/w)E(1/w)]$, i.e. assume

$$(1/w)E(1/w) = -C_E(wI_{n_1} - J_E)^{-1}B_E \tag{38}$$

with

$$\operatorname{rank}[wI_{n_1} - J_E, \ B_E] = \operatorname{rank}\begin{bmatrix} wI_{n_1} - J_E \\ -C_E \end{bmatrix} = n_1, \quad \text{for all } w \in \sigma(J_E) \tag{39}$$

Then (J_E, I_{n_1}, B_E, C_E) is a strongly irreducible realization of $E(s)$, i.e.

$$E(s) = C_E[sJ_E - I_{n_1}]^{-1}B_E \tag{40}$$

and the polynomial matrices

$$\text{(i)}\ [sJ_E - I_{n_1}, \ B_E] \quad \text{and} \quad \text{(ii)} \begin{bmatrix} sJ_E - I_{n_1} \\ C_E \end{bmatrix} \tag{41}$$

have no finite or infinite zeros.

Proof

From (38) it simply follows that $E(1/w) = C_E[(1/w)J_E - I_{n_1}]^{-1}B_E$ which by the substitution $1/w = s$ gives (40). Consider (41) (i). Obviously rank $[sJ_E - I_{n_1}, B_E] = n_1$, for all finite s. Hence $[sJ_E - I_{n_1}, B_E]$ has no finite zeros. Consider now the point at infinity. Then $[sJ_E - I_{n_1}, B_E]$ has no zeros at $s=\infty$ if and only if the rational matrix $[(1/w)J_E - I_{n_1}, B_E]$ has no zeros at $w=0$ (Kailath 1980, Pugh and Ratcliffe 1979). Now

$$\left.\begin{aligned}[(1/w)J_E - I_{n_1}, B_E] &= [-J_E + wI_{n_1}, B_E]\begin{bmatrix} -wI_{n_1} & 0 \\ 0 & I_m \end{bmatrix}^{-1} \\ &= N(w)D(w)^{-1}\end{aligned}\right\} \tag{42}$$

and since

$$\operatorname{rank}\begin{bmatrix} D(w) \\ N(w) \end{bmatrix} = \operatorname{rank}\begin{bmatrix} -wI_{n_1} & 0 \\ 0 & I_m \\ wI_{n_1} - J_E & B_E \end{bmatrix} = n_1 + m, \quad \text{for all } w \in \mathbb{C} \tag{43}$$

Equation (42) is a right coprime matrix fraction description of $[(1/w)J_E - I_{n_1}, B_E]$ and hence the zeros of $[(1/w)J_E - I_{n_1}, B_E] \in \mathbb{R}_{pr}(w)$ at $w=0$ are given by the zeros of the 'numerator' $[wI_{n_1} - J_E, B_E]$ at $w=0$. But $[wI_{n_1} - J_E, B_E]$ has no zeros because of (39). An analogous procedure proves (41) (ii). □

We can now give a theorem which characterizes the structure of J_E in (38) or (40) in terms of the *pole* structure of $E(s)$ at infinity.

Theorem 4

Let $E(s) \in \mathbb{R}^{p \times m}[s]$ with $\operatorname{rank}_{\mathbb{R}(s)} E(s) = r$ have Smith–MacMillan form at infinity:

$$S_{E(s)}^{\infty} = \operatorname{diag}\,[s^{q_1}, s^{q_2}, \ldots, s^{q_k}, 1/s^{\bar{q}_{k+1}}, 1/s^{\bar{q}_{k+2}}, \ldots, 1/s^{\bar{q}_r}, 0, \ldots, 0] \tag{44}$$

with $q_1 \geq q_2 \geq \ldots \geq q_k \geq 0$; $\bar{q}_r \geq \bar{q}_{r-1} \geq \ldots \geq \bar{q}_{k+1} > 0$. Then all strongly irreducible realizations (J_E, I, B_E, C_E) of $E(s)$ are such that J_E has Jordan form given by:

$$\text{block diag}\,[J_{E1}, J_{E2}, \ldots, J_{Ek}] \tag{45}$$

where

$$J_{Ei} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & 1 \\ 0 & 0 & & \ldots & 0 \end{bmatrix}}_{q_i+1} \Bigg\updownarrow \; q_i + 1, \quad i \in \mathbf{k} \tag{46}$$

modulo block re-ordering.

Proof

Let $(J_E, B_E, -C_E)$ be a minimal realization of the strictly proper rational matrix $(1/w)E(1/w)$ so that $E(s) = C_E(sJ_E - I)^{-1}B_E$. Now since $E(s)$ is polynomial, all its poles are at infinity. This implies that $(1/w)E(1/w)$ has all its poles at $w=0$ which in turn implies that all eigenvalues of J_E are at 0 and so J_E can be decomposed as

$$J_E = W \text{ block diag } [\bar{J}_1, \bar{J}_2, \ldots, \bar{J}_t] W^{-1} \tag{47}$$

where $\bar{J}_i$, $i \in \mathbf{t}$ are Jordan blocks

$$\bar{J}_i = \underbrace{\begin{bmatrix} 0 & 1 & 0 \ldots 0 \\ 0 & 0 & 1 \ldots 0 \\ \vdots & & \ddots \quad \ddots \vdots \\ \vdots & & \ddots \quad 1 \\ 0 & 0 & \ldots\ldots \quad 0 \end{bmatrix}}_{v_i} \Big\updownarrow v_i \quad i \in \mathbf{t} \tag{48}$$

with (say) $v_1 \geqslant v_2 \geqslant \ldots \geqslant v_t > 0$, and W is a non-singular matrix whose columns are eigenvectors and generalized eigenvectors of J_E. Now the v_i's give the degrees of the non-trivial invariant polynomials of $(wI - J_E)$ or equivalently the degrees of the denominator polynomials $\psi_i(w) = w^{v_i}$ appearing in the Smith–MacMillan form

$$M(w) = \text{diag} \left\{ \frac{\epsilon_1(w)}{\psi_1(w)}, \ldots, \frac{\epsilon_t(w)}{\psi_t(w)}, \epsilon_{t+1}(w), \ldots, \epsilon_r(w), 0 \ldots 0 \right\} \tag{49}$$

of $(1/w)E(1/w)$. Therefore $v_i - 1$, $i \in \mathbf{t}$ give the degrees of the denominator polynomials $\psi'(w) = w^{v_i - 1}$ of the Smith–MacMillan form of $E(1/w)$, i.e. the orders of the poles at $w=0$ of $E(1/w)$ or equivalently the orders of the poles of $E(s)$ at infinity. But by (44) the orders of the poles of $E(s)$ at infinity are equal to the q_i's, $i \in \mathbf{k}$, i.e. $k = t$ and $v_i = q_i + 1$, $i \in \mathbf{k}$. □

If $E(s)$ is a $p \times m$ column (row) proper polynomial matrix with $p \geqslant m$ ($p \leqslant m$) then a strongly irreducible realization (J_E, I, B_E, C_E) of $E(s)$ and the Smith–MacMillan form $S_{E(s)}{}^{\infty}$ of $E(s)$ at infinity can be obtained by inspection. These results are formally stated in Theorem 5 and Corollary 3 below.

Theorem 5

Let $E(s) \in \mathbb{R}^{p \times m}[s]$, $p \geqslant m$, be column proper and $v_j = \deg e_j(s)$ where $e_j(s) = \sum_{l=0}^{v_j} e_{jl} s^l \in \mathbb{R}^{p \times 1}[s]$, $j \in \mathbf{m}$ are the m columns of $E(s)$. Then a *strongly irreducible* realization (J_E, I_{n_1}, B_E, C_E) of $E(s)$ is given by:

$$J_E = \text{block diag } (J_{E1}, J_{E2}, \ldots, J_{Em}) \tag{50}$$

with

$$J_{Ej} = \begin{bmatrix} 0 & 1 & 0 \ldots 0 \\ 0 & 0 & 1 \ldots 0 \\ \vdots & & \ddots \quad \ddots \vdots \\ \vdots & & \ddots \quad 1 \\ 0 & 0 & 0 \ldots 0 \end{bmatrix} \in \mathbb{R}^{(v_j+1) \times (v_j+1)}, \quad j \in \mathbf{m} \tag{51}$$

and

$$n_1 = \sum_{i=1}^{m} (v_i + 1), \quad B_E = \text{block diag}\,(b_1, b_2, \ldots, b_m) \tag{52}$$

with

$$b_j = [0, 0, \ldots, 0, 1]^{\mathrm{T}} \in \mathbb{R}^{(v_j-1)\times 1} \tag{53}$$

and

$$C_E = -[e_{1v_1}, \ldots, e_{10} | e_{2v_2}, \ldots, e_{20} | \ldots | e_{mv_m}, \ldots, e_{m0}] \tag{54}$$

Proof

Write

$$E(s) = [E(s)]_c^h [s^v] + E_c S_c(s) \tag{55}$$

where $[E(s)]_c^h \in \mathbb{R}^{p\times m}$ denotes the highest column degree coefficient matrix of the matrix inside the brackets, $[s^v] = \text{diag}\,(s^{v_1}, \ldots, s^{v_m})$

$$S_c(s) = \begin{array}{c} \left[\begin{array}{ccc} s^{v_1-1} & & \\ s^{v_1-2} & & \\ \vdots & & \bigcirc \\ s & & \\ 1 & & \\ \hline & \ddots & \\ \hline & & s^{v_m-1} \\ & & s^{v_m-2} \\ \bigcirc & & \vdots \\ & & s \\ & & 1 \end{array}\right] \end{array} \begin{array}{l} \left.\vphantom{\begin{array}{c} s \\ s \\ \vdots \\ s \\ 1 \end{array}}\right\} v_1 \\ \vphantom{\ddots} \\ \left.\vphantom{\begin{array}{c} s \\ s \\ \vdots \\ s \\ 1 \end{array}}\right\} v_m \end{array} \tag{56}$$

and E_c a $p \times \left(\sum_{j=1}^{m} v_j \right)$ real matrix. Then

$$E(1/w) = [E(s)]_c^h \,\text{diag}\,(1/w^{v_1}, \ldots, 1/w^{v_m}) + E_c S_c(1/w) \tag{57}$$

and

$$\begin{aligned}(1/w)E(1/w) &= [E(s)]_c^h \,\text{diag}\,(1/w^{v_1+1}, \ldots, 1/w^{v_m+1}) + (1/w)E_c S_c(1/w) \\ &= [[E(s)]_c^h + (1/w)E_c S_c(1/w)\,\text{diag}\,(w^{v_1+1}, \ldots, w^{v_m+1})] \\ &\qquad \times \begin{bmatrix} w^{v_1+1} & & \\ & \ddots & \\ & & w^{v_m+1} \end{bmatrix}^{-1} \\ &= N(w)D(w)^{-1} \end{aligned} \tag{58}$$

where $N(w) = [E(s)]_c^h + E_c(1/w)S_c(1/w)\,\text{diag}\,(w^{v_1+1}, \ldots, w^{v_m+1}) \in \mathbb{R}^{p\times m}[w]$ can also be written as

$$N(w) = C_E S(w) \tag{59}$$

where

$$S(w)=\left[\begin{array}{ccc} 1 & & \\ w & & \bigcirc \\ \vdots & & \\ w^{v_1} & & \\ & \ddots & \\ & & 1 \\ \bigcirc & & w \\ & & \vdots \\ & & w^{v_m}\end{array}\right]\begin{array}{l} \left.\vphantom{\begin{array}{c}1\\w\\\vdots\\w^{v_1}\end{array}}\right\} v_1+1 \\ \vdots \\ \left.\vphantom{\begin{array}{c}1\\w\\\vdots\\w^{v_m}\end{array}}\right\} v_m+1\end{array} \tag{60}$$

Now since for $w=0$, $\operatorname{rank}\begin{bmatrix} D(0) \\ N(0) \end{bmatrix}=\operatorname{rank}\begin{bmatrix} 0 \\ [E(s)]_c^h \end{bmatrix}=m$ and for $w\neq 0$, $\operatorname{rank}\begin{bmatrix} D(w) \\ N(w) \end{bmatrix}=m$ we have that (58) is a right coprime matrix fraction description (MFD) of $(1/w)E(1/w)\in\mathbb{R}_{pr}{}^{p\times m}(w)$ with $D(w)$ column proper and by virtue of the Wolovich–Falb structure theorem (Wolovich and Falb 1969, Wolovich 1974) (J_E, B_E, C_E) is a minimal realization of $(1/w)E(1/w)$. Furthermore by Proposition 2 $(J_E, I_{n_1}, B_E, -C_E)$ is also a strongly irreducible realization of $E(s)$. □

Remark

A dual result can be obtained for every $p\times m$ $(p\leqslant m)$ and *row proper* polynomial matrix $E(s)$ by transposing $E(s)$ and applying Theorem 5.

The proof of Theorem 5 leads to an interesting corollary regarding the Smith–MacMillan form of a $p\times m$, $p\geqslant m$, column proper $(p\leqslant m$, row proper) polynomial matrices at infinity. This is :

Corollary 4

Let $E(s)\in\mathbb{R}^{p\times m}[s]$ be column (row) proper with $p\geqslant m$ $(p\leqslant m)$. Then $E(s)$ has no zeros at infinity and its column (row) degrees give the orders of its infinite poles.

Proof

Consider the column proper, $p\geqslant m$, case. The infinite pole–zero structure of $E(s)$ is given by the finite pole–zero structure of $E(1/w)\in\mathbb{R}^{p\times m}(w)$ at $w=0$. From (57)

$$\begin{aligned} E(1/w) &= [[E(s)]_c^h + E_c S_c(1/w)\operatorname{diag}(w^{v_1},\ldots,w^{v_m})]\begin{bmatrix} w^{v_1} & & 0 \\ & \ddots & \\ 0 & & w^{v_m}\end{bmatrix}^{-1} \\ &= \tilde{N}(w)\tilde{D}(w)^{-1} \end{aligned} \tag{61}$$

where $\tilde{N}(w)\in\mathbb{R}^{p\times m}[w]$, $\tilde{D}(w)\in\mathbb{R}^{m\times m}[w]$. Now it is easily seen that (61) is a right coprime MFD of $E(1/w)$. Hence the pole–zero structure of $E(1/w)$ at $w=0$ is given by the zero structure of the numerator and denominator matrices $\tilde{N}(w)$ and $\tilde{D}(w)$ respectively. Since $\operatorname{rank}\tilde{N}(0)=\operatorname{rank}[E(s)]_c^h=m$, $E(1/w)$ has no zeros at $w=0$ therefore $E(s)$ has no zeros at infinity. On the other hand, since

w^{v_i}, $i \in \mathbf{m}$ are elementary divisors of $D(w)$, $E(1/w)$ has poles at $w=0$ of orders equal to the v_i's, and therefore $E(s)$ has poles at infinity with the same orders. The row proper, $p \leqslant m$ case follows by duality.

Example 5

Find the Smith–MacMillan form at infinity of the following column proper polynomial matrix

$$E(s) = \begin{bmatrix} s^2+s & s \\ 2s+1 & s+1 \\ 3s & 1 \end{bmatrix}$$

and then write down a strongly irreducible realization of $E(s)$.

By Corollary 4, $E(s)$ has no infinite zeros and its infinite pole structure is given by its column degrees, i.e.

$$S_{E(s)}{}^{\infty} = \begin{bmatrix} s^2 & 0 \\ 0 & s \\ 0 & 0 \end{bmatrix}$$

$S_{E(s)}{}^{\infty}$ can be checked using elementary bi-proper operations or the valuation approach. Theorem 5 allows us to write down a strongly irreducible realization of $E(s)$ as

$$J_E = \left[\begin{array}{ccc:cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad I_5 B_E = \left[\begin{array}{c:c} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ \hdashline 0 & 0 \\ 0 & 1 \end{array}\right] \quad \text{and} \quad C_E = -\left[\begin{array}{ccc:cc} 1 & 1 & 0 & 1 & 0 \\ \hdashline 0 & 2 & 1 & 1 & 1 \\ \hdashline 0 & 3 & 0 & 0 & 1 \end{array}\right]$$

Note how the block structure of J_E vindicates Theorem 4.

5. Conclusions

The Smith–MacMillan form at infinity is dealt with using a new set of elementary operations which are similar in form to the classical unimodular ones. The essential property of this new set of elementary operations is that their matrix representations have no poles or zeros at infinity. The standard algorithm for computing the Smith–MacMillan form at finite frequencies using elementary unimodular operations is generalized to deal with infinite frequencies using bi-proper operations. The connection between these operations and the valuation approach is established. The orders of the infinite poles and infinite zeros of a rational matrix constitute a complete set of invariants for equivalence at infinity. Links between the poles at infinity and strongly irreducible realizations of polynomial matrices are derived. Finally, a simple graphical procedure for finding the Smith–MacMillan form at infinity from a Bode magnitude array is proposed.

ACKNOWLEDGMENTS

The authors would like to thank M. C. Smith and Y. S. Hung for many helpful discussions which added considerably to the quality of this paper. D. J. N. Limebeer would like to thank the C.S.I.R. in South Africa and the University of Cambridge for financial support.

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