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THE SEQUENTIAL CONSTRUCTION OF MINIMAL PARTIAL REALIZATIONS FROM FINITE INPUT–OUTPUT DATA*

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Abstract. Any strictly proper transfer function matrix of a continuous or discrete, linear, constant, multivariable system can be written as the product of a numerator polynomial matrix with the inverse of another polynomial matrix, the denominator. Since a realization is easily constructed from the polynomial matrix representation, the minimal partial realization problem is translated to that of extracting a minimal order partial denominator polynomial matrix from a finite length matrix sequence. It is shown that minimal partial denominator matrices evolve recursively; that is, a minimal partial denominator matrix for any finite length sequence is a combination of the minimal partial denominator matrices of its proper subsequences. A computationally efficient algorithm that sequentially constructs a minimal partial denominator matrix for a given finite length sequence is presented. A theorem by Anderson and Brasch leads to a definition of uniqueness for the resulting denominator matrix based upon its invariant factors. Parameters used during execution of the algorithm are shown to be sufficient for enumerating all invariant factor sets in the equivalence class of minimal partial realizations. The results apply to continuous and discrete linear systems including finite state machines.

1. Introduction. Consider the following discrete, linear, constant dynamical system:

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k), \\
y(k) &= Cx(k), \qquad\qquad k = 0, 1, 2, \cdots .
\end{aligned}
\tag{1}
$$

The vectors and matrices have real-valued elements or, if (1) represents a finite state machine, the vectors and matrices may be defined over a finite field. The state is denoted by the n-vector x; u, an m-vector, and y, an r-vector, are the external input and output, respectively. Thus A, B and C are constant matrices of dimension $n \times n$, $n \times m$ and $r \times n$, respectively, over some appropriate but fixed field $\mathscr{F}$. For a continuous, linear, constant system,

$$
\begin{aligned}
\frac{dx(t)}{dt} &= Ax(t) + Bu(t), \\
y(t) &= Cx(t).
\end{aligned}
\tag{2}
$$

Here the vectors and matrices have real-valued elements.

Both systems are characterized externally by a strictly proper rational matrix $M(z)$ called a transfer function and given by

$$
M(z) = C(zI - A)^{-1}B. \tag{3}
$$

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When dealing with discrete-time systems, the polynomial indeterminate z can be thought of as the z-transform variable. For continuous systems z can be thought of as the Laplace transform variable.

For either the discrete or continuous system $M(z)$ is said to have a *realization* [6] given by the matrix triple $\Sigma = (A, B, C)$ if A, B and C satisfy (3). Let (3) be expanded in a Laurent series,

$$M(z) = \sum_{i=1}^{\infty} M_i z^{-i}, \tag{4}$$

where the $r \times m$ matrices M_i are called Markov parameters [6] and

$$M_i = CA^{i-1}B, \qquad i = 1, 2, \cdots. \tag{5}$$

Alternatively Σ is a realization if A, B and C satisfy (5) for all i. If the dimension of A is minimized over all matrix triples satisfying (3), then Σ is called a *minimal realization.*

The transfer function may also be expressed as the product of a numerator polynomial matrix, $Q(z) \in \mathscr{F}^{r \times m}(z)$, with the inverse of a denominator polynomial matrix, $P(z) \in \mathscr{F}_m^{m \times m}(z)$, i.e.,

$$M(z) = Q(z)P^{-1}(z). \tag{6}$$

The columns of $P(z)$ are called *column annihilating polynomials* (CAP) and are written

$$p_i(z) = \sum_{j=0}^{n_i} p_{ji} z^{n_i - j}, \qquad 1 \leqq i \leqq m. \tag{7}$$

The matrix $P_0 = [p_{01} \quad p_{02} \cdots p_{0m}]$ is called the *leading coefficient matrix*, provided $|P_0| \neq 0$. The nonnegative integers n_i, $1 \leqq i \leqq m$, will be called *column degrees*; their sum is the *composite degree* of $P(z)$. Then from (6),

$$M(z)p_i(z) = q_i(z) = \begin{cases} \sum_{j=1}^{n_i} q_{ji} z^{n_i - j}, & n_i > 0, \\ 0, & n_i = 0, \end{cases} \tag{8}$$

where the r-vector coefficients of column i in $Q(z)$ are given by

$$q_{ji} = \sum_{l=0}^{j-1} M_{j-l} p_{li}. \tag{9}$$

Since $|P_0| \neq 0$, let

$$S(z) = P_0^{-1} P(z). \tag{10}$$

For any $k \geqq 1$,

$$\sum_{j=0}^{n_i} M_{k+n_i-j} p_{ji} = 0. \tag{11}$$

The vector π_i of length $(n_i + 1)m$ constructed from the coefficients of $p_i(z)$ as

$$\pi_i = \begin{bmatrix} p_{n_i,i} \\ p_{n_i-1,i} \\ \vdots \\ p_{1,i} \\ p_{0,i} \end{bmatrix} \tag{12}$$

will be called a *column annihilating vector* (CAV).

It has been established [4] that a realization for $M(z)$ having dimension equal to the composite degree of $P(z)$ is easily constructed from a representation $[Q(z), P(z)]$ satisfying (6). Moreover, a minimal representation [13] exists for any proper transfer function $M(z)$; that is, there exists a $P(z)$ with composite degree equal to the dimension of a minimal realization and satisfying (6). A property of $P(z)$, originally proved in [4], is given by the following theorem.

THEOREM 1. *Let $\Sigma = (A, B, C)$ be a minimal realization and let $[Q(z), P(z)]$ be a minimal representation of $M(z)$. Then the m highest degree invariant factors of $(zI - A)$ are identical to the invariant factors of $P(z)$.*

The invariant factors [5] of $P(z)$ are monic polynomials denoted $\gamma_{Pi}(z)$, with the property that $\gamma_{Pi}(z)$ divides $\gamma_{Pi+1}(z)$, $1 \leqq i \leqq m - 1$. Thus the $m \times m$ matrix $P(z)$ contains the same information about the system dynamics that is contained in the system matrix A.

Let $\{M_i\}$ denote any infinite sequence of $r \times m$ matrices, and consider the finite Markov parameter sequence of length N, $\{M_i\}_N$. This finite sequence is said to have a *partial realization* Σ if (5) is satisfied for $i = 1, 2, \cdots, N$. That is, a partial realization of $\{M_i\}_N$ has a transfer function with a Laurent expansion whose first N terms correspond identically with $\{M_i\}_N$; remaining coefficients of the Laurent expansion are called the *extension sequence.* It has been shown [12] that every finite sequence has a *minimal partial realization* that is computed from the elements $M_1, M_2, \cdots, M_N$. It will be convenient to arrange the Markov parameters in an array called a Hankel matrix given by

$$\sigma^k H(i,j) = \begin{bmatrix} M_{k+1} & M_{k+2} & & M_{k+j} \\ M_{k+2} & M_{k+3} & \cdots & M_{k+j+1} \\ \vdots & \vdots & & \vdots \\ M_{k+i} & M_{k+i+1} & \cdots & M_{k+i+j-1} \end{bmatrix}, \tag{13}$$

where the shift operator σ^k, $k \geqq 0$, effectively adds k to the subscript of each block element. If σ^k is omitted, k is understood to be zero.

Since $\{M_i\}_N$ has a minimal partial realization, it also has a pair of minimal partial numerator and denominator matrices, $Q_N(z)$ and $P_N(z)$, that satisfy (6) for the transfer function of any minimal partial realization.

THEOREM 2. *Let $P_N(z)$ be a minimal partial denominator matrix for $\{M_i\}_N$. Then the column degrees of $P_N(z)$ are bounded by*

$$n_{Ni} \leqq N, \qquad 1 \leqq i \leqq m. \tag{14}$$

Proof. Since the extension sequence may be arbitrarily chosen, the degree of any column of $P_N(z)$ need not exceed N. In fact, if for any i, $n_{Ni} = N$, coefficients for $p_{Ni}(z)$, excepting the leading coefficient, may be specified arbitrarily. □

Since (i) a minimal (partial) realization is easily constructed from a minimal (partial) representation [4], [13], and (ii) given a minimal $P(z)$ and the Markov parameter sequence, $Q(z)$ is easily obtained from (8)–(9), the minimal (partial) realization problem [12], [8] translates to the problem of extracting a minimal (partial) denominator matrix from a given finite sequence of Markov parameters.

For an arbitrary infinite sequence of $r \times m$ matrices this paper will establish that minimal partial denominator polynomial matrices for the finite length subsequences evolve recursively. This is a straightforward and useful approach to solving the multivariable minimal partial realization problem on the digital computer. These results are motivated by the work of Massey [9] where Massey's minimal length shift register synthesis algorithm is seen as a recursive means of constructing a minimal partial realization from a scalar, i.e., single-input, single-output sequence.

The main result of § 2 forms the theoretical basis for the sequential realization algorithm presented in § 3. In § 4 uniqueness is defined for sequentially generated denominator polynomial matrices. The sequential realization method is evaluated and compared with some existing realization techniques in § 5.

Notation. All scalar elements and polynomial coefficients belong to an arbitrary but fixed field $\mathscr{F}$; 0 (zero) denotes the additive and 1 the multiplicative identity elements of $\mathscr{F}$. An $m \times n$ matrix X having rank r over $\mathscr{F}$ is written $X \in \mathscr{F}_r^{m \times n}$. I_n represents the $n \times n$ identity matrix, 0 is any matrix of zeros, and the transpose of X is written X'. The range and null space of a matrix X are denoted $\mathscr{R}(X)$ and $\mathscr{N}(X)$, respectively. Elements of the integral domain $\mathscr{F}(z)$ are polynomials of degree l, $0 \leqq l < \infty$, with coefficients in $\mathscr{F}$. If $X(z)$ is an $m \times n$ matrix of rank r over $\mathscr{F}(z)$, $X(z) \in \mathscr{F}_r^{m \times n}(z)$. The units of $\mathscr{F}(z)$ are the nonzero elements of $\mathscr{F}$; an element is monic if its leading, i.e., highest degree, coefficient is 1. Additional notation will be presented as needed.

2. The sequential realization theorem. This section establishes the theoretical basis for the sequential realization algorithm. A lower bound on the dimension of a minimal partial realization is now given.

THEOREM 3. *The dimension of a minimal partial realization of the sequence* $\{M_i\}_N$, $1 \leqq N < \infty$, *is zero if and only if*

$$M_i = 0, \qquad i = 1, 2, \cdots, N. \tag{15}$$

Proof. Sufficiency. Assume (15) holds. Then $P_N(z)$ is any nonsingular matrix, and the composite degree is zero. Hence $n = 0$.

Necessity. Assume that $M_j \neq 0$ for any $j = 1, 2, \cdots, N$. Then from (11) at least one column of $P_N(z)$ has degree $n_i > 0$, implying $n > 0$. □

This theorem may obviously be extended to include minimal realizations of the infinite sequence of zero matrices. Any sequence for which the dimension of a minimal (partial) realization is zero will be called the *zero sequence*. Any nonsingular $m \times m$ matrix is a minimal denominator polynomial matrix for the zero sequence.

DEFINITION 1. The *zero length sequence*, denoted $\{M_i\}_0$, is the sequence having no elements. Since every infinite sequence, including the zero sequence, is an extension of the zero length sequence, a minimal partial realization of $\{M_i\}_0$ has dimension zero.

DEFINITION 2. The sequence $\{\overline{M}_i\}_j$ is said to be a *subsequence* of $\{M_i\}_k$ if $j \leqq k$ and $\overline{M}_i = M_i$, $i = 1, 2, \cdots, j$; $\{\overline{M}_i\}_j$ is a *proper subsequence* if $j < k$. Thus for any N, $1 \leqq N < \infty$, $\{M_i\}_N$ has N distinct proper subsequences, $\{M_i\}_0, \{M_i\}_1, \cdots, \{M_i\}_{N-1}$.

The main result of this section is the following theorem.

THEOREM 4 (The sequential realization theorem). *Let* $\{M_i\}_0, \{M_i\}_1, \cdots, \{M_1\}_j, \cdots$ *be the distinct proper subsequences of an arbitrary infinite sequence* $\{M_i\}$. *Then for any* $N \geqq 0$, *the information contained in the minimal partial denominator matrices for* $\{M_i\}_0, \{M_i\}_1, \cdots, \{M_i\}_N$, *denoted by* $P_0(z), P_1(z), \cdots, P_N(z)$, *respectively, is sufficient to calculate a minimal partial denominator matrix* $P_{N+1}(z)$ *for* $\{M_i\}_{N+1}$.

Proof. This theorem is proved in a vector space rather than a polynomial matrix formulation. Some notational preliminaries and lemmas are required before proceeding with the main proof.

To column i, $1 \leqq i \leqq m$, of $P_j(z)$, $0 \leqq j \leqq N$, there corresponds a CAV π_{ji} with degree n_{ji}; by Theorem 2, $0 \leqq n_{ji} \leqq j$. Applying π_{ji} to the shifted Hankel matrix of $\{M_i\}_{j+1}$ below yields a vector d_{ji}, where

$$d_{ji} = \sigma^{j-n_{ji}} H(1, n_{ji} + 1)\pi_{ji}. \tag{16}$$

DEFINITION 3. The $r \times m$ matrix D_j given by

$$D_j = [d_{j1} \quad d_{j2} \cdots d_{jm}] \tag{17}$$

is called the *j-th discrepancy matrix*. If D_j is zero, it is clear from (11) that $P_j(z)$ is also a minimal partial denominator matrix for $\{M_i\}_{j+1}$. More likely, $D_j \neq 0$, so an alternative, less obvious method for finding $P_{j+1}(z)$ is required. This is the topic of the remainder of this section.

The next two definitions are made primarily for notational convenience.

DEFINITION 4. For the vector π_{ji} of degree n_{ji} define the *augmented column annihilating vector* $\pi_{ji}(l, n_{ji} + 1, k)$ as

$$\pi_{ji}(l, n_{ji} + 1, k) = \begin{bmatrix} 0 \\ --- \\ \pi_{ji} \\ --- \\ 0 \end{bmatrix}. \tag{18}$$

That is, the vector $\pi_{ji}(l, n_{ji} + 1, k)$ consists of π_{ji} embedded between two zero vectors of lengths lm and km, respectively. The length of the augmented vector is $m(l + m_{ji} + 1 + k) \geqq m(n_{ji} + 1)$. Thus $\pi_{ji}(0, n_{ji} + 1, 0) = \pi_{ji}$. The degree associated with an augmented CAV is the same as the degree of the CAV being augmented.

DEFINITION 5. Associated with π_{ji} and $p_{ji}(z)$ is an integer k_{ji} called the *accumulation index relative to* N and given by

$$k_{ji} = (N - j) + n_{ji}, \qquad 0 \leqq j \leqq N, \quad 1 \leqq i \leqq m. \tag{19}$$

It will be shown that $p_{N+1,i}(z)$ is a polynomial combination of columns in $P_j(z)$, $0 \leqq j \leqq N$; the degree of $p_{N+1,i}(z)$ is determined from the largest accumulation index of elements comprising the combination. Equation (16) becomes

$$d_{ji} = \sigma^{j-n_{ji}} H(1, k_{ji} + 1)\pi_{ji}(0, n_{ji} + 1, N - j). \tag{20}$$

A slight rearrangement of (19) substituted into (20) yields

$$d_{ji} = H(1, N + 1)\pi_{ji}(N - k_{ji}, n_{ji} + 1, k_{ji} - n_{ji}). \tag{21}$$

LEMMA 1. *For any i and j, $1 \leqq i \leqq m$, $0 \leqq j \leqq N$, to each $p_{ji}(z)$ there corresponds a set of $N - k_{ji}$ linearly independent augmented CAV's in $\mathcal{N}[H(1, N + 1)]$.*

The proof follows from observing that for every $n_{ji} < j$ the structure of the Hankel matrices for $\{M_i\}_{N+1}$ implies

$$\sigma^{j-n_{ji}-l} H(1, k_{ji} + 1)\pi_{ji}(0, n_{ji} + 1, N - j) = 0, \qquad 1 \leqq l \leqq j - n_{ji}. \tag{22}$$

Since $N - k_{ji} = j - n_{ji}$, (22) becomes

$$H(1, N + 1)\pi_{ji}(N - k_{ji} - l, n_{ji} + 1, N - j + l) = 0, \qquad 1 \leqq l \leqq N - k_{ji}, \tag{23}$$

which proves the lemma.

Now let augmented CAV's from $P_j(z)$ form the columns of $\theta_j(N + 1) \in \mathscr{F}_m^{(N+1)m \times m}$, i.e.,

$$\theta_j(N + 1) = [\pi_{j1}(N - k_{j1}, n_{j1} + 1, N - j) \quad \cdots \quad \pi_{jm}(N - k_{jm}, n_{jm} + 1, N - j)] \tag{24}$$

so that from (21),

$$H(1, N + 1)\theta_j(N + 1) = D_j, \qquad 0 \leqq j \leqq N. \tag{25}$$

By letting

$$\Theta(N + 1) = [\theta_0(N + 1) \quad \theta_1(N + 1) \quad \cdots \quad \theta_N(N + 1)] \tag{26}$$

and

$$\Delta(N + 1) = [D_0 \quad D_1 \quad \cdots \quad D_N] = [\Delta(N) \vdots D_N], \tag{27}$$

equation (25) yields

$$H(1, N + 1)\Theta(N + 1) = \Delta(N + 1). \tag{28}$$

The matrix $\Theta(N + 1)$ is square and block upper triangular of size $(N + 1)m$; by construction the $m \times m$ diagonal blocks are the nonsingular leading coefficient matrices of $P_0(z), P_1(z), \cdots, P_N(z)$. Thus $\Theta(N + 1)$ is nonsingular.

LEMMA 2. $\mathscr{R}[H(1, N + 1)] = \mathscr{R}[\Delta(N + 1)]$.

The proof of this lemma is obvious from (28) and the nonsingularity of $\Theta(N + 1)$. In accordance with Definition 1 define $\Delta(0)$ and $H(1, 0)$ to be the zero vector so $\mathscr{R}[H(1, 0)] = \mathscr{R}[\Delta(0)]$, the space spanned by the zero vector.

The number of columns of $P_{N+1}(z)$ having degree equal to $N + 1$ may be determined from the Nth discrepancy matrix D_N and $\Delta(N)$. Let $\hat{\theta}_N(N + 1)$ and $\hat{D}_N$ with columns $\hat{d}_{Ni}$ be given by

$$\begin{aligned} \hat{D}_N &= D_N \hat{U}_N, \\ \hat{\theta}_N(N + 1) &= \theta_N(N + 1)\hat{U}_N, \end{aligned} \tag{29}$$

where $\hat{U}_N$ is obtained as follows. Define D_N^* with columns

$$d_{Ni}^* = \begin{cases} d_{Ni} & \text{if } d_{Ni} \notin \mathscr{R}[\Delta(N)], \\ 0 & \text{otherwise}, \end{cases} \qquad 1 \leqq i \leqq m. \tag{30}$$

Then form $\hat{U}_N$ unit upper triangular so that the nonzero columns of

$$\bar{D}_N = D_N^* \hat{U}_N \tag{31}$$

are linearly independent.

LEMMA 3. *There is a one-to-one correspondence between the nonzero columns of $\bar{D}_N$ and the columns of $P_{N+1}(z)$ having degree $N + 1$.*

By Theorem 2 each column of $P_{N+1}(z)$ may be placed in one of two categories: those with degree equal to $N + 1$, and those with degree less than $N + 1$. By Lemma 2 and (29), any linear combination of the columns of $H(1, N + 1)$ is also a linear combination of the columns of $[\Delta(N) \vdots \hat{D}_N]$. No combination of the columns of $[\Delta(N) \vdots \hat{D}_N]$ that includes a nonzero column of $\bar{D}_N$ can equal zero because the nonzero columns of $\bar{D}_N$ are linearly independent and are independent of the columns of $[\Delta(N) \vdots (\hat{D}_N - \bar{D}_N)]$. Since the leading coefficient matrix of $P_{N+1}(z)$ is nonsingular and every column is a CAP, to every nonzero column of $\bar{D}_N$ there must correspond a column of $P_{N+1}(z)$ with degree $N + 1$. It remains to show that to every zero column of $\bar{D}_N$ there corresponds a column of $P_{N+1}(z)$ with degree less than $N + 1$. For any i, $1 \leqq i \leqq m$, suppose $\bar{d}_{Ni} = 0$. Then there exists a vector t such that

$$\Delta(N)t + \hat{d}_{Ni} = 0.$$

By Lemma 2 there exists a vector x such that

$$H(1, N)x + \hat{d}_{Ni} = 0.$$

Substituting for $\hat{d}_{Ni}$,

$$H(1, N + 1)\left[\begin{bmatrix} x \\ \hdashline 0 \end{bmatrix} + \hat{\pi}_{Ni}(N - \hat{n}_{Ni}, \hat{n}_{Ni} + 1, 0)\right] = 0, \tag{32}$$

where $\hat{\pi}_{Ni}$ is the ith column of $\hat{\theta}_N(N + 1)$. The vector postmultiplying $H(1, N + 1)$ in (32) is clearly an augmented CAV for $\{M_i\}_{N+1}$. Thus an augmented CAV satisfying (32) exists for every zero column of $\bar{D}_N$. Since $P_{N+1}(z)$ is minimal, Lemma 3 is proved.

If there are r linearly independent columns in $\Delta(N + 1)$, it is of full row rank. By Lemma 3 under the restrictions imposed by (29)–(31), there can be no more than r columns in all of the $P_j(z)$, $0 \leqq j \leqq N + 1$, for which $n_{ji} = j$. Aside from these at most r columns, the degree associated with each column of $P_j(z)$ must be strictly less than j, $0 \leqq j \leqq N + 1$.

Let the $(N + 1)m$-vector ϕ represent any augmented CAV of degree $n_\phi < N + 1$ associated with the finite sequence $\{M_i\}_{N+1}$. Moreover, assume ϕ is partitioned in m-vector segments as

$$\phi = \begin{bmatrix} \phi_N \\ \phi_{N-1} \\ \vdots \\ \phi_1 \\ \phi_0 \end{bmatrix}, \qquad \phi_0 \neq 0.$$

Then

$$H(1, N + 1)\phi = \sum_{j=0}^{n_\phi} M_{N+1-j}\phi_j = 0. \tag{33}$$

Since $\Theta(N + 1)$ is nonsingular, there exists a vector x, partitioned like ϕ and satisfying,

$$\phi = \Theta(N + 1)x = \sum_{j=0}^{N} \theta_{N-j}(N + 1)x_j. \tag{34}$$

LEMMA 4. *Let ϕ and x be as given above and let x_{ji}, $1 \leqq i \leqq m$, denote the elements of x_j. Then ϕ is a linear combination of augmented CAV's from $P_0(z)$, $P_1(z), \cdots, P_N(z)$, and its degree is*

$$n_\phi = \max_{0 \leqq j \leqq N} \left\{ \max_{1 \leqq i \leqq m} \{k_{N-j,i} | x_{ji} \neq 0\} \right\}. \tag{35}$$

The first part of the lemma follows from (34); it remains to prove (35). First assume that $x_0 \neq 0$ and that

$$n_\phi < \max_{1 \leqq i \leqq m} \{k_{Ni} | x_{0i} \neq 0\}.$$

Then replace the column of $\theta_N(N + 1)$ with nonzero x_{0i} and the largest accumulation index with ϕ. Note that $k_{Ni} = n_{Ni}$. The result is a new matrix, $\tilde{\theta}_N(N + 1)$, of augmented CAV's with a lower composite degree than $\theta_N(N + 1)$. This would contradict the minimality of $P_N(z)$ so

$$n_\phi \geqq \max_{1 \leqq i \leqq m} \{k_{Ni} | x_{0i} \neq 0\}. \tag{36}$$

The proof is continued for $j = 1, 2, \cdots, N$ on the vectors ϕ^j given by

$$\phi^j = \phi - \sum_{l=0}^{j-1} \theta_{N-l}(N + 1)x_l. \tag{37}$$

Assume $x_j \neq 0$ and that

$$n_\phi < \max_{1 \leqq i \leqq m} \{k_{N-j,i} | x_{ji} \neq 0\}.$$

Then ϕ^j replaces the column of $\theta_{N-j}(N+1)$ having the highest accumulation index and $x_{ji} \neq 0$ to yield a matrix of augmented CAV's with a lower composite degree than $\theta_{N-j}(N+1)$. This contradicts the assumed minimality of $P_{N-j}(z)$. Hence combining with (36),

$$n_\phi \geqq \max_{0 \geqq j \geqq N} \left\{ \max_{1 \geqq i \geqq m} \{k_{N-j,i} | x_{ji} \neq 0\} \right\}.$$

But by Lemma 1 every column of $\Theta(N+1)$ for which $x_{ji} \neq 0$ can, by appropriate internal shifts, generate $N - k_{N-j,i}$ linearly independent vectors in $\mathscr{N}[H(1, N+1)]$. By the same argument ϕ can be shifted internally to produce a total of $N + 1 - n_\phi$ augmented CAV's in $\mathscr{N}[H(1, N+1)]$. Hence Lemma 4 is proved.

Now consider the matrix $\theta_{N+1}(N+2) \in \mathscr{F}_m^{(N+2)m \times m}$ with columns

$$\pi_{N+1,i}(N + 1 - n_{N+1,i}, n_{N+1,i} + 1, 0), \quad 1 \leqq i \leqq m,$$

constructed from $P_{N+1}(z)$. The first m rows in any column of $\theta_{N+1}(N+2)$ having degree less than $N+1$ are zero; columns with degree $N+1$ can have the first m row elements set arbitrarily to zero. Thus $\theta_{N+1}(N+2)$ may be written

$$\theta_{N+1}(N+2) = \begin{bmatrix} 0 \\ \hline F_N \end{bmatrix}, \tag{38}$$

where $F_N \in \mathscr{F}_m^{(N+1)m \times m}$

Certain nonsingular matrices specified in the following lemma postmultiply $\theta_{N+1}(N+2)$ to yield a matrix of augmented CAV's having the same composite degree.

LEMMA 5. *The composite degree of* $\theta_{N+1}(N+2)$ *remains invariant under post-multiplication by a nonsingular matrix* R *if* (i) R *is a diagonal matrix,* (ii) R *is a permutation matrix or* (iii) R *is lower triangular and the columns of* $\theta_{N+1}(N+2)$ *are ordered from the left by decreasing degree, i.e., any column of* $\theta_{N+1}(N+2)$ *has degree less than or equal to the degree of every column to its left.*

Now to proceed with the proof of Theorem 4. From Lemmas 3 and 4 it is clear that

$$H(1, N+1)F_N = E_N, \tag{39}$$

where the nonzero columns of E_N are linearly independent and are not elements of $\mathscr{R}[H(1, N)]$. Moreover, for every nonzero column of E_N there is a column of $P_{N+1}(z)$ with degree equal to $N+1$. Since $\Theta(N+1)$ is nonsingular, there exist matrices $W_N \in \mathscr{F}_m^{m \times m}$ and $V_N \in \mathscr{F}^{Nm \times m}$ satisfying

$$F_N = \Theta(N+1) \begin{bmatrix} V_N \\ \hline W_N \end{bmatrix}. \tag{40}$$

From Lemma 2 and (27)–(28),

$$H(1, N+1)F_N = [\Delta(N) \vdots D_N] \begin{bmatrix} V_N \\ \hline W_N \end{bmatrix}. \tag{41}$$

It is now claimed, without loss of generality, that W_N is unit upper triangular. In its most general form the nonsingular W_N may be factored [3] as follows:

$$W_N = U_N R_N L_N, \tag{42}$$

where U_N is unit upper triangular, L_N is lower triangular and R_N is a permutation matrix. Should W_N have the form (42) with the columns of $\theta_{N+1}(N + 2)$ ordered from left to right by decreasing degree, $\theta_{N+1}(N + 2)$ can be postmultiplied first by L_N^{-1} and then by R_N'. According to Lemma 5, the resulting matrix of augmented CAV's has the same composite degree as the original and is thus minimal. Hence $W_N = U_N$. Note also that U_N contains $\hat{U}_N$ of (29)–(31) as a factor, i.e., U_N, $\hat{U}_N$, $\hat{U}_N^{-1}U_N$ are all unit upper triangular and

$$H(1, N + 1)F_N = [\Delta(N) \vdots \hat{D}_N]\left[\begin{array}{c} V_N \\ \hdashline \hat{U}_N^{-1}U_N \end{array}\right]. \tag{43}$$

Combining (38)–(40) expresses in augmented column annihilating vector form $P_{N+1}(z)$ as a combination of the columns of $P_0(z), P_1(z), \cdots, P_N(z)$, i.e.,

$$\theta_{N+1}(N + 2) = \left[\begin{array}{c} 0 \\ \hdashline \Theta(N + 1) \end{array}\right]\left[\begin{array}{c} V_N \\ \hdashline U_N \end{array}\right]. \tag{44}$$

This proves Theorem 4. □

The form of (44), i.e., U_N upper triangular with unity diagonal elements, indicates that, regardless of the associated degree, column i in $\theta_{N+1}(N + 2)$ is a shifted linear combination of column i in $\theta_N(N + 1)$ with columns to its left in $\Theta(N + 1)$.

3. The sequential realization algorithm. The results of the previous section will be used in this section to develop an algorithm for the recursive construction of the minimal partial denominator matrices of the finite length subsequences of a given infinite sequence. The proof of Theorem 4 is constructive because it demonstrates that column i of $\theta_{N+1}(N + 2)$ equals column i of $\theta_N(N + 1)$ shifted and added to a linear combination of shifted columns to the left of i in $\Theta(N + 1)$. The exact numerical form of this linear combination is dependent upon column i of the Nth discrepancy matrix, and it is selected to minimize $n_{N+1,i}$. Let

$$\Delta_i(N) = \begin{cases} \Delta(N), & i = 1, \\ [\Delta(N) \vdots d_{N1} \quad d_{N2} \quad \cdots \quad d_{N,i-1}], & 1 < i \leqq m. \end{cases} \tag{45}$$

By Lemma 3 and construction, $n_{N+1,i} = N + 1$ if and only if $d_{Ni} \notin \mathscr{R}[\Delta_i(N)]$. If $d_{Ni} \notin \mathscr{R}[\Delta_i(N)]$, every row element in column i of $\theta_{N+1}(N + 2)$ is arbitrary except for the last $m + 1 - i$; hence the linear combination is also arbitrary.

If $d_{Ni} \in \mathscr{R}[\Delta_i(N)]$, $n_{N+1,i} < N + 1$ and the selected linear combination must satisfy two conditions: when applied to the columns of $\Delta_i(N)$, it must yield $-d_{Ni}$; and when applied to the columns of $\Theta(N + 1)$, $n_{N+1,i}$, given by (35) as a function of the accumulation indices of columns included in the combination, must be minimized. If $n_{N+1,i}$ is minimized in this manner column by column, (44) clearly indicates that the composite degree of $\theta_{N+1}(N + 2)$ is also minimized.

Let $\rho_{Ni} = \operatorname{rank}[\Delta_i(N)]$. Since a column basis for $\Delta_i(N)$ is formed from ρ_{Ni} linearly independent columns, $0 \leqq \rho_{Ni} \leqq r$, usually only a subset of the columns of $\Delta_i(N)$ need be considered for any minimal linear combination that produces $-d_{Ni} \in \mathscr{R}[\Delta_i(N)]$; viz. a subset of ρ_{Ni} linearly independent columns generated by a corresponding subset of columns in $\Theta(N + 1)$ whose largest accumulation index is minimized. More precisely, let K_{Ni} be a spanning set of columns of $\Delta_i(N)$ generated by

$$H(1, N + 1)\Phi_{Ni} = K_{Ni}, \tag{46}$$

where

$$\Phi_{Ni} = \begin{cases} [\phi_{Ni1} \quad \phi_{Ni2} \quad \cdots \quad \phi_{Ni\rho_{Ni}}], & \rho_{Ni} \geqq 1, \\ 0, \quad \rho_{Ni} = 0. \end{cases} \tag{47}$$

The columns ϕ_{Nij} are columns of $\Theta(N + 1)$ with degree $n_{\phi Nij}$ and accumulation index $k_{\phi Nij}$ selected so as to minimize $\max_{1 \leqq j \leqq \rho_{Ni}} \{k_{\phi Nij}\}$. By the previous discussion, if $n_{N+1,i} < N + 1$, either $n_{N+1,i} = n_{Ni}$ or $n_{N+1,i}$ equals the largest accumulation index of the columns of Φ_{Ni} included in the linear combination. In either case $n_{N+1,i}$ is minimized. Finally, for notational consistency, let $b_{Nij}(z)$ denote the column of $P_0(z), P_1(z), \cdots, P_N(z)$ from which ϕ_{Nij} was constructed, and let

$$B_{Ni}(z) = \begin{cases} [b_{Ni1}(z) \quad b_{Ni2}(z) \quad \cdots \quad b_{Ni\rho_{Ni}}(z)], & \rho_{Ni} \geqq 1, \\ 0, \quad \rho_{Ni} = 0. \end{cases} \tag{48}$$

The following algorithm is a procedure for computing a minimal partial denominator polynomial matrix for a finite matrix sequence of length $N_0 \geqq 0$. Initially, I_m is assumed as the minimal denominator for the zero length sequence; the procedure halts with the minimal denominator for $\{M_i\}_{N_0}$. The notation is consistent with that presented previously except that the subscript Ni has been dropped for convenience. Also the algorithm is presented in polynomial rather than vector space notation.

THE SEQUENTIAL REALIZATION ALGORITHM.

Step 1. *Initialization.* Set $N = \rho = 0$, $B(z) = 0$, $K = 0$, $P(z) = I_m$, $n_i = 0$, $1 \leqq i \leqq m$.

Step 2. If $N = N_0$, stop.

Step 3. Otherwise perform Steps 4–14 for $i = 1, 2, \cdots, m$.

Step 4. From column i of $P(z)$ compute

$$d = \sum_{j=0}^{n_i} M_{N+1-j} p_{ji}.$$

Step 5. If $d = 0$, go to Step 14.

Step 6. If $d \neq 0$ and $\rho = 0$, set $\rho = 1$, $K = d$, $B(z) = b_1(z) = p_i(z)$, $k_{\phi 1} = n_{\phi 1} = n_i$, $p_i(z) = p_i(z)z^{N+1-n_i}$, $n_i = N + 1$ and go to Step 14.

Step 7. If $d \neq 0$ and $\rho > 0$, compute $x = K^{\#}d$,

$$\bar{d} = d - Kx = (I - KK^{\#})d,$$

where $K^{\#}$ is a pseudoinverse of K. That is, $K = KK^{\#}K$. (See [3] for a discussion of pseudoinverses of finite field matrices.)

Step 8. Set

$$n_t = \begin{cases} N + 1 & \text{if } \bar{d} \neq 0, \\ \max\left\{ n_i, \max\limits_{1 \leqq j \leqq \rho} \{k_{\phi j} | x_j \neq 0\} \right\} & \text{if } \bar{d} = 0 \end{cases}$$

and

$$t(z) = p_i(z) z^{(n_t - n_i)} - \sum_{j=1}^{\rho} b_j(z) x_j z^{(n_t - k_{\phi j})}.$$

Step 9. If $n_t = n_i$, go to Step 13.

Step 10. If $n_t > n_i$, set $K = [K \vdots d]$, $b_{\rho+1}(z) = p_i(z)$, $B(z) = [B(z) \vdots p_i(z)]$, $k_{\phi,\rho+1} = n_{\phi,\rho+1} = n_i$.

Step 11. Order the columns of $B(z)$ from left to right by increasing accumulation index; place the corresponding columns of K in the same order.

Step 12. Set $\rho = \operatorname{rank}(K)$. Discard the linearly dependent column of K, if any, with the largest accumulation index, i.e., the right-most linearly dependent column. Remove from $B(z)$ the column corresponding to any discarded column of K. If necessary, renumber the columns of K and $B(z)$ from 1 through ρ.

Step 13. Set $p_i(z) = t(z)$ and $n_i = n_t$.

Step 14. If $i < m$, increment i by 1 and go to Step 4.

Step 15. If $i = m$, set $N = N + 1$. If $\rho > 0$, set $k_{\phi j} = k_{\phi j} + 1$, $1 \leqq j \leqq \rho$, and go to Step 2.

Before demonstrating the algorithm on some examples it should be noted that the following values hold at Step 2 for any $N \leqq N_0$:

$$P(z) = P_N(z) \quad \text{with column degrees} \quad n_i = n_{Ni}, \quad 1 \leqq i \leqq m, \quad n = n_N;$$

$$K = K_{N1} \quad \text{of rank} \quad \rho_{N1} = \operatorname{rank}[\Delta(N)] = \rho;$$

$$B(z) = B_{N1}(z) \quad \text{with accumulation indices} \quad k_{\phi j} = k_{\phi N1j}, \quad 1 \leqq j \leqq \rho_{N1}.$$

The algorithm is now applied to an example from [12] where the sequence is defined over the real number field. Parameter values are given at Step 2 of the algorithm for $N = 0, 1, 2, 3, 4$.

Example 1. Let $N_0 = 4$ and $\{M_i\}_4 = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 10 & 7 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 22 & 15 \\ 3 & 3 \end{pmatrix} \right\}$.

$$N = 0: \quad P(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad n_1 = n_2 = n = 0,$$

$$K = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \rho = 0,$$

$$B(z) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$N = 1$: $\quad P(z) = \begin{bmatrix} z & -1 \\ 0 & 1 \end{bmatrix}, \quad n_1 = 1, \; n_2 = 0, \; n = 1,$

$$K = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \rho = 1,$$

$$B(z) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad k_{\phi 1} = 1.$$

$N = 2$: $\quad P(z) = \begin{bmatrix} z - 4 & -z + 1 \\ 0 & z \end{bmatrix}, \quad n_1 = n_2 = 1, \; n = 2,$

$$K = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \rho = 1,$$

$$B(z) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad k_{\phi 1} = 1.$$

$N = 3$: $\quad P(z) = \begin{bmatrix} z^3 + 2z^2 & -z \\ -6z^2 & z + 1 \end{bmatrix}, \quad n_1 = 3, \; n_2 = 1, \; n = 4,$

$$K = \begin{bmatrix} -1 & -6 \\ 0 & 1 \end{bmatrix}, \quad \rho = 2,$$

$$B(z) = \begin{bmatrix} -1 & z - 4 \\ 1 & 0 \end{bmatrix}, \quad k_{\phi 1} = k_{\phi 2} = 2.$$

$N = 4$: $\quad P(z) = \begin{bmatrix} z^3 + 3z^2 + 2z & -z^2 - z - 2 \\ -6z^2 - 6z & z^2 + z + 6 \end{bmatrix}, \quad n_1 = 3, \; n_2 = 2, \; n = 5,$

$$K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \rho = 2,$$

$$B(z) = \begin{bmatrix} -z & -1 \\ z + 1 & 1 \end{bmatrix}, \quad k_{\phi 1} = 2, \; k_{\phi 2} = 3.$$

4. Uniqueness of minimal partial denominator matrices. In this section uniqueness of a minimal partial denominator polynomial matrix is defined, and a criterion for determining the uniqueness of a sequentially computed denominator matrix is presented. The information carried along while applying the sequential realization algorithm, viz. N, ρ, $P(z)$, $B(z)$, K, $\{n_i, i = 1, \cdots, m\}$ and $\{n_{\phi j}, k_{\phi j}, j = 1, \cdots, \rho\}$, is shown to be sufficient for constructing the equivalence class of minimal partial denominator matrices after the algorithm terminates.

First consider the set of all minimal partial denominator matrices for $\{M_i\}_{N+1}$, i.e., let

$$\Omega_{N+1} = \{\tilde{P}_{N+1}(z) | \tilde{P}_{N+1}(z) \text{ is a minimal partial denominator matrix for } \{M_i\}_{N+1}\}. \tag{49}$$

It is useful to define uniqueness for a minimal partial denominator matrix in terms of its invariant factors.

DEFINITION 6. The minimal partial denominator matrix $P_{N+1}(z)$ for $\{M_i\}_{N+1}$ is *unique* if it and every other element $\tilde{P}_{N+1}(z) \in \Omega_{N+1}$ has the same set of invariant factors.

The above is consistent with the definition given in [8] for uniqueness of minimal realizations modulo a choice of basis for the state. That is, a minimal realization is said to be unique modulo an equivalence class of similar system matrices. This definition has now been extended to partial realizations via the polynomial matrix formulation.

THEOREM 5. *Let $P_{N+1}(z)$ and $B_{N+1,1}(z)$ be computed as above and let $\tilde{P}_{N+1}(z) \in \Omega_{N+1}$. Then there exist polynomial matrices $X_{N+1}(z)$ and $Y_{N+1}(z)$ such that*

$$\tilde{P}_{N+1}(z) = P_{N+1}(z)X_{N+1}(z) + B_{N+1,1}(z)Y_{N+1}(z), \tag{50}$$

where $X_{N+1}(z)$ is an elementary matrix.

Proof. Let $\tilde{\theta}_{N+1}(N+2)$ be a set of augmented CAV's constructed from $\tilde{P}_{N+1}(z)$ according to (24). Then $\tilde{\theta}_{N+1}(N+2)$ generates a discrepancy matrix when postmultiplying $H(1, N+2)$. Since $\Theta(N+2)$ is nonsingular and upper block triangular,

$$\begin{aligned} \tilde{\theta}_{N+1}(N+2) &= \Theta(N+2)\begin{bmatrix} T_{N+1} \\ \hdashline R_{N+1} \end{bmatrix} \\ &= \begin{bmatrix} \Theta(N+1) \\ \hdashline 0 \end{bmatrix} T_{N+1} + \theta_{N+1}(N+2)R_{N+1}, \end{aligned} \tag{51}$$

where $\theta_{N+1}(N+2)$ is given by (38). Note that elements of R_{N+1} and T_{N+1} are chosen so that the composite degree of $\tilde{\theta}_{N+1}(N+2)$ equals that of $\theta_{N+1}(N+1)$. Lemma 5 gives the restrictions on the nonsingular matrix R_{N+1}. Without loss of generality, column permutations will be ignored so that R_{N+1} must have nonzero diagonal elements, and there is a column degree equivalence between corresponding columns of $\tilde{\theta}_{N+1}(N+2)$ and $\theta_{N+1}(N+2)$. Moreover, for off-diagonal elements

$$\|R_{N+1}\|_{ji} \neq 0 \quad \text{only if } n_{N+1,j} \leqq n_{N+1,i}, \qquad 1 \leqq i \neq j \leqq m. \tag{52}$$

That is, only those columns of $\theta_{N+1}(N+2)$ for which the degree is not greater than $n_{N+1,i}$ can be combined with column i of $\theta_{N+1}(N+2)$ to form column i of $\theta_{N+1}(N+2)$. The matrix T_{N+1} has elements restricted for $1 \leqq i, l \leqq m, 0 \leqq j \leqq N$, by

$$\|T_{N+1}\|_{jm+l,i} \neq 0 \quad \text{only if } k_{jl} \leqq n_{N+1,i}, \tag{53}$$

where k_{jl} is the accumulation index *relative to* $N+1$. The linear combinations of (51) allow every variation of $\tilde{\theta}_{N+1}(N+2)$ that leaves the composite degree unchanged. Thus (51) under the restrictions of (52)–(53) specifies the equivalence class of minimal sets of augmented CAV's for $\{M_i\}_{N+1}$.

Now for the matrix $\theta_{N+1}(N+2)$ define the shift matrix $\sigma^{-1}[\theta_{N+1}(N+2)]$ whose columns consist of every distinct augmented CAV of length $(N+2)m$

associated with $P_{N+1}(z)$ except those in $\theta_{N+1}(N+2)$. Thus by Lemma 1 there are $N+1-n_{N+1,i}$ columns in $\sigma^{-1}[\theta_{N+1}(N+2)]$ for every column $p_{N+1,i}(z)$, $1 \leqq i \leqq m$. Let $\sigma^{-1}[\theta_{N+1}(N+2)]$ be the zero vector if every column of $P_{N+1}(z)$ has degree $N+1$.

Let $\Phi_{N+1,1}$, given by (46)–(47), be the subset of columns of $\Theta(N+2)$ that generates discrepancies forming a column basis for $\Delta_1(N+1) = \Delta(N+1)$ with minimum largest accumulation index relative to $N+1$. Define the shift matrix $\sigma^{-1}(\Phi_{N+1,1})$ whose columns are every distinct augmented CAV of length $(N+2)m$ associated with $B_{N+1,1}(z)$ with the last m rows zero and not in $\Phi_{N+1,1}$. Thus for every column $b_{N+1,1j}(z)$, $1 \leqq j \leqq \rho_{N+1,1}$, there are $N+1-k_{\phi,N+1,1j}$ linearly independent columns in $\sigma^{-1}(\Phi_{N+1,1})$; the columns of $\Phi_{N+1,1}$ and $\sigma^{-1}(\Phi_{N+1,1})$ are all linearly independent by construction. If the accumulation index is $N+1$ for every column of $\Phi_{N+1,1}$ or if $\Phi_{N+1,1} = 0$, let $\sigma^{-1}(\Phi_{N+1,1})$ be the zero vector.

LEMMA 6. *The columns of* $\theta_{N+1}(N+2)$, $\sigma^{-1}[\theta_{N+1}(N+2)]$, $\Phi_{N+1,1}$ *and* $\sigma^{-1}(\Phi_{N+1,1})$ *are a basis for the column space of* $\Theta(N+2)$.

First from the algorithm presented in § 3 it is readily verified that

$$\theta_N(N+2), \quad \sigma^{-1}[\theta_N(N+2)], \quad \begin{bmatrix} \Phi_{N,1} \\ \hdashline 0 \end{bmatrix} \quad \text{and} \quad \sigma^{-1}\begin{bmatrix} \Phi_{N,1} \\ \hdashline 0 \end{bmatrix}$$

are generated as linear combinations of

$$\sigma^{-1}[\theta_{N+1}(N+2)], \Phi_{N+1,1} \quad \text{and} \quad \sigma^{-1}(\Phi_{N+1,1}).$$

Reversing the algorithm it may be shown by successive induction that the column space of $\Theta(N+2)$ is spanned by columns of $\theta_{N+1}(N+2), \sigma^{-1}[\theta_{N+1}(N+2)]$, $\Phi_{N+1,1}$ and $\sigma^{-1}(\Phi_{N+1,1})$. It remains to show that a mutual linear independence exists between the columns of these four matrices.

Since the last m rows of $\theta_{N+1}(N+2)$ form a nonsingular matrix, its columns are obviously independent of $[\Phi_{N+1,1} \vdots \sigma^{-1}[\theta_{N+1}(N+2)] \vdots \sigma^{-1}(\Phi_{N+1,1})]$. Let x and y be arbitrary with $x \in \mathscr{R}[\sigma^{-1}[\theta_{N+1}(N+2)] \vdots \sigma^{-1}(\Phi_{N+1,1})]$ and $y \in \mathscr{R}[\Phi_{N+1,1}]$. Then $H(1, N+2)x = 0$, but $H(1, N+2)y = d \neq 0$ implying $x \neq y$. Thus the columns of $\Phi_{N+1,1}$ are linearly independent of $[\sigma^{-1}[\theta_{N+1}(N+2)] \vdots \sigma^{-1}(\Phi_{N+1,1})]$. To show that columns of $\sigma^{-1}(\Phi_{N+1,1})$ and $\sigma^{-1}[\theta_{N+1}(N+2)]$ are mutually independent, assume that some element $t \in \mathscr{R}[\sigma^{-1}(\Phi_{N+1,1})]$ is a linear combination of columns in $\sigma^{-1}[\theta_{N+1}(N+2)]$. By the structure of the shift matrix $\sigma^{-1}(\Phi_{N+1,1})$, t can be shifted internally. This yields the contradictory implication that an element in $\mathscr{R}(\Phi_{N+1,1})$ is a linear combination of columns in $\sigma^{-1}[\theta_{N+1}(N+2)]$ and $\sigma^{-1}(\Phi_{N+1,1})$. The proof of Lemma 6 is now complete.

From Lemma 6 it is possible to rewrite (51) as

$$\begin{aligned} \tilde{\theta}_{N+1}(N+2) = \theta_{N+1}(N+2)R_{N+1,1} + \sigma^{-1}[\theta_{N+1}(N+2)]R_{N+1,2} \\ + \Phi_{N+1,1}T_{N+1,1} + \sigma^{-1}(\Phi_{N+1,1})T_{N+1,2} \end{aligned} \tag{54}$$

with $R_{N+1,1} = R_{N+1}$ and appropriate restrictions on the elements of $R_{N+1,2}$, $T_{N+1,1}$ and $T_{N+1,2}$. But since an internally shifted CAV has an alternative representation as a CAP multiplied by the polynomial indeterminate to a nonnegative power, (54) has the more compact polynomial representation of (50). If column permutations are again ignored, $X_{N+1}(z)$ has nonzero diagonal elements; the elements of $X_{N+1}(z)$ are given for $1 \leqq i, j \leqq m$ by

$$(55)\quad x_{N+1,ji}(z) = \begin{cases} \displaystyle\sum_{l=0}^{n_{N+1,i}-n_{N+1,j}} x_{N+1,jil} z^{n_{N+1,i}-n_{N+1,j}-l}, & n_{N+1,i} \geqq n_{N+1,j}, \\ 0, \quad n_{N+1,i} < n_{N+1,j}. \end{cases}$$

Elements of $Y_{N+1}(z)$ are given for $1 \leqq j \leqq \rho_{N+1,1}$, $1 \leqq i \leqq m$, by

$$(56)\quad y_{N+1,ji}(z) = \begin{cases} \displaystyle\sum_{l=0}^{n_{N+1,i}-k_{\phi N+1,1j}} y_{N+1,jil} z^{n_{N+1,i}-k_{\phi N+1,1j}-l}, & \\ & n_{N+1,i} \geqq k_{\phi N+1,1j}, \\ 0, \quad n_{N+1,i} < k_{\phi N+1,1j}. \end{cases}$$

Suppose that the columns of $P_{N+1}(z)$ are ordered from the left by decreasing degree. It is then clear from (55) that $X_{N+1}(z)$ is block lower triangular with diagonal blocks that are nonsingular matrices of elements in $\mathscr{F}$. Hence $X_{N+1}(z)$ is an elementary matrix, and Theorem 5 is proved. □

THEOREM 6. *$P_{N+1}(z)$ is a unique (by Definition 6) minimal partial denominator matrix for $\{M_i\}_{N+1}$ if and only if*

$$(57)\qquad \min_{0 \leqq j \leqq N} \left\{ \min_{1 \leqq i \leqq m} \{k_{ji} | d_{ji} \neq 0\} \right\} > \max_{1 \leqq l \leqq m} \{n_{N+1,l}\},$$

where the accumulation indices k_{ji} are relative to $N+1$.

Proof. By Theorem 5, $X_{N+1}(z)$ is an elementary matrix, and by Lemma 6, no column of $B_{N+1,1}(z)$ is a polynomial combination of columns in $P_{N+1}(z)$. Hence $P_{N+1}(z)$ is unique by Definition 6 if and only if

$$(58)\qquad Y_{N+1}(z) \equiv 0.$$

But inspection of (56) shows that (58) holds if and only if

$$(59)\qquad \min_{1 \leqq j \leqq \rho_{N+1,1}} \{k_{\phi N+1,1j}\} > \max_{1 \leqq i \leqq m} \{n_{N+1,i}\}.$$

By the process of selection the columns of $\Phi_{N+1,1}$, (59) and (57) are equivalent conditions. This completes the proof of Theorem 6. □

Now (50) with (55)–(56) enumerate the entire equivalence class Ω_{N+1} given the parameters required by the sequential realization algorithm. Also (58) provides a criterion for uniqueness. If (58) does not hold, $P_{N+1}(z)$ is not unique, and it may be desirable to examine the range of variation in the set of invariant factors over the elements in Ω_{N+1}. This task can be simplified considerably by considering

$$(60)\qquad \bar{P}_{N+1}(z) = P_{N+1,0}^{-1} \tilde{P}_{N+1}(z) X_{N+1}^{-1}(z).$$

Since $P_{N+1,0}$ and $X_{N+1}(z)$ are elementary matrices, $\bar{P}_{N+1}(z)$ and $\tilde{P}_{N+1}(z)$ are equivalent and thus have the same set of invariant factors. Moreover, $\bar{P}_{N+1}(z)$ can be written

$$(61)\qquad \bar{P}_{N+1}(z) = S_{N+1}(z) + \bar{B}_{N+1,1}(z) \bar{Y}_{N+1}(z),$$

where $S_{N+1}(z)$ is obtained from $P_{N+1}(z)$ by (10), $\bar{B}_{N+1,1}(z) = P_{N+1,0}^{-1}B_{N+1,1}(z)$ and $\bar{Y}_{N+1}(z) = Y_{N+1}(z)X_{N+1}^{-1}(z)$. Corresponding columns of $\bar{B}_{N+1,1}(z)$ and $B_{N+1,1}(z)$ have the same degree so that elements of $\bar{Y}_{N+1}(z)$ must obey the same constraints, viz. (56), as $Y_{N+1}(z)$. Every column of $B_{N+1,1}(z)Y_{N+1}(z)$ has degree strictly less than the corresponding column of $P_{N+1}(z)X_{N+1}(z)$ by construction; a similar relation holds between columns of $\bar{B}_{N+1,1}(z)\bar{Y}_{N+1}(z)$ and $S_{N+1}(z)$. Although (61) and (56) simplify the task of enumerating the invariant factor sets of a nonunique minimal partial denominator matrix, in general complicated, nonlinear, algebraic manipulations may be required as the following example serves to illustrate.

Example 2. Continuing with the sequence of Example 1, $P_4(z)$ is clearly not unique by Definition 6. It is the purpose of this example to enumerate all invariant factor sets for $P_4(z)$.

$$P_4(z) = \begin{bmatrix} z^3 + 3z^2 + 2z & -z^2 - z - 2 \\ -6z^2 - 6z & z^2 + z + 6 \end{bmatrix},$$

$$B_{4,1}(z) = \begin{bmatrix} -z & -1 \\ z + 1 & 1 \end{bmatrix}, \qquad k_{\phi 1} = 2, \quad k_{\phi 2} = 3,$$

$$P_{4,0} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \qquad P_{4,0}^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$S_4(z) = \begin{bmatrix} z^3 - 3z^2 - 4z & 4 \\ -6z^2 - 6z & z^2 + z + 6 \end{bmatrix}, \qquad \bar{B}_{4,1}(z) = \begin{bmatrix} 1 & 0 \\ z + 1 & 1 \end{bmatrix}.$$

Then $\bar{Y}_4(z)$ has the form

$$\bar{Y}_4(z) = \begin{bmatrix} uz + v & w \\ t & 0 \end{bmatrix}$$

with parameters t, u, v and w. In general,

$$\bar{P}_4(z) = \begin{bmatrix} z^3 - 3z^2 + (u - 4)z + v & w + 4 \\ (u - 6)z^2 + (u + v - 6)z + (v + t) & z^2 + (w + 1)z + (w + 6) \end{bmatrix}.$$

Case 1. $w \neq -4$. Clearly $\gamma_{\bar{P}1} = 1$ and $\gamma_{\bar{P}2}$ is the characteristic equation. Thus

$$\begin{aligned}\gamma_{\bar{P}2} = z^5 &+ (w - 2)z^4 + (u - 2w - 1)z^3 + (v - 3u - w + 2)z^2 \\ &+ (2u - 3v + 2w)z + 2v - t(w + 4).\end{aligned}$$

If the desired characteristic equation is

$$\gamma_{\bar{P}2} = z^5 + \beta_1 z^4 + \beta_2 z^3 + \beta_3 z^2 + \beta_4 z + \beta_5,$$

then all but one of the coefficients may be independently specified;

$$\beta_4 = -(15\beta_1 + 7\beta_2 + 3\beta_3 + 31)$$

and

$$w = \beta_1 + 2, \qquad t = \frac{1}{\beta_1 + 6}(14\beta_1 + 6\beta_2 + 2\beta_3 - \beta_5 + 30),$$

$$u = 2\beta_1 + \beta_2 + 5, \qquad v = 7\beta_1 + 3\beta_2 + \beta_3 + 15.$$

Case 2. $w = -4$.

$$\bar{P}_4(z) = \begin{bmatrix} z^3 - 3z^2 + (u - 4)z + v & 0 \\ (u - 6)z^2 + (u + v - 6)z + (v + t) & z^2 - 3z + 2 \end{bmatrix}.$$

(a) $t \neq 0, \gamma_{\bar{P}1} = 1, \gamma_{\bar{P}2} = [z^3 - 3z^2 + (u - 4)z + v](z^2 - 3z + 2)$.

(b) $t = 0, v = 2(6 - u)$.

$$\bar{P}_4(z) = \begin{bmatrix} [z^2 - z + (u - 6)](z - 2) & 0 \\ [(u - 6)z + (u - 6)](z - 2) & (z - 1)(z - 2) \end{bmatrix},$$

$$\gamma_{\bar{P}1} = z - 2, \quad \gamma_{\bar{P}2} = [z^2 - z + (u - 6)](z^2 - 3z + 2).$$

(c) $t = 0, v = 6 - u$.

$$\bar{P}_4(z) = \begin{bmatrix} [z^2 - 2z + (u - 6)](z - 1) & 0 \\ [(u - 6)z + (u - 6)](z - 1) & (z - 2)(z - 1) \end{bmatrix},$$

$$\gamma_{\bar{P}1} = z - 1, \qquad \gamma_{\bar{P}2} = [z^2 - 2z + (u - 6)](z^2 - 3z + 2).$$

(d) $t = v = 0, u = 6$.

$$\bar{P}_4(z) = \begin{bmatrix} z^3 - 3z^2 + 2z & 0 \\ 0 & z^2 - 3z + 2 \end{bmatrix},$$

$$\gamma_{\bar{P}1} = z^2 - 3z + 2, \quad \gamma_{\bar{P}2} = (z^2 - 3z + 2)z.$$

5. Discussion of the sequential realization method. The method of realizing linear constant dynamical systems derived from the sequential realization algorithm has much to recommend it over previous techniques. It may be used to construct linear realizations from transfer function matrices whose elements are ratios of not necessarily co-prime polynomials that need not be in factored form. The method is also useful in constructing a minimal linear model from a set of empirical measurements represented by a finite output data sequence. Hence it is a practical solution to the black box problem.

The sequential realization method is recursive so that the algorithm may be halted at any point in a matrix sequence with a minimal partial denominator matrix from which a minimal partial realization can be easily obtained. It is an improvement over Massey's algorithm because it is applicable to multivariable systems.

Rissanen's recursive realization procedure is essentially a variation of the Ho algorithm [6], [8] where in [10] and [11] the Hankel matrix of a finite sequence is factored as the product of a unit lower triangular matrix with a matrix whose bottom row is zero. The block symmetric structure of the Hankel matrix is exploited to yield a recursive factorization. In order to maintain this recursive property, it is necessary that the row dimension of the Hankel matrix not exceed the column dimension and that the bottom row be linearly dependent. Thus Rissanen's approach will not always produce a minimal partial realization for an arbitrary finite sequence. For example, no minimal partial realization can be obtained using the method in [10] and [11] for the scalar sequence of length $N \geqq 2$ given by $M_1 = M_2 = \cdots = M_{N-1} = 0$, $M_N = 1$. In contrast it has been shown that the sequential realization algorithm yields a minimal partial realization of *any* finite sequence. Another comparison to be made is based upon the amount of storage required if implementing the method on the computer. Since it is necessary to store and manipulate a Hankel matrix and its factors in order to execute Rissanen's algorithm, the required storage grows rapidly with the length of the sequence, particularly for matrix sequences. To use the sequential realization algorithm it is only necessary to store the given sequence, N, $P(z)$, $B(z)$, $\{n_i, 1 \leqq i \leqq m\}$, ρ, $\{k_{\beta j}$ and $n_{\beta j}$, $1 \leqq j \leqq \rho\}$ and K.

The sequential realization method applies to the broad class of systems of (1)–(2) including finite state machines. It is a solution to the partial realization problem for arbitrary finite sequences including the zero sequence and the zero length sequence. Within this framework the initialization of $B(z)$ in the sequential realization algorithm is handled in a more natural way than by Massey [9].

Finally the sequential realization method provides a complete answer to an open question in Kalman [7]. He points out that the whole question of invariant factors, cyclicity and so forth of minimal partial realizations is entirely open. In the preceding section the uniqueness of a minimal partial realization was based upon the invariant factors of a minimal partial denominator matrix. Upon termination the parameters generated by the sequential realization algorithm yield the entire range of variation on the invariant factors of the minimal partial realizations. This certainly provides answers to questions like: Do there exist any minimal partial realizations for a given finite sequence having a cyclic system matrix? The sequence of Examples 1 and 2 has been used by Ackerman [1] to point out the superiority of the realization method in [2] over that of [12] because a coupling parameter results that is not available using the method in [12]. However, a complete enumeration of the available minimal partial realizations given in Example 2 reveals that the realization obtained by Ackerman in [1], although an improvement over Tether's, is still more restricted than it needs to be. In particular, Ackerman's realization corresponds to Case 1 in Example 2 with $t = 0$.

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