

Problem Set 2: Normal Subgroup and Quotient Group

1. The kernel of an endomorphisms is a normal subgroup

Given a group G , not necessarily commutative, with binary operation $\perp$, with neutral element $e \in G$

$$\forall g \in G \quad g \perp e = e \perp g = g$$

A normal subgroup N is a subgroup of G , written $N \subset G$, for which, for any $g \in G$, it holds that

$$g \perp N = N \perp g \quad \forall g \in G$$

In other words, one cannot leave the subgroup N when commuting with any element of G (with g^{-1} denote the inverse of $\perp$ in G)

$$g^{-1} \perp N \perp g = N \quad \forall g \in G$$

Let $\phi : G \rightarrow G$ denote an endomorphism of G , i.e., with $g \in G$ and $f \in G$

$$\phi(g \perp f) = \phi(g) \perp \phi(f)$$

Consider the kernel of the endomorphism, which means the elements of G that are sent to the neutral element e

$$\ker \phi = \{g \in G \mid \phi(g) = e\}$$

Problem 1: Prove that $\ker \phi = N$ is a normal subgroup of the group G of endomorphisms.

2. Quotient Group

Let G designate a group as in question 1. Consider $g_1 \in G$ and $g_2 \in G$. Let N be a subgroup of G and consider the sets

$$\begin{aligned} \bar{g}_1 &= \{g \mid \exists n \in N, g = g_1 \perp n\} \\ \bar{g}_2 &= \{g \mid \exists n \in N, g = g_2 \perp n\} \end{aligned}$$

These sets are well defined whether N is normal or not.

Problem 2: Show that if $N \subset G$ is a normal subgroup then the operation $\perp$ can be extended to the sets defined giving a meaning to

$$\bar{g}_1 \perp \bar{g}_2$$