Problem Set 1: Algebraic Introduction

1. Polynomials in many variables

1. Going back to the example of Lecture 1 given by
$$f = x^2y + xy^2 + y^2 \in \mathbb{R}(x, y)$$
check using Mathematica (or equivalent having a Groebner basis package) the remainders are different when considering the remainder of polynomial division using the two dividends $f_1 = y^2 - 1$ and $f_2 = xy - 1$. Consider lexicographic ordering with $x > y$. (Use `PolynomialReduce` command if you are using Mathematica.)

2. Compute a Groebner basis and then check that the remainders are the same whether dividing by f_1 and f_2 in that order or in the reverse order f_2 and f_1. In each case compute the divisors. Are they the same in the two case even though the remainder is identical? (Use `GroebnerBasis` and `PolynomialReduce` if you are using Mathematica.)

3. Using the Groebner basis construction, check that in $\mathbb{R}(x, y, z)$ with $x > y > z$
$$-4x^2y^2z^2 + y^6 + 3z^5 \in \langle xz - y^2, x^3 - z^2 \rangle = \mathcal{I}$$
by showing that the remainder upon division is zero after computing a suitable Groebner basis.

2. Transitive relation

This exercise can be solved either after Lecture 1 as an introduction to Lecture 2, or after Lecture 2.
A relation $\sim$ is called strongly transitive when the following proposition is true
$$(a \sim b) \ \& \ (b \sim c) \Rightarrow a \sim c$$
for elements a, b, and c different from each other. It is called transitive when true for any elements a, b and c (not necesarily distinct). Complete the following table so that it becomes the table of a strongly transitive relation by introducing the least possible additional crosses x.

$\sim$	a	b	c	d
a		x		x
b				x
c		x		
d	x			

The first element a_1 of the relation $a_1 \sim a_2$ is given in the first column and the second element a_2 is given in the first row. A cross x means that the two corresponding elements are in relation with each other. For example, the first cross x in the table means $a \sim b$.