Proofs of LEO approximations

Under the assumption that we are in LEO and that all changes are negligible with respect to their magnitude (ie $\frac{\Delta A}{A} < 10^{-3}\ \forall A \in (r, x, v)$). We will prove the following approximations:

A) $\Delta x \approx 3\pi\Delta r$

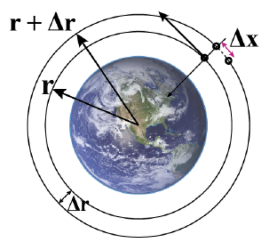
B) $\frac{\Delta r}{r} \approx 4\frac{\Delta v}{v}$

Hints:

- $\sqrt{1+x} \approx 1 + \frac{x}{2}\ \forall x | x \ll 1$
- $\frac{1}{1+x} \approx 1 - x\ \forall x | x \ll 1$

Solution.

A) **Approach 1.** In the expression $\Delta x \approx v\Delta T$, one can determine the ΔT easily assuming that the change in x produces a small change in r. We can therefore only keep the first order in Δr:



$$\begin{aligned} T(r) &= 2\pi\sqrt{\frac{r^3}{\mu}} \\ T(r+\Delta r) &= 2\pi\sqrt{\frac{(r+\Delta r)^3}{\mu}} \approx 2\pi\sqrt{\frac{r^3+3r^2\Delta r}{\mu}} = 2\pi\sqrt{\frac{r^3}{\mu}}\sqrt{1+3\frac{\Delta r}{r}} \approx T(r)\left(1+\frac{3}{2}\frac{\Delta r}{r}\right) \\ \implies \Delta T &= T(r+\Delta r) - T(r) \approx \frac{3}{2}\frac{\Delta r}{r}T(r) \\ \implies \Delta x &\approx v \cdot \frac{3}{2}\frac{\Delta r}{r}T(r) \end{aligned}$$

Moreover, knowing the expression of the orbital speed $v = \sqrt{\frac{\mu}{r}}$, we can replace in the above last statement:

$$\Delta x \approx \sqrt{\frac{\mu}{r}} \cdot \frac{3}{2}\frac{\Delta r}{r} 2\pi\sqrt{\frac{r^3}{\mu}} = 3\pi\Delta r$$

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Approach 2. The horizontal distance of two spacecrafts, after each orbit, is equal to the difference between the orbital paths travelled in one orbit by the two spacecrafts.

$$\begin{aligned}
\Delta x &= C_{\text{circ}} - Tv = 2\pi(r+\Delta r) - 2\pi\sqrt{\frac{r^3}{\mu}}\sqrt{\frac{\mu}{r+\Delta r}} \\
&= 2\pi\left(r+\Delta r - r\sqrt{\frac{r}{\Delta r + r}}\right) \approx 2\pi\left(r+\Delta r - r\sqrt{\frac{1}{1+\frac{\Delta r}{r}}}\right) \approx 2\pi\left(r+\Delta r - r\sqrt{1-\frac{\Delta r}{r}}\right) \\
&\approx 2\pi\left(r+\Delta r - r\left\{1-\frac{\Delta r}{2r}\right\}\right) = 2\pi\left(\Delta r + \Delta r/2\right) = 3\pi\Delta r
\end{aligned}$$

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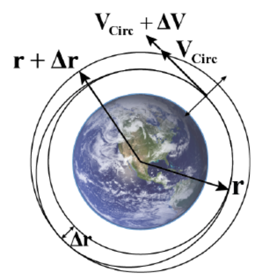
B) The semi-major axis of an orbit with a perigee at r and an apogee at $r+\Delta r$ is given by:



$$a = \frac{r+(r+\Delta r)}{2} = r + \frac{\Delta r}{2}$$

Therefore, the expression of the velocity at the perigee is given by :

$$\begin{aligned}
v_p &= \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}} = \sqrt{\frac{2\mu}{r} - \frac{\mu}{r+\frac{\Delta r}{2}}} = \sqrt{\frac{\mu}{r}}\sqrt{2-\frac{1}{1+\frac{\Delta r}{2r}}} \approx \underbrace{\sqrt{\frac{\mu}{r}}}_{=v_{\text{circ}}}\sqrt{2-\left(1-\frac{\Delta r}{2r}\right)} \\
&= v_{\text{circ}}\sqrt{1+\frac{\Delta r}{2r}} \approx v_{\text{circ}}\left(1+\frac{\Delta r}{4r}\right)
\end{aligned}$$

where v_{circ} is the orbital velocity for a circular orbit. The speed at the perigee of any given orbit is greater than the circular orbit ($v_p \geq v_{\text{circ}}$) and therefore for a slightly eccentric orbit we have: $v_p = v_{\text{circ}} + \Delta v$. This yields :

$$\frac{v_p}{v_{\text{circ}}} = \frac{v_{\text{circ}} + \Delta v}{v_{\text{circ}}} \approx 1 + \frac{\Delta r}{4r} \implies \frac{\Delta r}{r} \approx 4\frac{\Delta v}{v}$$

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