Solution to Exercise Session 4

Problem 1 Multiple Choice Questions

A) *You observe a satellite 30 seconds later than you would expect from the Two Line Elements orbital data at your disposal. Which parameter of the trajectory has changed and how?*

(1) Semi-major axis a, decrease
(2) Semi-major axis a, increase
(3) Inclination i, increase
(4) It is a measurement error, surely.

The orbital period of a satellite is $T = 2\pi\sqrt{\frac{a^3}{\mu}}$. If the observed period $\hat{T} > T_{\mathrm{TLE}}$, then $\hat{a}$ should be larger than a_{TLE}. The altitude of the satellite has *increased* since its previous orbit measurement.

B) *The satellite that you observed at the previous point was previously on an orbit at an altitude of 400 km and an inclination of 52°. You are told that it manoeuvred 12 orbits before your observation of the previous point. What is the value of the changed parameter of its orbit?*

(1) $i = 52.04°$
(2) $a = 398$ km
(3) $a = 402$ km
(4) $a = 408$ km

The orbital velocity of the satellite is given by $v_{\mathrm{circ}} = \sqrt{\frac{\mu}{r}}$. A $\Delta t = 30$ s lead translates into a $\Delta t \cdot v_{\mathrm{circ}} \approx 230$ km lead distance. This corresponds to a Δr from $\Delta x = 3\pi\Delta r \cdot n_{\mathrm{orbits}} \implies \Delta r \approx 2$ km, which in turns gives $\hat{a} \approx 402$ km. Note that since the difference in altitude is small, we can make the approximation $\Delta x = v_{\mathrm{circ}}(z = 400 \text{ km}) \cdot \Delta t$.

C) *($\star$) A small satellite is placed in LEO, and is hence in the shadow of the Earth for 40% of its orbit. It has a nominal electric consumption of 500 W. What is the size of the solar arrays necessary to produce the required average electrical power over the full orbit, knowing that the conversion efficiency of the solar arrays is 25%? We assume that during the sunlit portion of the orbit, excess electrical energy is stored in batteries and fully recovered during the eclipse.*

(1) 0.8 m^2
(2) 1.5 m^2
(3) 2.5 m^2

(4) 6.7 m^2

The radiation power per unit surface arriving from the sun is equal to the solar constant $S = 1367$ W/m^2. The solar array efficiency is $\eta = 0.25$ and the available time is $\tau = 1 - 40\% = 0.6$. Averaged over a full orbit, the effective irradiance is then :

$$H = S \cdot \eta \cdot \tau \simeq 205 \text{ W/m}^2$$

The total area is then

$$A = \frac{P}{H} = \frac{500}{205} \simeq \mathbf{2.5\ m^2}$$

D) *STS-75 was a Space Shuttle mission (in 1996) whose primary objective was to carry a tethered satellite system (TSS-1R) into orbit and to deploy it on a conducting tether. The Shuttle circled the Earth at an altitude of 296 kilometers, placing the 20 kilometers tether system upwards within the rarefied, electrically charged ionosphere. As the Space Shuttle orbiter/tethered satellite system orbit the Earth, the tether rapidly cuts across the Earth's magnetic field lines.*
The interaction creates a voltage across the tether. However, for an electric current to be produced, a complete circuit must be formed. This is accomplished by using an electron generator on the orbiter to return charged particles back into the ionosphere. Knowing that the Earth's magnetic field is equal to $3 \cdot 10^{-5}$ *T, what is the produced voltage along the conductor of the tethered system at the full deployed length of 20 kilometers? Assume that the velocity vector, the tether between the Shuttle and the satellite, and the magnetic field lines are all perpendicular to each another.*

(1) 5.50 kV
(2) 4.64 kV
(3) 3.66 kV
(4) 4.80 kV

The circular orbit velocity of the center of gravity at 296 km is $v_{\text{CM}} = \sqrt{\frac{\mu_\oplus}{r_{\text{CM}}}} = 7.728$ km/s. As $\boldsymbol{B}$ (Earth's magnetic field), $\boldsymbol{v}$ (velocity of conductive tether) and $\boldsymbol{\ell}$ (tether length) are all perpendicular to each other, the produced voltage is

$$U_i = (\boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{\ell} = vB\ell = \mathbf{4.64\ kV} \tag{1}$$

Problem 2 "Houston, we have a problem"

You are in the STS-75, in the same conditions as described in Question 1D).

A) *The tether is aligned with the Earth's gravitational field. The Space Shuttle has a mass of 100 tons and the satellite, of 500 kg. The tethered satellite is deployed upwards (away from the Earth). What are the exact altitude of both the Space Shuttle and the tethered satellite when the system is deployed (the tether being considered massless)?*

B) *The orbit of the Space Shuttle is inclined by* $i = 28.5°$*. At some point, the tether breaks at the attachment point to the satellite. What will be the resulting orbit for the satellite that has suddenly become free?*

C) *Will the new orbit of the satellite remain coplanar with the orbit of the Shuttle? If not, in which direction (East or West) will the satellite orbit line of nodes drift vs the Shuttle orbit line of nodes? Why is this deviation taking place?*

Solution.

A) The center of gravity of the system will stay at 296 km altitude. Assuming a uniform gravity field over the 20 km distance, the ratio of the distance from the center of gravity is given by

$$m_{\text{sat}} \cdot d_{\text{sat}} = m_{SS} \cdot d_{SS} \quad \Rightarrow \quad d_{\text{sat}} = d_{SS} \frac{m_{SS}}{m_{\text{sat}}} = 200 \cdot d_{SS}$$

with $m_{SS} = 100'000$ kg and $m_{\text{sat}} = 500$ kg. As $d_{\text{sat}} + d_{SS} = d = 20$ km, it leads to

$$d_{SS} = \frac{d}{201} \simeq 100 \text{ m}, \qquad d_{\text{sat}} \simeq 19.9 \text{ km}$$

The altitudes are then

$$h_{SS} = h_{\text{CM}} - 0.1 = \mathbf{295.9\ km}, \qquad h_{\text{sat}} = h_{\text{CM}} + 19.9 = \mathbf{315.9\ km}$$

The final radii of the orbits are then

$$\begin{aligned} r_{\text{CM}} &= R_\oplus + h_{\text{CM}} = 6'674 \text{ km} \\ r_{\text{sat}} &= R_\oplus + h_{\text{CM}} + 19.9 = 6'693.9 \text{ km} \\ r_{SS} &= R_\oplus + h_{\text{CM}} - 0.1 = 6'673.9 \text{ km} \end{aligned}$$

with $h_{\text{CM}} = 296$ km.

B) The orbital velocity of the system is given as the orbital velocity at the center of gravity.

$$v_{\text{CM}} = \sqrt{\frac{\mu_\oplus}{r_{\text{CM}}}} = 7'728.15 \text{ m/s}$$

The angular velocity of the complex is

$$\omega = \omega_{\text{CM}} = \frac{v_{\text{CM}}}{r_{\text{CM}}}$$

The tangential velocity of the spacecraft can then be deduced as

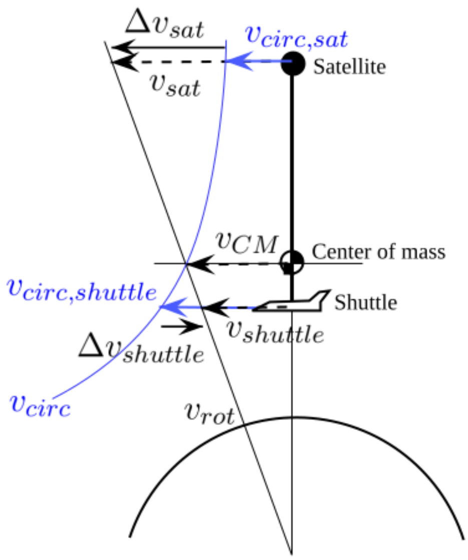
$$v_{\text{sat}} = \omega \cdot r_{\text{sat}} = v_{\text{CM}} \frac{r_{\text{sat}}}{r_{\text{CM}}} = 7'751.19 \text{ m/s}$$

The circular orbital velocity of a satellite at that altitude is given as

$$v_{\text{circ,sat}} = \sqrt{\frac{\mu_\oplus}{r_{\text{sat}}}} = 7'716.65 \text{ m/s}$$

At the instant of release, this difference in velocity can be assimilated to a instantaneous impulse of

$$\Delta v = v_{\text{sat}} - v_{\text{circ,sat}} = 34.5 \text{ m/s}$$



The satellite is then in an elliptical orbit. The raise of apogee compared to the circular orbit can be estimated using the LEO approximation

$$\Delta r_a \text{ [km] } = 3.5 \cdot \Delta v \text{ [m/s] } \simeq 121 \text{ km}$$

The satellite it then on an elliptical orbit of perigee $r_p = r_{\text{sat}} = 6'693.9$ km and apogee $r_a = r_{\text{sat}} + \Delta r_a = 6'814.8$ km.

C) The two orbits will not remain coplanar. The satellite orbit line of nodes will drift to the east of the shuttle orbit line of nodes. The inclination of the two orbits will remain unchanged.

The nodal regression for the satellite orbit will be lower than for the shuttle orbit because the satellite orbit is higher for the same inclination. For an inclination $< 90°$, applicable for us, the nodal regression always causes a drift of the line of nodes to the West, and more so for the shuttle orbit that for the higher satellite orbit, so that the satellite orbit line of nodes will drift to the East of the shuttle orbit line of nodes.

Problem 3 Falcon 9 Staging

Falcon 9 is a two-stage rocket designed and manufactured by SpaceX, which is a private space transport company headquartered in Hawthorne, California. It was founded in 2002 by the former PayPal entrepreneur Elon Musk. Falcon 9 is capable of carrying up to 10.45 tones of payload to LEO. To optimise the m_i/m_f ratio in the Tsiolkovsky formula, the Falcon 9 propulsion system included two stages (Table 1).

Stage	Dry mass [t]	Wet mass [t]	Propellant
First stage	20	260	LOX/Kerosene (RP-1) $I_{\text{sp}} = 282$ s at sea level $I_{\text{sp}} = 311$ s in vacuum
Second stage	3	53	LOX/Kerosene (RP-1) $I_{\text{sp}} = 342$ s in vacuum
Payload	10		

Table 1

A) *If the atmospheric effects and the gravity effect are* ***not*** *taken in consideration, what is the total Δv produced by Falcon 9?*

B) *In comparison, what would be the Δv produced by an equivalent single-stage launch system with $I_{sp} = 342$ s (Table 2)?*

Stage	Dry mass [t]	Wet mass [t]
Single stage	23	313
Payload	10	

Table 2

Solution. In the following solution, the symbols $\Delta v_1, \Delta v_2$ and Δv_F mean respectively the Δv for the first and second stage of the Falcon 9 rocket as well as the total Δv. The initial mass is denoted by m_i and is the wet mass. The final mass is the dry mass m_f. The subscripts $1, 2, *$ represent the first stage, the second stage and the payload masses respectively. The I_{sp} used for the first stage is $311s$ as the atmospheric effects are neglected.

A) Using Tsiolkovsky formula, we have:

$$\begin{aligned}
\Delta v_1 &= g_0 I_{\mathrm{sp}} \ln\left(\frac{m_{i1}+m_{i2}+m_*}{m_{f1}+m_{i2}+m_*}\right) = 4.15 \text{ km/s} \\
\Delta v_2 &= g_0 I_{\mathrm{sp}} \ln\left(\frac{m_{i2}+m_*}{m_{f2}+m_*}\right) = 5.29 \text{ km/s} \\
\Delta v_F &= \Delta v_1 + \Delta v_2 = \mathbf{9.44 \text{ km/s}}
\end{aligned}$$

B) For the single stage:

$$\Delta v_s = g_0 I_{\mathrm{sp}} \ln\left(\frac{m_i+m_*}{m_f+m_*}\right) = \mathbf{7.65 \text{ km/s}}$$

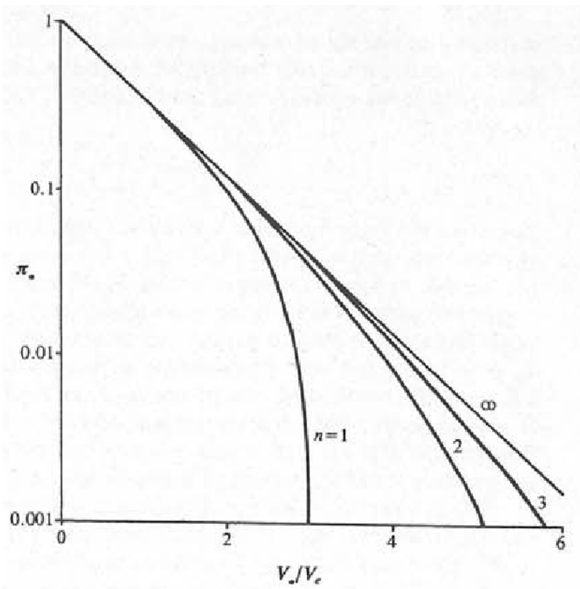
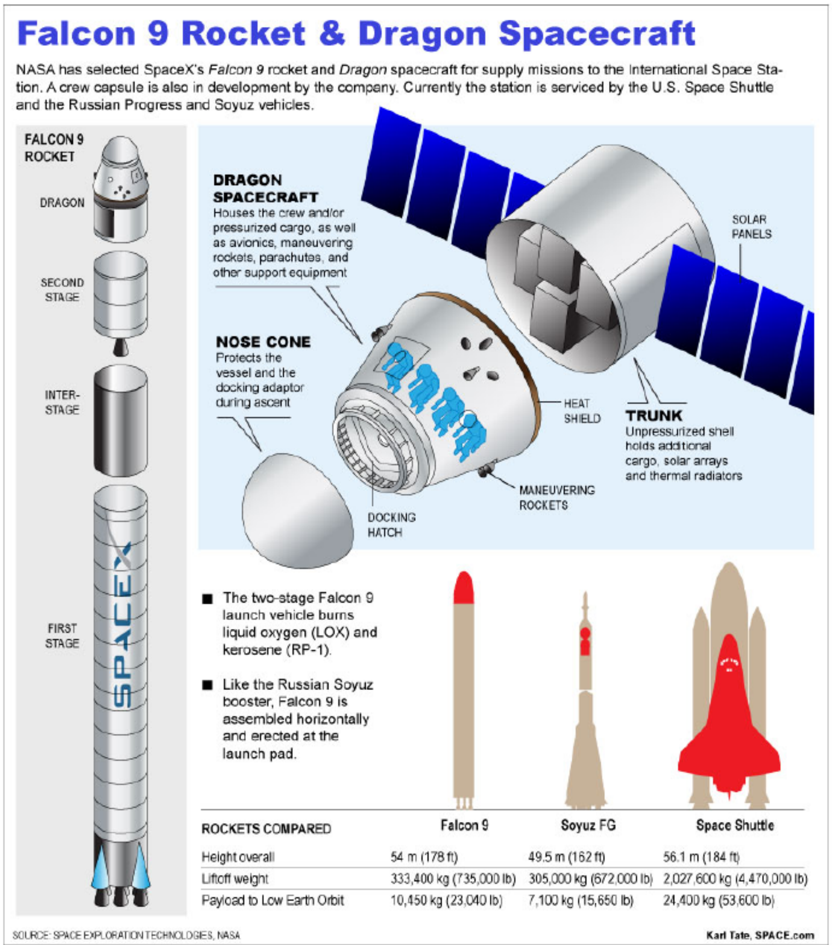


Figure 1: *Payload mass to mass of n stages ratio $\pi_\star = \Pi_{k=1}^{n} \frac{m_*}{m_{ik}}$ as a function of the ratio of the final velocity to the exhaust velocity v_0/v_e. An infinite number of stages would represent a rocked made only of propellant plus the payload. This optimum can be approached by the use of 3 or even 2 stages. We can see that a two-stage rocket can achieve a greater final velocity than a single stage rocket.*



Problem 4 ($\star$) Orbit precession

A satellite is launched into a circular Sun-Synchronous orbit at an altitude of 1000 km. During the launch, the last propulsion stage failed to deliver the satellite to the correct orbit. The satellite is now into the correct inclination plan for the original 1000 km circular orbit, has an orbit apogee of 1000 km but a perigee of only 400 km.

A) *At which rate will this new orbit deviate (node regression) compared to the planned Sun-Synchronous orbit ?*

B) *A way to recover the synchronicity with the Sun is to perform a boost with the satellite propulsion system to change the orbit inclination. What is the Δv needed to get this failed orbit to a Sun-Synchronous condition? The line of apsides of the degraded orbit is aligned with the Equator.*

Solution.

A) The node regression of the orbit is given by the following relation

$$\dot{\Omega} = -2.06474 \cdot 10^{14} \, \frac{\cos(i)}{a^{3.5}(1-e^2)^2} \quad [^\circ/\text{day}] \tag{2}$$

with a the semi-major axis in [km]. The rotation rate of the Earth compared to the Sun the sun-centered inertial frame is equal to the progression of the Earth along its orbit:

$$\dot{\Omega}_{\oplus} = \frac{360}{365 \cdot 24 \cdot 3600 + 6 \cdot 3600 + 9 \cdot 60 + 10} = 1.141 \cdot 10^{-5} \text{ °/s} \equiv 0.986 \text{ °/day}$$

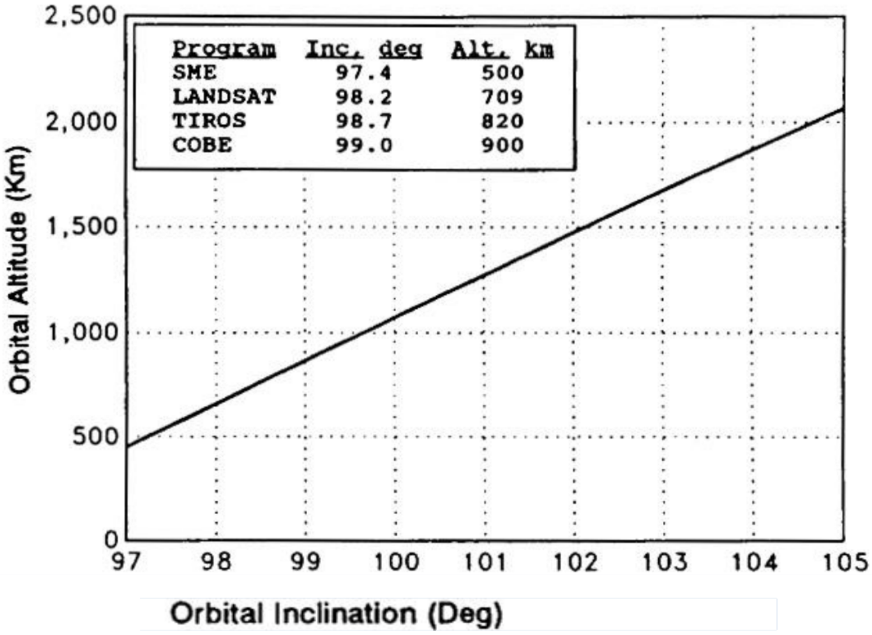


Figure 2: *Inclination vs altitude for sun-synchronous orbits*

Graphical approximation: Using figure 2, we can see that at an altitude of 1'000 km, the inclination of the orbit should be $i_d \sim 99.5°$ to remain sun-synchronous. The semi-major axis of the orbit is hence 700 km. Graphically (or using the table in the graph), at 700 km, the orbital inclination should be $i_f \sim 98.2°$ for a Sun-Synchronous orbit. The error on the inclination of the orbit is thus $i_f - i_d = 1.3°$. The node regression is a function of the cosine of the inclination plane. As $\dot{\Omega} \propto \cos i$ as seen in figure 3, we have

$$\frac{\dot{\Omega}_{\oplus}}{\cos i_f} = \frac{\dot{\Omega}_f}{\cos i_d} \implies \Omega_f = \Omega_{\oplus} \frac{\cos i_f}{\cos i_d} = 1.141 \text{ °/day}$$

The difference in node regression is thus $\Delta\dot{\Omega} = \dot{\Omega}_f - \dot{\Omega}_{\oplus} = \mathbf{0.155 \text{ °/day}}$.

Precise solution: By inverting (2) and using $\dot{\Omega}_{\oplus}$ we can define the Sun-Synchronous inclination for an orbit as

$$i = \arccos\left(\frac{\dot{\Omega}_{\oplus} a^{3.5}(1-e^2)^2}{-2.06474 \cdot 10^{14}}\right) \tag{3}$$

The designed orbit inclination is given by (3) as

$$i_d = 99.48°$$

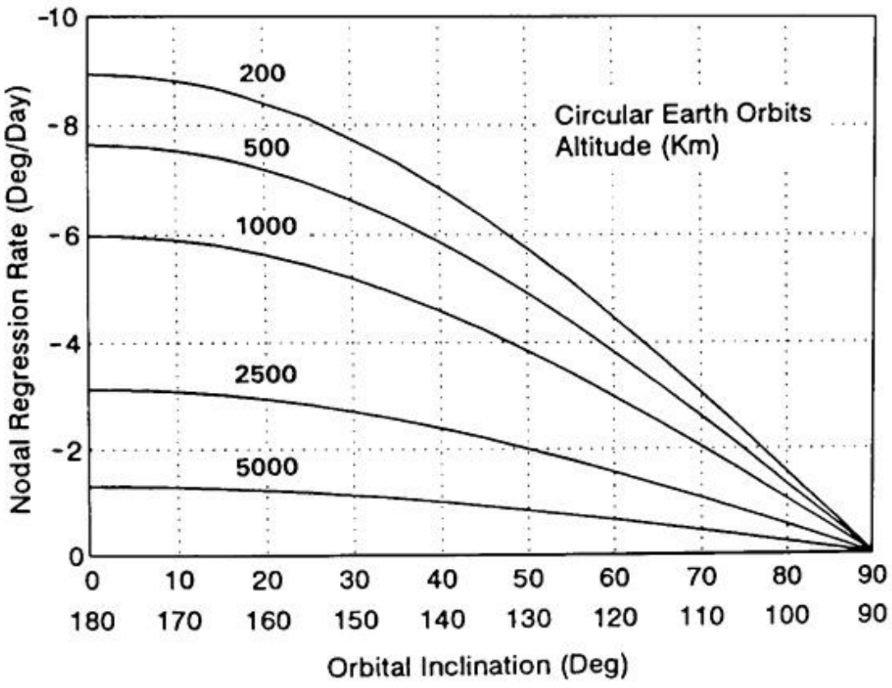


Figure 3: *Inclination vs nodal regression rate*

with $a = R_{\oplus} + 1000 = 7378.14$ km and $e = 0$. The failed orbit being at a lower altitude, its regression rate can be computed using $i = i_d$ in (2) as

$$\dot{\Omega}_f = -2.06474 \cdot 10^{14} \ \frac{\cos(i_d)}{a^{3.5}(1-e^2)^2} = 1.144 \ ^\circ/\text{day}$$

with $a = R_{\oplus} + \frac{400+1000}{2} = 7078.14$ km and $e = \frac{r_a - r_p}{r_a + r_p} = 0.429$. The difference with the Earth rotation rate is then

$$\Delta\dot{\Omega} = \dot{\Omega}_f - \dot{\Omega}_{\oplus} = \mathbf{0.724 \ ^\circ/day}$$

The orbit is deviating from a Sun-Synchronous condition of more than 1° per week.

At this new failed orbit, the Sun-Synchronous inclination is given by (3) as

$$i_f = 98.16^\circ$$

B) **Correction of the orbit.** The difference in inclination is $\Delta i = i_f - f_d = 1.32^\circ$. The Δv required to change this inclination is

$$\Delta v = 2v \sin(\Delta i/2) = \mathbf{172.8 \ m/s}$$

The velocity v is the velocity of the satellite at the moment of the burn. It is thus most economical to perform the maneuver when at apogee, *i.e.* when the velocity is the lowest. In this case the velocity is $v = \sqrt{\frac{2\mu_{\oplus} r_p}{r_p(r_p+r_a)}} = 7\,189$ m/s. In order to change the inclination plane (with regard to the equator) without affecting other parameters, the burn has to be performed at the equator. This is the case here since the line of apsides is in the equatorial plane.