4.1 Electroacoustic analogies

H. Lissek

November 4, 2020

Introduction

Outcome of the lecture

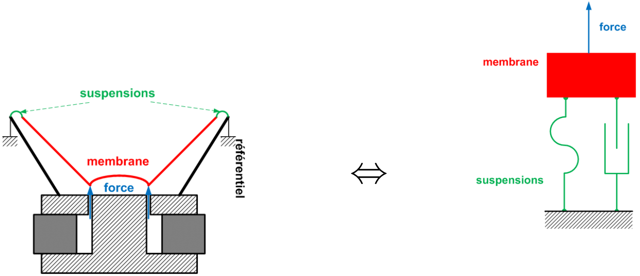
This lecture reminds the analogies between mechanical, acoustical and electrical systems.
The main learning outcome is the realization of analogue electrical schemes accounting for mechanical or acoustical phenomena.

Pre-requisite

- point mechanics
- physical acoustics
- electrical engineering

Mechanical systems

Many mechanical or acoustical systems can be accurately modeled with a finite number of discrete components.



In this course, the mechanical systems are supposed as only moving in translation, but the same approach can be used for rotating movements, or a combination of multiple translations and rotations.

Mechanical components

We are about to represent the main mechanical phenomena by individual components:

- the inertia of a mass,
- the deformation of an elastic object,
- the dissipation through viscous losses,
- the transformation by a lever.

Inertia of a mass

Ponctual rigid mass

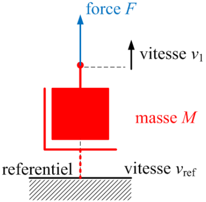
The resulting external forces F applied to a rigid body (without deformation and without damping) leads to

Inertia of a mass

Ponctual rigid mass

The resulting external forces F applied to a rigid body (without deformation and without damping) leads to an acceleration of the body.
The inertia linked to a mass M is proportional to the acceleration, within a Galilean referential.

Fundamental dynamics law



$$F(t) = M\frac{d}{dt}(v_1(t) - v_{ref})$$
$$F = j\omega M(v_1 - v_{ref})$$

- In this lecture, the mechanical referential is not moving: $v_{ref} = 0$.
- The inertia of the mass corresponds to the kinetic energy of the system.

Deformation of an object

Axial linear suspension without mass and without dissipation

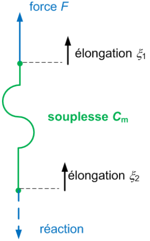
The resulting external forces F applied to an elastic object (here without a mass and without dissipation) leads to

Deformation of an object

Axial linear suspension without mass and without dissipation

The resulting external forces F applied to an elastic object (here without a mass and without dissipation) leads to deform this object.

Behavioral law



$$
\begin{aligned}
F(t) &= \frac{1}{C_m}(\xi_1(t) - \xi_2(t)) \\
&= \frac{1}{C_m}\int (v_1(t) - v_2(t))dt \\
F &= \frac{1}{j\omega C_m}(v_1 - v_2)
\end{aligned}
$$

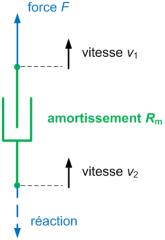
- In the frame of linear elasticity, the deformation $(\xi_1 - \xi_2)$ is proportional to the force.
- In this lecture, the elasticity is expressed as a "compliance" C_m (rather than the stifness $K_m = 1/C_m$).
- The elastic deformation corresponds to a potential energy "storage".

Damping

Linear "dashpot"

The resulting external forces F applied to an object without mass and without stiffness might deform it. The reaction of the object to this deformation leads to a dissipation of energy.

Behavioral law



$$
\begin{aligned}
F(t) &= R_m(v_1(t) - v_2(t)) \\
F &= R_m(v_1 - v_2)
\end{aligned}
$$

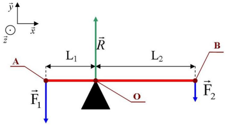
- The deformation velocity $(v_1 - v_2)$ is supposed proportional to F.
- It signifies an irreversible transformation due to a linear viscosity.

Mechanical lever

Ideal mechanical (without losses)

A lever is an example of ideal mechanisms linking two pairs of mechanical quantities (F_1, v_1) et (F_2, v_2).

Coupling equations



$$\begin{aligned} F_1 \ell_1 &= F_2 \ell_2 \\ \frac{v_1}{\ell_1} &= \frac{v_2}{\ell_2} \end{aligned}$$

- The lever plays the role of a transformer : $\frac{F_1}{v_1} = \left(\frac{\ell_2}{\ell_1}\right)^2 \frac{F_2}{v_2}$.
- This ideal transformation preserves the energy.

Mechanical-electrical analogies

We will represent now the basic mechanical phenomena with analogue electrical schemes. This is based to formal analogies between mechanical and electrical equations.

The analogies can take the 2 following forms:

- inverse analogy (or admittance analogy),
- direct analogy (or impedance analogy).

Inverse analogy

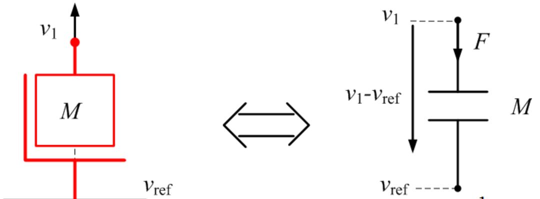
Conventionally, the "inverse" analogy consists in "identifying" the velocity of a mechanical mass with an electrical voltage.

On an energetic viewpoint, it yields that the kinetic energy of a mechanical system is modeled by the potential energy stored in a condenser.
On the same way, the potential energy stored by the deformation of a mechanical system is modeled by the kinetic energy linked to the circulation of a courant in an inductance.

The "inverse" analogy leads to associate an electrical admittance to a mechanical impedance : it is then also called "admittance analogy".

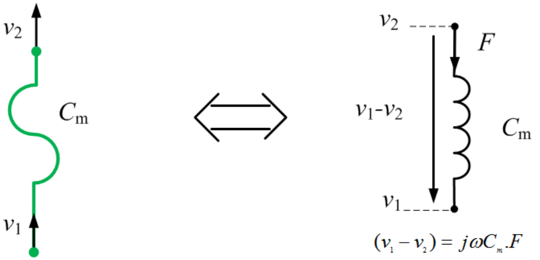
Inverse analogies of mechanical ideal components (1)

Inertie



$$(v_1 - v_{ref}) = \frac{1}{j\omega M}.F$$

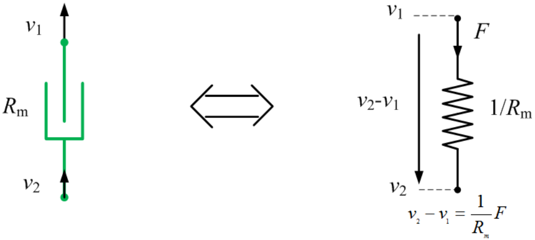
Souplesse



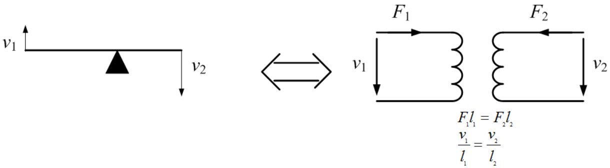
$$(v_1 - v_2) = j\omega C_m.F$$

Inverse analogies of mechanical ideal components (2)

Amortissement



Levier



Direct analogy

Conventionally, the "direct" analogy consists in "identifying" the velocity of a mechanical mass with an electrical current.

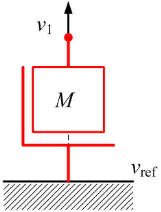
On an energetic viewpoint, it yields that the kinetic energy of a mechanical system is modeled by the same type of energy in an electrical system.
On the same way, the potential energy stored by the deformation of a mechanical system under differential velocities is modeled by the potential energy stored in a capacitance under a voltage difference.

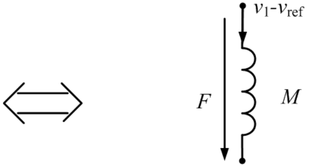
The "direct" analogy leads then to associate an électrical impedance àto a mechanical impedance : it is then also called "impedance analogy"..

The representations of the dash-pot and the lever are the same in the two analogies, apart that the roles of force and velocity are interchanged.

Direct analogies of ideal mechanical components (1)

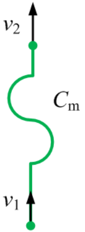
Inertie

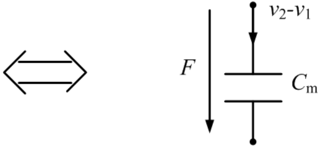




$$F = j\omega M(v - v_{ref})$$

Souplesse

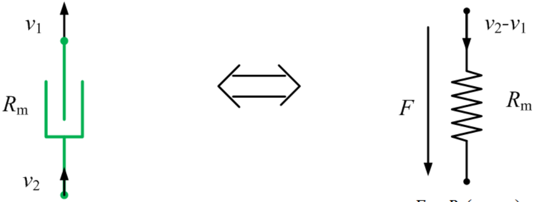




$$F = \frac{1}{j\omega C_m}(v_2 - v_1)$$

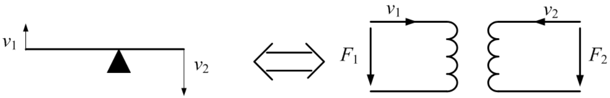
Direct analogies of ideal mechanical components (2)

Amortissement



$$F = R_m(v_2 - v_1)$$

Levier



$$F_1 l_1 = F_2 l_2$$

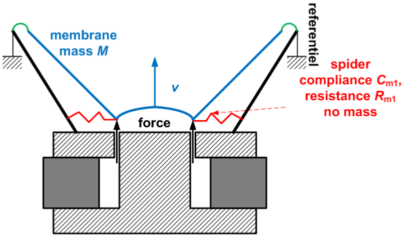
$$\frac{v_1}{l_1} = \frac{v_2}{l_2}$$

Simple mechanical system

We will now illustrate the mechanical-electrical analogies (inverse and direct analogies) with the mechanical scheme of a simple "mass-spring-losses" system.

"Mass-spring-losses" (1/2)

Let's consider the mechanical system schematically illustrated on the figure, consisting of a loudspeaker diaphragm (mass-spring-losses system) subject to the electromechanical force applied by the driver.



The dynamics of this system is described with only one degree of freedom (dof), velocity ($v - v_{ref}$), and is subject to the total external forces F (the driver), to which it reacts.

"Mass-spring-losses" system (2/2)

The Newton's law reads:

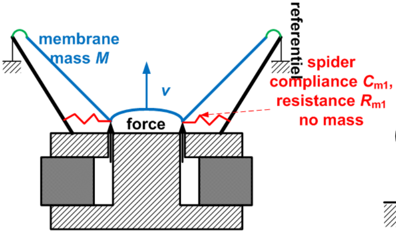
$$F(t) - \frac{1}{C_m}(\xi(t) - \xi_{ref}) - R_m(v(t) - v_{ref}) \quad = \quad M\partial_t(v(t) - v_{ref}),$$

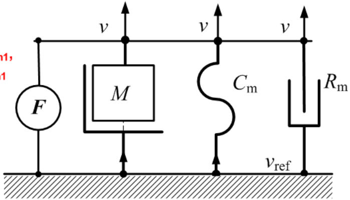
or in harmonic regim:

$$F - \frac{1}{jC_m\omega}(v - v_{ref}) - R_m(v - v_{ref}) \quad = \quad j\omega M(v - v_{ref}).$$

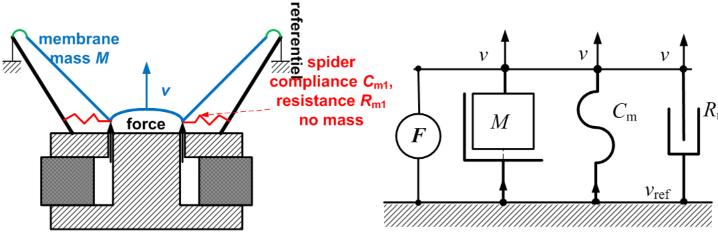
Considering the action is the exerted force (input) and the effect is the movement of the mass (output), it is natural to consider that **velocity v is the dof (observable)** in mechanics.

Mass-spring-losses system in inverse analogy (1)

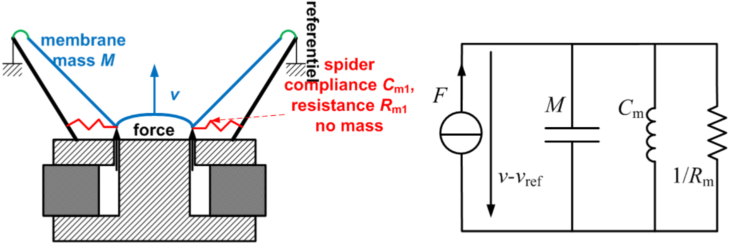




Mass-spring-losses system in inverse analogy (1)

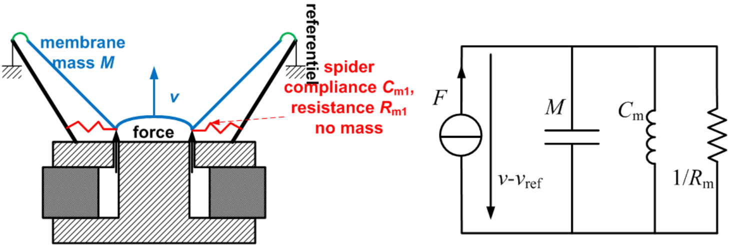


Deduction of the inverse analogy scheme



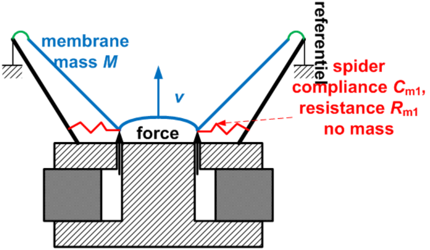
Mass-spring-losses system in inverse analogy (2)

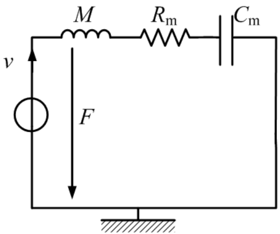
- The action of a mechanical force corresponds to an electric current generator.
- The resulting velocity, common to the three mechanical elements, corresponds to the electric voltage at the terminals of the three corresponding electrical components (admittances!)
- The graphical structure of the electrical scheme resembles the symbolic representation of the mechanical system (simple substitution of symbols with electrical admittance).



Mass-spring-losses system in direct analogy

- The action of the mechanical force corresponds to an electric voltage generator.
- The resulting velocity, common to the 3 mechanical components, corresponds to the electrical current circulating through the 3 corresponding electrical components (impedances).
- The graphical structure of the electrical scheme is not similar anymore to the symbolic mechanical scheme.

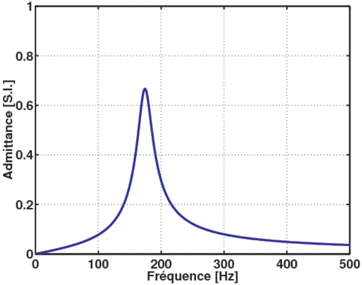




Mechanical admittance

$$Y_m = \frac{v}{F} = \frac{j\omega C_m}{[1 - \omega^2(MC_m) + j\omega(R_m C_m)]} = \frac{j\omega C_m}{[1 - (\frac{\omega}{\omega_0})^2 + j\frac{\omega}{Q_m \omega_0}]}$$

$$\text{with } \omega_0 = \frac{1}{\sqrt{MC_m}} \text{ and } Q = \frac{M\omega_0}{R_m}$$

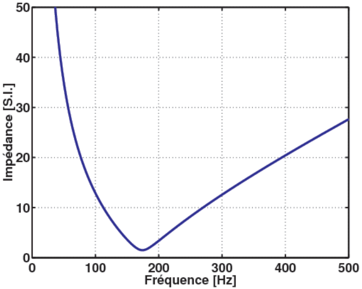


The mechanical admittance Y_m is maximal at resonance, which is somehow an intuitive way to represent the system dynamic response (input=force, output=velocity).

Mechanical impedance

$$Z_m = \frac{F}{v} = \left[\frac{1}{j\omega C_m} + R_m + j\omega M\right] = \frac{\left[1 - \left(\frac{\omega}{\omega_0}\right)^2 + j\frac{\omega}{Q\omega_0}\right]}{j\omega C_m}$$

$$\text{with } \omega_0 = \frac{1}{\sqrt{MC_m}} \text{ and } Q = \frac{M\omega_0}{R_m}$$

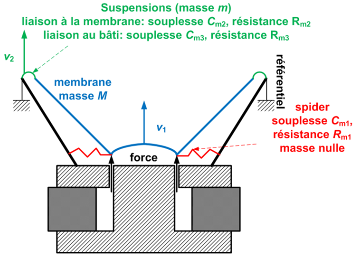


Whereas the mechanical system presents a maximal reaction at resonance (at the frequency $f_0 = \frac{1}{2\pi\sqrt{MC_m}}$), it corresponds to a minimum of the impedance Z_m.

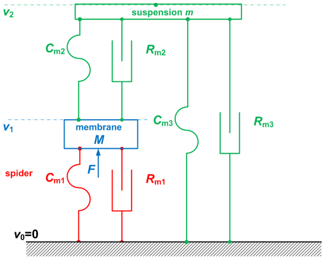
Method for drawing a scheme in direct analogy

1. inspect the mechanical system:
   - identify all velocities (assign them a value v_i). Usually, a velocity should be attributed to every mass in the mechanical system
   - identify all components connecting the different velocities (mass, spring, dash-pot) and assign them an identifier (M_i, C_{mi}, R_{mi})
   - do not forget any generator (force/velocity), and eventually any lever
2. draw the symbolic mechanical scheme
   - draw an horizontal line for each velocity (ie. "velocity potentials"), including the reference v_{ref}
   - draw the corresponding mechanical components between each potential of velocity (do not forget masses, which should always connect to the ground reference v_{ref})
   - draw any generator (it should also connect to the ground reference)
3. convert the symbolic scheme into an inverse electric scheme, by substituting the symbolic components with the corresponding admittance (beware: the dash-pots components should be assigned the value $1/R_{mi}$ in this analogy!)
4. Convert the inverse scheme into a direct scheme
   - all mesh becomes a node and vice-versa,
   - all admittance is converted in the corresponding impedance ($C \to L$, $L \to C$, $1/R \to R$,)
   - all generator is inverted (force generator $\to$ velocity generator and vice-versa)

1. Inspect the mechanical system

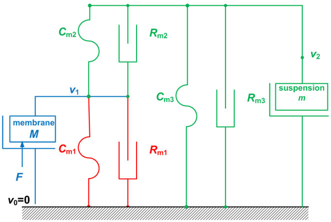


Considered mechanical system

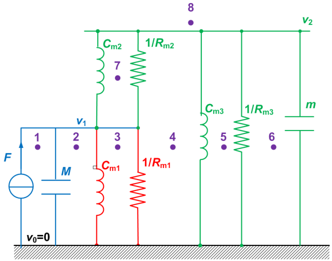


Identification of velocities and mechanical components

2. draw the symbolic mechanical scheme and 3. conversion in inverse analogy

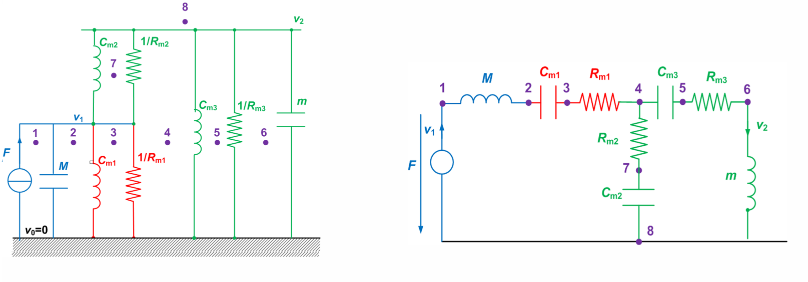


Arranged symbolic mechanical scheme



Inverse analogy scheme

4. conversion in a direct scheme



left: inverse analogy scheme. right: direct analogy scheme.

Conclusion

This section has provided a methodology to model mechanical systems with simple electric equivalent components, allowing to derive the main characteristics in a straightforward manner.
We will now see how acoustic systems can also be modelled through simple electric equivalent components.

Acoustic waveguide

Goals

- To remind acoustic propagation phenomena in a 1D medium (acoustic waveguide) and illustrate the equations with analogue electrical schemes
- To present the equivalent acoustical elements based on electrical-acoustical analogies
- To provide a methodology to derive an analogue scheme representing the main acoustic phenomena is small acoustic systems

Sound : vibratory movement of a fluid

(Onde particules.avi)

Sound : vibratory movement of a fluid

- The sound corresponds to an oscillating movement of fluid particles.
- The material medium particles movement can be characterized, eg. by the particle velocity v.
- It is important to precise that there is no matter transport within the acoustic movement (each particle fluctuates around the same position).
- There is an instantaneous energy transport (with a certain propagation velocity called the celerity).
- The sound celerity depends only on the medium properties.

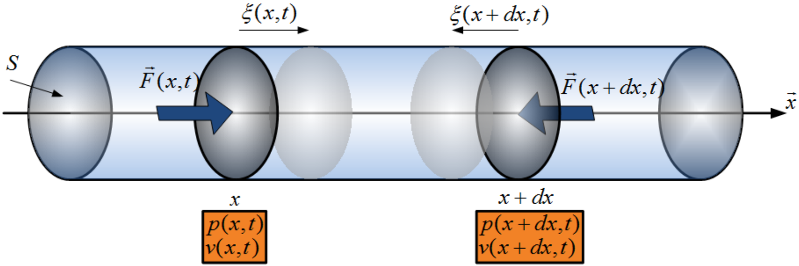
Hypotheses

Mechanical properties:

- Fluids, as opposed to solid matter, are easily deformable materials (media): we consider in the following that the fluid is compressible and that the fluid molecules are weakly linked together (in perfect gazes, we even consider there is no mutual interaction between molecules),
- We also consider the fluid is homogeneous (physical properties, at rest, are the same everywhere), continuous and isotropic, and unlimited,
- There is no external force applied to the fluid (gravity will be neglected),
- We only consider the acoustic pressure forces,
- Last, we suppose (at least in the beginning) there is no dissipation phenomenon.

Presentation of the physical problem

Let's consider a cylindrical acoustic waveguide along axis x, of section S and infinite transversal length, subject to an acoustic excitation. We will focus on a fluid portion of length dx



Acoustic quantities

QUANTITIES DESCRIBING THE MOVEMENT :

- Displacement of the fluid sections $\xi(x, t)$
- Particle velocity $v(x, t) = \dfrac{\partial \xi}{\partial t}$
- Particle acceleration $a(x, t) = \dfrac{\partial v}{\partial t}$

QUANTITIES LINKED TO FORCES :

- Acoustic pressure $p(x, t)$ Represents the small fluctuation of pressure around the static pressure (atmospheric pressure $p_s = P_{atm}$)

INTRINSIC PROPERTIES OF THE FLUID :

- Mass density ρ_0 at rest, locally fluctuating ($\rho(x, t)$) under the fluid particles oscillation
- Thermodynamic quantities :
  - static pressure p_s
  - ratio of heat capacity at constant pressure (C_p) and the heat capacity at constant volume (C_v): $\Gamma = C_p/C_v$ ($\Gamma \approx 1,402$ for diatomic gazes)

  $\rightarrow$ adiabatic compressibility $\chi_s = (\Gamma p_s)^{-1}$, represents the capacity of the fluid to deform under an external force without heat exchange (no losses)

Linear hypothesis: small fluctuations

In acoustics, sound pressure, particle displacement and velocity, and mass density are supposed to oscillate with a low magnitude compared to the static values.

$$\begin{cases} P_T = p_s + p(x,t) \\ \rho_T = \rho_0 + \rho \\ v_T = 0 + v(x,t) \\ p(x,t) \ll P_0 \\ \rho \ll \rho_0 \\ p_s = cste \\ \rho_0 = cste \end{cases}$$

Fluid at rest : p_s,ρ_0

Acoustic disturbance: $p(x,t)$, $\rho(x,t)$, et $v(x,t)$

Linear hypothesis: small fluctuations

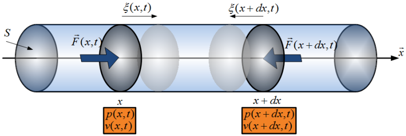
In acoustics, sound pressure, particle displacement and velocity, and mass density are supposed to oscillate with a low magnitude compared to the static values.

$$\begin{cases} P_T = p_s + p(x,t) \\ \rho_T = \rho_0 + \rho \\ v_T = 0 + v(x,t) \\ p(x,t) \ll P_0 \\ \rho \ll \rho_0 \\ p_s = cste \\ \rho_0 = cste \end{cases}$$

Fluid at rest : p_s,ρ_0

Acoustic disturbance: $p(x,t)$, $\rho(x,t)$, et $v(x,t)$

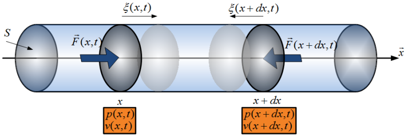
Equations of the acoustic wave: inertia effects



- Let's consider for this part the fluid portion is not compressible. We only consider the acceleration of a non deformable mass of fluid $m_0 = \rho_0 S dx$ under the sound pressure forces.
- Newton's law $\rightarrow m_0 a(x,t) = \sum F = F(x,t) - F(x+dx,t)$
- it yields $S(-p(x+dx,t) + p(x,t)) = -S\frac{\partial p}{\partial x}dx$
- We finally derive the linearized Euler law:

$$\frac{\partial p}{\partial x} = -\rho_0 \frac{\partial v}{\partial t}$$

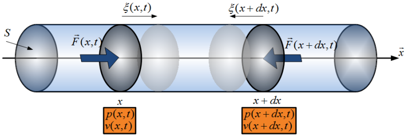
Equations of the acoustic wave: compressibility effects



- Let's now consider the fluid portion is only compressible, without inertia.
- The relationship between the particle volume variation and the pressure is derived from the state equation of the fluid :
- $\dfrac{\delta V(x,t)}{V_0} = -\chi_s p(x,t)$, where $\chi_s = (\Gamma p_s)^{-1}$ is the adiabatic compressibility, and $V_0 = Sdx$ the volume at rest
- By expressing the fluid volume variation δV to the elongation of the faces of the fluid portion ($\xi(x,t)$ and $\xi(x+dx,t)$), and deriving this equation over t, we get :

$$\frac{\partial v}{\partial x} = -\chi_s \frac{\partial p}{\partial t}$$

Wave equation (1D)



$$\begin{cases} \dfrac{\partial p}{\partial x} = -\rho_0 \dfrac{\partial v}{\partial t} & \texttt{Inertia} \\ \dfrac{\partial v}{\partial x} = -\chi_s \dfrac{\partial p}{\partial t} & \texttt{Compressibility} \end{cases}$$

This system yields the following wave equation:

$$\frac{\partial^2 p}{\partial x^2} - \rho_0 \chi_s \frac{\partial^2 p}{\partial t^2} = 0$$

Note: $c = \sqrt{\dfrac{1}{\rho_0 \chi_s}}$ in $m.s^{-1}$ is the sound celerity.

Solutions of the wave equation

$$\frac{\partial^2 p}{\partial x^2} - \rho_0 \chi_s \frac{\partial^2 p}{\partial t^2} = 0$$

The solutions of this equation are propagatives harmonic solutions, propagating

- towards increasing x at velocity c: $p_+(x,t) = p_{0+} \cdot e^{j(\omega t - kx)}$
- towards decreasing x at velocity c: $p_-(x,t) = p_{0-} \cdot e^{j(\omega t + kx)}$

Amplitudes p_{0+} and p_{0-} (can be complex values) depend on the boundary conditions of the waveguide.

Electrical-acoustical analogies

It is noticeable that the pair of equations resembles the telegraphist's equation for an electric transmission line:

Electrical-acoustical analogies

It is noticeable that the pair of equations resembles the telegraphist's equation for an electric transmission line:

Electric telegraphist's equations

$$\begin{cases} \dfrac{\partial u}{\partial x} = -L'\dfrac{\partial i}{\partial t} & \textit{Meshes} \\ \dfrac{\partial i}{\partial x} = -C'\dfrac{\partial u}{\partial t} & \textit{Nodes} \end{cases}$$

where L' is the lineic line inductance and C' is the lineic line capacitance.

Electrical-acoustical analogies

It is noticeable that the pair of equations resembles the telegraphist's equation for an electric transmission line:

Electric telegraphist's equations

$$\begin{cases} \dfrac{\partial u}{\partial x} = -L' \dfrac{\partial i}{\partial t} & \textit{Meshes} \\ \dfrac{\partial i}{\partial x} = -C' \dfrac{\partial u}{\partial t} & \textit{Nodes} \end{cases}$$

where L' is the lineic line inductance and C' is the lineic line capacitance.

Acoustic waveguide equations

$$\begin{cases} \dfrac{\partial p}{\partial x} = -\rho_0 \dfrac{\partial v}{\partial t} & \textit{Inertia} \\ \dfrac{\partial v}{\partial x} = -\chi_s \dfrac{\partial p}{\partial t} & \textit{Compressibility} \end{cases}$$

where ρ_0 is the fluid mass density and χ_s is the fluid compressibility.

Representation by schemes (direct) (1)

Substituting the flow velocity $q(x,t) = Sv(x,t)$ for the velocity $v(x,t)$ (Reminder: $\mathcal{P}_a = p.q$):

$$\begin{cases} \dfrac{\partial p}{\partial x} = -\dfrac{\rho_0}{S}\dfrac{\partial q}{\partial t} & \texttt{Meshes} \\ \dfrac{\partial q}{\partial x} = -S\chi_s\dfrac{\partial p}{\partial t} & \texttt{Nodes} \end{cases}$$

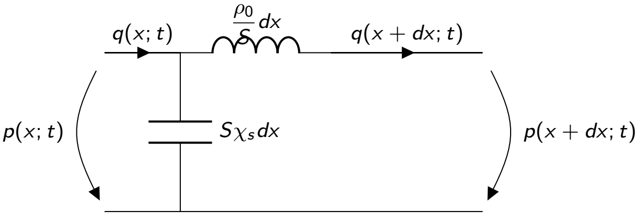
How can we represent these equations with an "electric" scheme?

Representation by schemes (direct) (1)

Substituting the flow velocity $q(x,t) = Sv(x,t)$ for the velocity $v(x,t)$ (Reminder: $\mathcal{P}_a = p.q$):

$$\begin{cases} \dfrac{\partial p}{\partial x} = -\dfrac{\rho_0}{S}\dfrac{\partial q}{\partial t} & \texttt{Meshes} \\ \dfrac{\partial q}{\partial x} = -S\chi_s\dfrac{\partial p}{\partial t} & \texttt{Nodes} \end{cases}$$

How can we represent these equations with an "electric" scheme?



We notice on the scheme that the inductance is associated with the mass density, and the capacitance is associated with the compressibility.
This is called the direct analogue circuit, where

- inertia effects are represented by a self-inductance symbol
- compressibility effects are represented by a capacity symbol

Representation by schemes (inverse) (2)

Substituting the flow velocity $q(x,t) = Sv(x,t)$ for the velocity $v(x,t)$ (Reminder: $\mathcal{P}_a = p.q$):

$$\begin{cases} \dfrac{\partial q}{\partial x} = -S\chi_s \dfrac{\partial p}{\partial t} & \texttt{Meshes} \\ \dfrac{\partial p}{\partial x} = -\dfrac{\rho_0}{S}\dfrac{\partial q}{\partial t} & \texttt{Nodes} \end{cases}$$

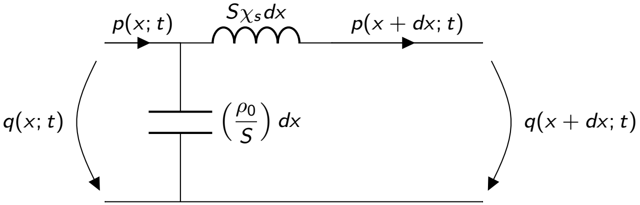
How can we represent these equations with an "electric" scheme?

Representation by schemes (inverse) (2)

Substituting the flow velocity $q(x,t) = Sv(x,t)$ for the velocity $v(x,t)$ (Reminder: $\mathcal{P}_a = p.q$):

$$\begin{cases} \dfrac{\partial q}{\partial x} = -S\chi_s \dfrac{\partial p}{\partial t} & \texttt{Meshes} \\ \dfrac{\partial p}{\partial x} = -\dfrac{\rho_0}{S}\dfrac{\partial q}{\partial t} & \texttt{Nodes} \end{cases}$$

How can we represent these equations with an "electric" scheme?

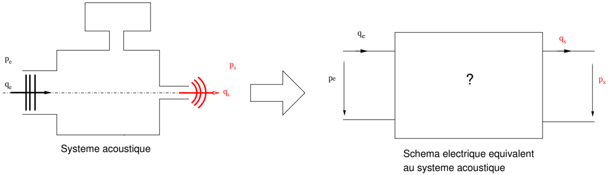


We notice on the scheme that the inductance is associated with the compressibility, and the capacitance is associated with the mass density.
This is called the inverse analogue circuit, where

- compressibility effects are represented by a self-inductance symbol
- inertia effects are represented by a capacity symbol

Small acoustic components and systems

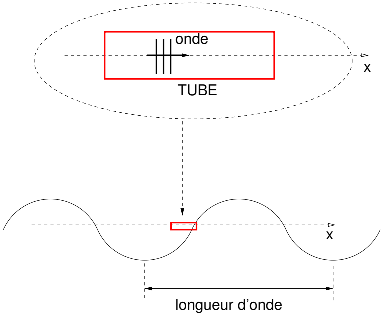
In this section, we will see how to model small (compared to the wavelength) acoustic sytems, and identify basic acoustic components, in an electric representation of the acoustic phenomena.



Hypotheses

In the following we will consider different air-filled ducts. We will assume here that

- the duct length is larger than its radius (plane wave assumption)
- the wavelength is much larger than the duct length.



In the following, we will need to use analogue electric quadripoles, on which we apply the antisymmetrical convention (the "input" is in receiver convention, and the "output" is in generator convention) - see illustration on slide 43

Lumped elements assumption

Definition

The former hypotheses are named lumped-element (or "local constants") hypotheses. In the case of a duct of length L, they read $kL << 1$, where k is the wavenumber, then $\lambda >> \frac{2\pi}{L}$, where λ is the wavelength. It is then a "low-frequency" assumption.

Example

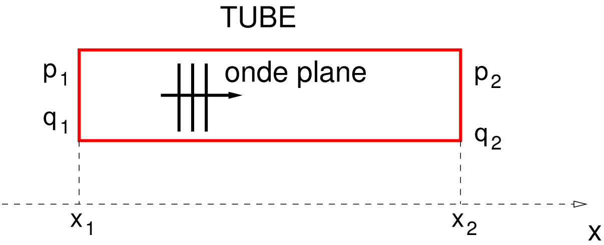
- Example 1 : for a length $L = 10$ cm, the assumption is valide within the range 0 - 3000 Hz.
- Example 2 : for a length $L = 1$ m, the assumption is valide within the range 0 - 300 Hz.

Consequences

- The following problems are 1 dimensional, all wave structures are planar.
- The physical quantities are linearly varying between the duct input and output.
  The partial derivatives in the wave equations can be written as finite differences.
- We can consider the physical quantities only at the input (p_1,q_2) and at the output (p_2,q_2).

Portion de tube: description

Let's consider a duct of length $L = x_2 - x_1$ and section S.



The physical variables considered here are:

- pressure p_1 and p_2 at the duct input and output,
- flow velocities q_1 and q_2 at the duct input and output.

The linear differential equations within the fluid generally read:

$$\begin{cases} \dfrac{\partial p}{\partial x} = -\dfrac{\rho_0}{S}\dfrac{\partial q}{\partial t} \\ \dfrac{\partial q}{\partial x} = -S\chi_s\dfrac{\partial p}{\partial t} \end{cases} .$$

Behaviorial laws

$$\begin{cases} \dfrac{\partial p}{\partial x} = -\dfrac{\rho_0}{S}\dfrac{\partial q}{\partial t} \\ \dfrac{\partial q}{\partial x} = -S\chi_s\dfrac{\partial p}{\partial t} \end{cases} .$$

Behaviorial laws

$$\begin{cases} \dfrac{\partial p}{\partial x} = -\dfrac{\rho_0}{S}\dfrac{\partial q}{\partial t} \\ \dfrac{\partial q}{\partial x} = -S\chi_s\dfrac{\partial p}{\partial t} \end{cases}.$$

Considering lumped elements approximations

$$\begin{cases} \dfrac{\partial p}{\partial x} \simeq \dfrac{p_2 - p_1}{x_2 - x_1} \\ \dfrac{\partial q}{\partial x} \simeq \dfrac{q_2 - q_1}{(x_2 - x_1)} \end{cases},$$

the lumped-element equations of the duct read:

$$\begin{cases} \dfrac{p_2 - p_1}{x_2 - x_1} \simeq -\dfrac{\rho_0}{S}\dfrac{\partial q_1}{\partial t} \\ \dfrac{q_2 - q_1}{S(x_2 - x_1)} = -\chi_s\dfrac{\partial p_1}{\partial t} \end{cases} \quad (\text{Reminder: } \chi_s = \frac{1}{\Gamma p_s} = \frac{1}{\rho_0 c_0^2})$$

Behaviorial laws

$$\begin{cases} \dfrac{\partial p}{\partial x} = -\dfrac{\rho_0}{S}\dfrac{\partial q}{\partial t} \\ \dfrac{\partial q}{\partial x} = -S\chi_s\dfrac{\partial p}{\partial t} \end{cases} .$$

Considering lumped elements approximations

$$\begin{cases} \dfrac{\partial p}{\partial x} \simeq \dfrac{p_2 - p_1}{x_2 - x_1} \\ \dfrac{\partial q}{\partial x} \simeq \dfrac{q_2 - q_1}{(x_2 - x_1)} \end{cases} ,$$

the lumped-element equations of the duct read:

$$\begin{cases} \dfrac{p_2 - p_1}{x_2 - x_1} \simeq -\dfrac{\rho_0}{S}\dfrac{\partial q_1}{\partial t} \\ \dfrac{q_2 - q_1}{S(x_2 - x_1)} = -\chi_s\dfrac{\partial p_1}{\partial t} \end{cases} \quad (\text{Reminder: } \chi_s = \frac{1}{\Gamma p_s} = \frac{1}{\rho_0 c_0^2})$$

For a harmonic plane wave at pulsation ω:

$$p_1 - p_2 \simeq \frac{\rho_0 L}{S} j\omega q_1, \tag{1}$$

$$q_1 - q_2 = \frac{V}{\rho_0 c_0^2} j\omega p_1, \tag{2}$$

where V is the total volume of the duct ($V = S(x_2 - x_1)$).

Analogue electrical scheme

$$p_1 - p_2 \simeq \frac{\rho_0 L}{S} j\omega q_1$$
$$q_1 - q_2 = \frac{V}{\rho_0 c_0^2} j\omega p_1$$

- equation 1 expresses the inertia effects (drop of pressure) through the acoustic mass $m_a = \frac{\rho_0 L}{S}$,
- equation 2 expresses compressibility effects (drop of flow velocity) through the acoustic compliance $C_a = \frac{V}{\rho_0 c_0^2}$.

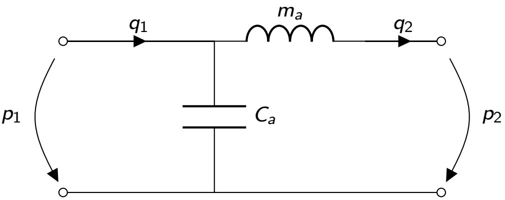
What could the analogue electrical scheme be?

Analogue electrical scheme

$$p_1 - p_2 \simeq \frac{\rho_0 L}{S} j\omega q_1$$
$$q_1 - q_2 = \frac{V}{\rho_0 c_0^2} j\omega p_1$$

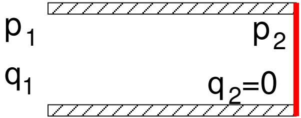
- equation 1 expresses the inertia effects (drop of pressure) through the acoustic mass $m_a = \frac{\rho_0 L}{S}$,
- equation 2 expresses compressibility effects (drop of flow velocity) through the acoustic compliance $C_a = \frac{V}{\rho_0 c_0^2}$.

What could the analogue electrical scheme be?



Duct closed at one extremity

Let's consider the duct is closed at the right-side termination.



The boundary conditions impose $q_2 = 0$ (null output flow velocity). Physically, it can be explained by the fact that the compressibility effect of the air is predominant (at low frequencies) in the whole tube.

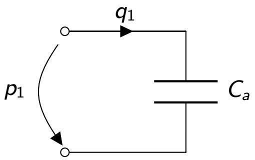
$$\text{Eq. (2)} \rightarrow q_1 = j\omega \frac{V}{\rho_0 c_0^2} p_1$$

Analogue scheme for the closed duct (1)

We can easily find the analogue scheme for the closed duct, after the compressibility law:

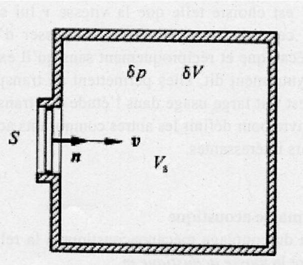
$$p_1 = \frac{1}{j\omega \frac{V}{\rho_0 c_0^2}} p_1$$

If we introduce the compliance $C_a = \frac{V}{\rho_0 c_0^2}$ it leads to the following scheme.



Analogue scheme for the closed duct (2)

Another way to derive this scheme is to consider the following system:



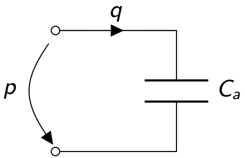
$$\delta p = p = -\frac{1}{\chi_s}\frac{\delta V}{V_s}$$

Since $\delta V = -\int Svdt = -\int qdt$,

it yields

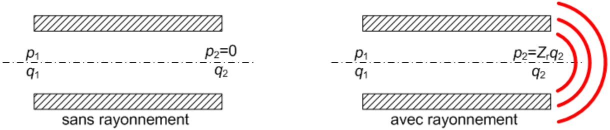
$$p = \frac{1}{V_s \chi_s}\int qdt = \frac{1}{C_a}\int qdt,$$

where $C_a = V_s\chi_s = \dfrac{V_s}{\rho_0 c_0^2}$



Duct open at both extremities

Let's consider the duct is open at the right-side extremity:



The boundary conditions read (in a 1^{st} order approximation) $p_2 = 0$ (null pressure at the output). Physically, it can be explained by the fact that the inertia effect of the air is predominant (at low frequencies) in the whole tube. The flow velocity is constant within the duct and there exists a pressure drop between the input and the output.

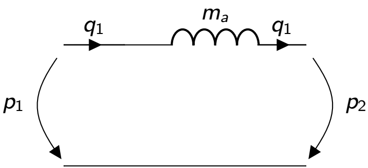
$$\text{Eq. (1)} \rightarrow p_1 - p_2 = j\omega \frac{\rho_0 L}{S} q_1$$

We will see later that the boundary condition becomes $p_2 = Z_{ar} q_2$ (where Z_{ar} is an "acoustic radiation impedance") when the duct is radiating sound in the fluid medium.

Analogue scheme for the closed duct

The tube is equivalent as a rigid mass of air with a certain acceleration.

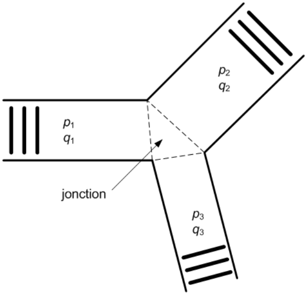
If we ignore sound radiation at the extremity $p_2 = 0$ (equivalent to a terminal impedance load $Z_a = 0$). The electric scheme analogue to the open duct is then an inductance $m_a = \frac{\rho_0 L}{S}$ in series.



Note: If we account for the acoustic radiation, we will show that this will result in a modified acoustic mass $m_a' = \frac{\rho_0 L}{S} + m_{ar}$, where m_{ar} is an additional "radiation mass".

Junctions/derivations

The objective is to draw the electrical scheme analogue to the derivation of ducts, through a junction (common small volume).
Let's consider the situation where an input duct derives into two side output channels.

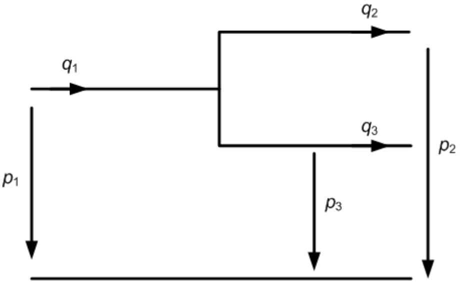


Considering the antisymmetrical convention, the physical laws impose at the junction:

- pressure balance : $p_1 = p_2 = p_3$,
- conservation of flow velocities $q_1 = q_2 + q_3$
  (or $q_1 + (-q_2) + (-q_3) = 0$).

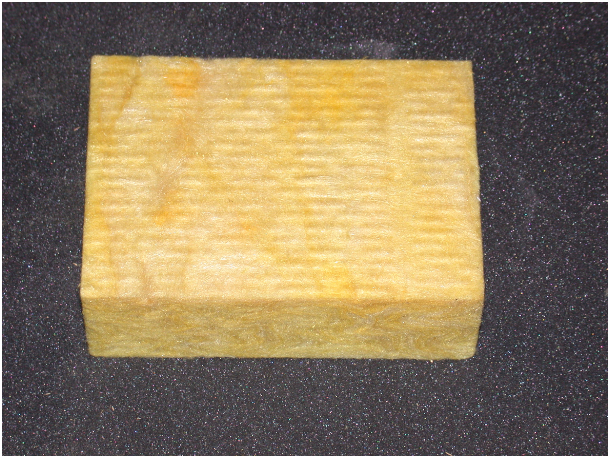
Analogue scheme for a derivation

The analogue scheme is then:



Acoustic losses: principle

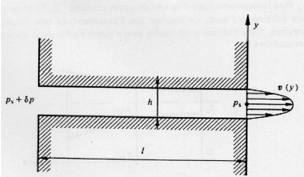
The main phenomena of dissipation of an acoustic wave along the propagation are the viscosity and the thermal conduction effects. Here, we will only consider the viscosity occurring at the boundary layer of lateral walls in a (thin) duct.



Acoustic losses: principle

The main phenomena of dissipation of an acoustic wave along the propagation are the viscosity and the thermal conduction effects. Here, we will only consider the viscosity occurring at the boundary layer of lateral walls in a (thin) duct.

The general principle of losses mechanisms is presented below:

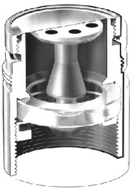


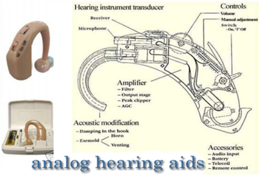
- **Losses due to viscous effects**. In a waveguide, the particles oscillate along the axis of the guide. At the lateral walls, the velocity is null: there exists a zone of transition (boundary layer) between walls and the zone of oscillation of the fluid, where the particle velocity increases rapidly. In this boundary layer, the viscosity of the fluid oppses to the tangential movement of the fluid along the wall. The energy lost by viscous effect is transformed into heat.

Losses in electroacoustics and audio engineering

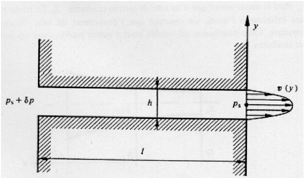
In electroacoustic design, it is sometimes mandatory to account for losses, such as in:

- loudspeaker cabinets assembly usually presenting leaks, even for careful realizations
- acoustic waveguides for which viscothermal losses are important due to the size
- small cavities (eg. in microphones) for which viscothermal losses condition the overall acoustic performances (damping factor of the membrane resonance).



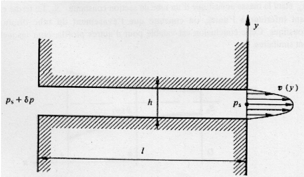


Acoustic resistances in thin tubes



- Effect of a leak/thin tube : acoustic resistance localized within the thin tube (or thin slot) of principal dimension l small compared to the wavelengthλ ($l < \lambda$). This resistance is mainly due to the viscous frictions against the tube walls. It is called viscous resistance.
- The acoustic resistance R_a unit is $kg.m^{-4}.s^{-1}$. It depends on the dynamic viscosity η of the fluid ($\eta = 18.610^{-6} kg.m^{-1}.s^{-1}$). This unit $kg.m^{-1}.s^{-1}$ is also called poise (named after the scientist Poiseuille).
  - in the case of a cylindrical thin duct of radius R and length l, the acoustic resistance is $R_a = \dfrac{8\eta l}{\pi R^4}$.
  - in the case of a parallelepiped slot of width b, height h and length l, the acoustic resistance is $R_a = \dfrac{12\eta l}{bh^3}$.

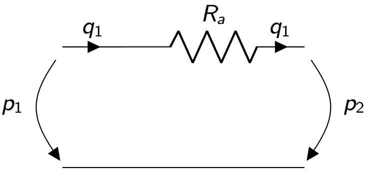
Representation of acoustic resistances



As seen before for the acoustic mass m_a associated to a duct, the acoustic resistance represents the circulation of a flow velocity q_1 through a thin tube subject to a pressure difference $p_1 - p_2$.

The acoustic losses will then be represented by a resistance symbol, in series between potentials p_1 and p_2, with flow velocity q_1.

$$p_1 - p_2 = R_a q_1$$



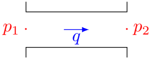
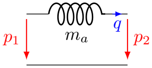
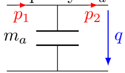
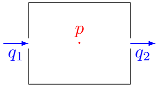
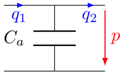
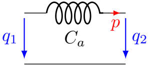
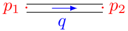
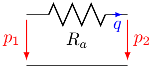
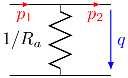
Assembling elements : methodology

It is possible now to draw analogue schemes (in "direct" or "inverse" conventions) for a systems of differents acoustic components. The methodology is the following:

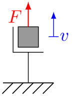
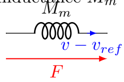
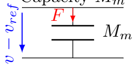
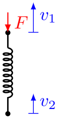
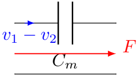
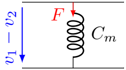
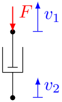
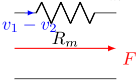
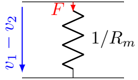
1. identify the source (pressure or flow velocity, assumed "ideal" here)
2. identify all volumes and attribute them a "node", with a corresponding acoustic pressure p_i
3. identify all junctions and attribute them also a "node", with a corresponding acoustic pressure p_i
4. add a node outside the system, representing the reference pressure $p_0 = 0$ (ie. the ground of the acoustic analogue circuit)
5. join all nodes (incl. p_0) through meshes passing through corresponding acoustic components:
   - an open duct is an acoustic mass $m_{ai} = \rho \frac{L_i}{S_i}$, represented by an "inductance" symbol
   - for each tube, we generally consider a term of losses $R_{ai} = \frac{8\eta\pi L_i}{S_i^2}$ (for a thin cylindrical tube), represented by a "resistance" symbol
   - a closed duct (or more generally a volume V) is an acoustic compliance $C_{ai} = \frac{V_i}{\rho_0 c_0^2}$, reprrsented by a "capacitance" symbol always connecting the pressure p_i to the ground pressure p_0

Note: an acoustic compliance is always connected to the ground p_0!

ACOUSTICAL-ELECTRICAL ANALOGIES

Acoustic domain	**Direct analogy**	**Inverse analogy**
Pressure p	Voltage p	Current p
Flow velocity q	Current q	Voltage q
Acoustic impedance $Z_a = \frac{p}{q}$	Electrical impedance Z_a	Electrical admittance Z_a
Mass $m_a = \frac{\rho_0 L}{S}$ 	Inductance m_a 	Capacity m_a 
Compliance $C_a = \frac{V}{\rho_0 c^2}$ 	Capacity C_a 	Inductance C_a 
Resistance $R_a = \frac{8\pi\eta L}{S^2}$ 	Resistance R_a 	Conductance R_a 

Mechanical-electrical analogies

Mechanical domain	**Direct analogy**	**Indirect analogy**
Force F	Voltage F	Current F
Velocity v	Current v	Voltage v
Mechanical impedance $Z_m = \frac{F}{v}$	Electrical impedance Z_m	Electrical admittance Z_m
Mass M_m 	Inductance M_m 	Capacity M_m 
Compliance C_m 	Capacity C_m 	Inductance C_m 
Dash-pot R_m 	Resistance R_m 	Conductance R_m 

bibliographie

M. Rossi, "Audio", Presses Polytechniques et Universitaires Romandes, 2007.

- Mechanical systems: chapter 6.2
- Acoustical systems: chapter 6.3
- Representation by schemes: chapter 6.4