5. Microwave network analysis

References:
F. E. Gardiol, "Hyperfréquences", volume XIII du Traité d'Electricité, Presses Polytechniques Romandes, Chap 6.
R.E. Collin, "Foundations for Microwave Engineering", Mc Graw Hill, 1992, Chap. 4.

5.1 Introduction

We will see in this chapter how the concepts of low-frequency circuit analysis can be extended to microwave circuits and networks. We will reconsider familiar concepts like current, voltage and impedance, find out if and when they can be used in microwave circuit analysis. We will learn to view currents and voltages as sums of incident and reflected waves. We will then introduce generalized waves and the scattering matrix as very efficient and practical tools for microwave circuit analysis.

5.2 Voltage, current and impedance

Currents and voltages are difficult to define in the microwave bands, excepted for the case of transmission lines supporting only a TEM wave. In all other cases, it is not possible to define these quantities in a univocal way. Moreover, they are extremely difficult to measure in a reliable way. Nevertheless, Kirchhoff's model is a very convenient tool for describing a circuit, and we would like to retain it. We will thus try to define equivalent currents and voltages on transmission line, remembering that excepted for the TEM case, these values are concepts without physical meaning and are not uniquely defined.
Each propagating mode will be described by a separate voltage current pair.

5.2.1 TEM Modes

The measurement of currents and voltages is very difficult if not impossible at microwave frequencies, excepted when access ports can be clearly defined. This is the case only for TEM or quasi TEM modes.
Figure 5.1 illustrates the electric and magnetic fields for an arbitrary TEM transmission line.

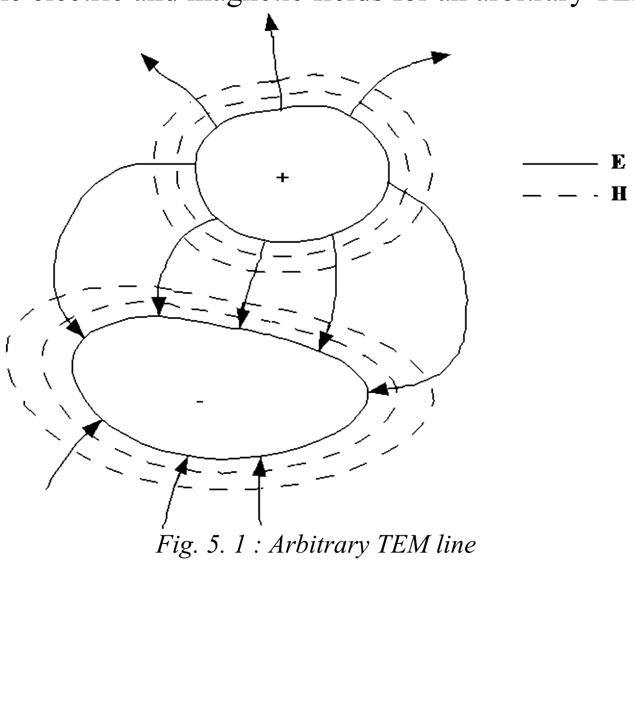

Fig. 5. 1 : Arbitrary TEM line

The voltage difference between the two conductors is defined as :

$$V = \int_{+}^{-} \mathbf{E} \cdot \mathbf{dl} \tag{5. 1}$$

In the case of a TEM wave, the field has a static behaviour, and the voltage will not depend on the integration path, as long as the latter goes from conductor + to conductor -. Thus, the voltage is uniquely defined and there is no ambiguity.
The total current in conductor + is defined by Ampere's law :

$$I = \oint_{C+} \mathbf{H} \cdot \mathbf{dl} \tag{5. 2}$$

where C+ is a closed integration path containing conductor +, but not conductor -. The characteristic impedance is the written as :

$$Z_c = \frac{V}{I} = \sqrt{\frac{L}{C}} \tag{5. 3}$$

Where L is the inductance per unit length of the TEM line and C its capacitance per unit length.

5.2.2 Non TEM modes

The situation is less clear for non-TEM modes, as a simple example can show :
The transverse fields of the TE_{10} mode of a rectangular waveguide are given by :

$$\begin{aligned} E_y(x,y,z) &= E_0 \frac{j\omega\mu a}{\pi} \sin\frac{\pi x}{a} e^{-j\beta z} = E_0\, e_y(x,y) e^{-j\beta z} \\ H_x(x,y,z) &= E_0 \frac{j\beta a}{\pi} \sin\frac{\pi x}{a} e^{-j\beta z} = E_0 h_y(x,y) e^{-j\beta z} \end{aligned} \tag{5. 4}$$

The voltage should thus be defined as

$$V = E_0 \frac{-j\omega\mu a}{\pi} \sin\frac{\pi x}{a} e^{-j\beta z} \int_y dy \tag{5. 5}$$

This voltage would depend on the x position we place the integration path in the guide, and of the geometry of this path. The result is clearly different if we choose a path 0<y<b at x=a/2 or at x=0. So what is the voltage ?
The answer is that in this case there is no "correct" voltage, which could be measured. We may however define a voltage and a current in many different ways for a non-TEM mode.
In order to obtain useful results, we will follow the following rules in our definition :

- The voltage and current are defined for one mode only. We decide (arbitrarily) that the voltage has to be proportional to the amplitude of the transverse electric field, while the current has to be proportional to the amplitude of the transverse magnetic field.

- In order to enable the use of Kirchhoff's model, the product of the current and the voltage should yield the power flux of the considered mode.
- The voltage divided by the current should be equal to the characteristic impedance of the line. The latter should also be equal to the mode impedance of the considered mode.

In an arbitrary guide, the transverse fields can be expressed as a function of an incident and a reflected wave. The voltage and current must thus be expressed in the same way :

$$\begin{aligned}
\mathbf{E_t}(x,y,z) &= \mathbf{e_t}(x,y)\left(E_o^+\,\mathrm{e}^{-j\beta z} + E_o^-\,\mathrm{e}^{j\beta z}\right) \\
&= \frac{\mathbf{e_t}(x,y)}{C_1}\left(V^+\,\mathrm{e}^{-j\beta z} + V^-\,\mathrm{e}^{j\beta z}\right) \\
\mathbf{H_t}(x,y,z) &= \mathbf{h_t}(x,y)\left(E_o^+\,\mathrm{e}^{-j\beta z} - E_o^-\,\mathrm{e}^{j\beta z}\right) \\
&= \frac{\mathbf{h_t}(x,y)}{C_2}\left(I^+\,\mathrm{e}^{-j\beta z} - I^-\,\mathrm{e}^{j\beta z}\right)
\end{aligned} \tag{5. 6}$$

We write thus :

$$\begin{aligned}
V(z) &= V^+\,\mathrm{e}^{-j\beta z} + V^-\,\mathrm{e}^{j\beta z} \\
I(z) &= I^+\,\mathrm{e}^{-j\beta z} - I^-\,\mathrm{e}^{j\beta z}
\end{aligned} \tag{5. 7}$$

The characteristic impedance of this wave is defined (by analogy to the TEM case) as

$$Z_c = \frac{V^+}{I^+} = \frac{V^-}{I^-} = \frac{C_1 E_o^+}{C_2 E_o^+} = \frac{C_1}{C_2} \tag{5. 8}$$

If we want moreover that the characteristic impedance is equal to the wave impedance of the mode, we get :

$$\frac{C_1}{C_2} = Z_{\mathrm{mod}} \tag{5. 9}$$

5.2.3 Impedance concepts

It is important to make the difference between :

- The characteristic impedance of the medium. It depends only on the material constituting the medium :

$$Z_o = \sqrt{\frac{\mu}{\varepsilon}} \tag{5. 10}$$

- The wave impedance of a mode. It will depend on the type of the mode (TE, TM, TEM), on the guide, and on the materials used. It is also dependent on the frequency and the geometry :

$$Z_{\text{mod}} = \frac{|\mathbf{E_t}|}{|\mathbf{H_t}|} \tag{5. 11}$$

- The characteristic impedance, defined as the voltage divided by the current. It is univocally defined only for a TEM transmission line :

$$Z_c = \frac{V^+}{I^+} = \frac{V^-}{I^-} = \sqrt{\frac{L}{C}} \tag{5. 12}$$

5.3 The impedance matrix

The concepts of voltage, current and impedance defined for transmission lines above can also be used to characterize microwave components, circuits and systems. The latter will then be defined by an impedance matrix, obtained from the voltage and current waves flowing on the transmission lines which are linked to the ports of the element.

References:

R.E. Collin, "Foundations for Microwave Engineering", Mc Graw Hill, 1992.

5.3.1 Impedance of a single port element

The simplest possible microwave component has only one access. Its impedance matrix reduces to a scalar, defined as the voltage divided by the current, both "measured" at the access of the component, the *reference plane*.

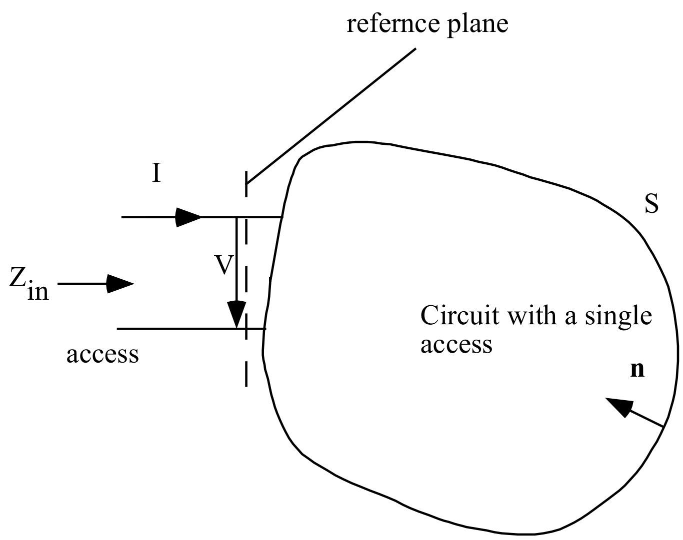

Fig. 5. 2 : Single port element

$$Z_{in} = \frac{V}{I} \tag{5. 13}$$

5.3.2 Impedance characteristics of a single port element

The complex power supplied to the element is given by Poynting's vector :

$$P = \oint_s \mathbf{E} \times \mathbf{H}^* \cdot \mathbf{ds} = P_r + 2j\omega\left(W_m - W_\mathrm{e}\right) \tag{5. 14}$$

The E and H fields on the transmission line are by definition linked to the voltage and the current :

$$\begin{aligned} \mathbf{E_t}(x,y,z) &= V(z)\frac{\mathbf{e_t}(x,y)}{C_1}\mathrm{e}^{-j\beta z} \\ \mathbf{H_t}(x,y,z) &= I(z)\frac{\mathbf{h_t}(x,y)}{C_2}\mathrm{e}^{-j\beta z} \end{aligned} \tag{5. 15}$$

Thus, with the chosen definition for voltage and current :

$$\frac{1}{C_1 C_2}\int_s \mathbf{e_t} \times \mathbf{h_t} \cdot \mathbf{ds} = 1 \tag{5. 16}$$

Thus

$$P = \frac{1}{C_1 C_2}\int_s VI^* \mathbf{e_t} \times \mathbf{h_t} \cdot \mathbf{ds} = VI^* \tag{5. 17}$$

Moreover, the input impedance can be written as a function of the mean power :

$$\begin{aligned} Z_{in} &= R + jX = \frac{V}{I} = \frac{VI^*}{|I|^2} = \frac{P}{|I|^2} \\ &= \frac{\left(P_r + 2j\omega\left(W_m - W_\mathrm{e}\right)\right)}{|I|^2} \end{aligned} \tag{5. 18}$$

Where P_r is the real mean power, W_m is the stored magnetic energy and W_e the stored electric energy. We can deduce from the above relation :

- R is proportional to the real power dissipated in the system (losses)
- X is proportional to the mean reactive energy stored in the system

5.3.3 Impedance and admittance matrices

Let us consider the generic element depicted in figure 5.3. It is characterized by a certain number of accesses defined by reference planes located on the transmission lines linking the component to the outside world. These planes, noted t_n, are the *reference planes* between which the component is defined.

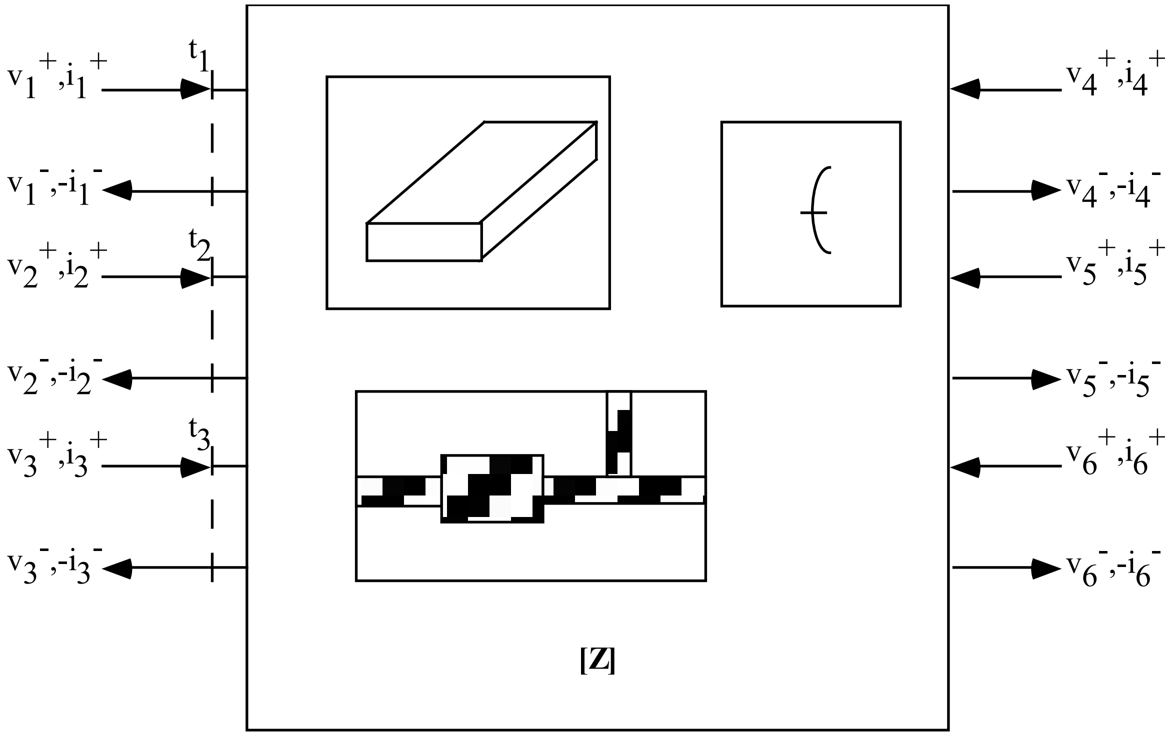

Fig. 5. 3 : Multi-port microwave component and its access ports

An axis of coordinates z_i is linked to each transmission line i. By definition, the origin of this axis is located in the reference plane. We have thus at ports t_1, t_2, ..., t_n

$$
\begin{aligned}
V_n &= V_n^+ + V_n^- \\
I_n &= I_n^+ - I_n^-
\end{aligned}
\qquad (5.\ 19)
$$

The impedance and admittance matrices characterizing the component are defined by :

$$
\begin{aligned}
[V] &= [Z][I] \\
[I] &= [Y][V]
\end{aligned}
\qquad (5.\ 20)
$$

with

$$
\begin{aligned}
Z_{ij} &= \left.\frac{V_i}{I_j}\right|_{I_k=0 \ for\ k\neq j} \\
Y_{ij} &= \left.\frac{I_i}{V_j}\right|_{V_k=0 \ for\ k\neq j}
\end{aligned}
\tag{5. 21}
$$

In consequence,
- The impedance matrix is obtained in open circuit conditions.
- The admittance matrix is obtained in short circuit conditions.

The impedance matrix is the inverse of the admittance matrix

$$[Y]=[Z]^{-1} \tag{5. 22}$$

5.3.4 Properties of the impedance and admittance matrix

5.3.4.1 Reciprocity

Let us consider the case where the basic conditions for Lorentz' reciprocity theorem are respected, thus the case where the component is isotropic, linear and passive. Consider the component depicted in figure 5.4, where all the accesses excepted for two are short circuited. Consider now E_a, H_a, E_b, et H_b which are due to independent sources located somewhere in the circuit. Lorentz' reciprocity theorem states that :

$$\oint_s \mathbf{E_a} \times \mathbf{H_b} \cdot \mathbf{ds} = \oint_s \mathbf{E_b} \times \mathbf{H_a} \cdot \mathbf{ds} \tag{5. 23}$$

where s is a closed integration surface enclosing the component.
We select the closed surface s as the external limit of the component passing through the reference planes, such that $E_{tan}=0$, excepted for reference planes 1 and 2. (If the transmission lines are made of conductors, this is always true. Otherwise, we can always select a surface sufficiently far away so that E_{tan} is negligible).

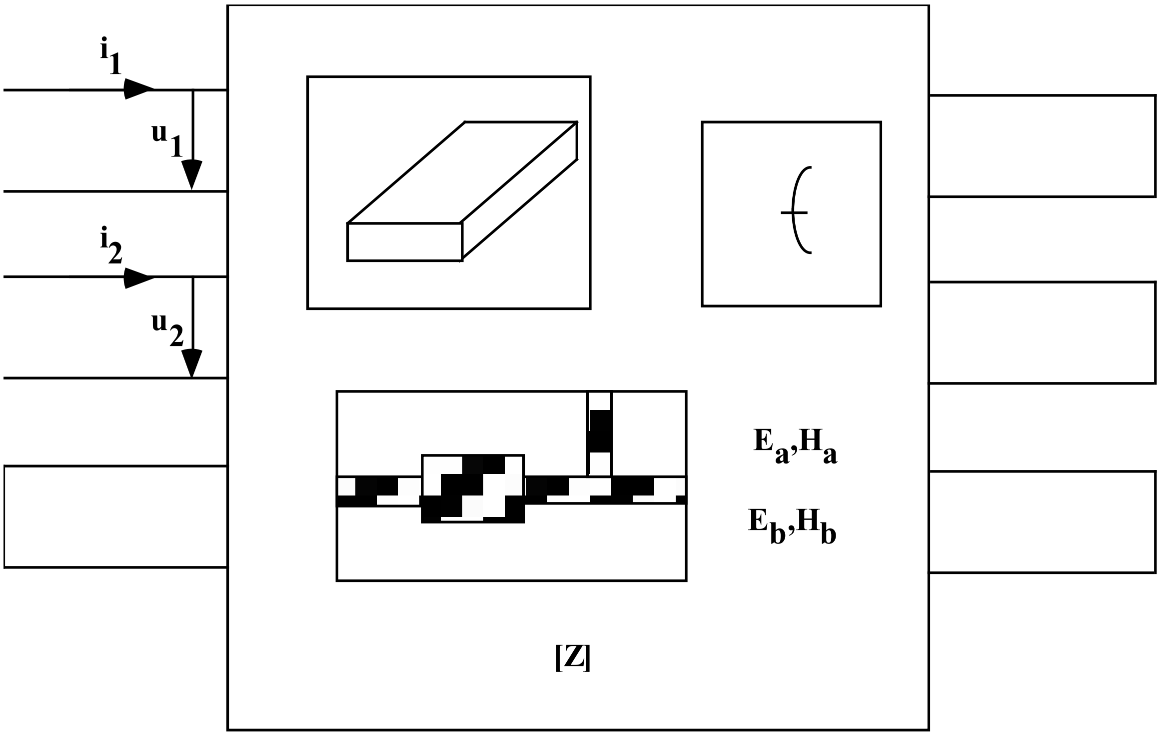

Fig. 5. 4 : Illustration of the reciprocity principle

The only contributions to the integrals come then from reference planes 1 and 2, the only ones which are not short circuited.
We write on these planes :

$$
\begin{aligned}
&\mathbf{E_{1a}} = V_{1a}\frac{\mathbf{e}_1}{C_1} \quad \mathbf{H_{1a}} = I_{1a}\frac{\mathbf{h_1}}{K_1} \\
&\mathbf{E_{1b}} = V_{1b}\frac{\mathbf{e}_1}{C_1} \quad \mathbf{H_{1b}} = I_{1b}\frac{\mathbf{h_1}}{K_1} \\
&\mathbf{E}_{2\mathbf{a}} = V_{2a}\frac{\mathbf{e}_2}{C_2} \quad \mathbf{H}_{2\mathbf{a}} = I_{2a}\frac{\mathbf{h}_2}{K_2} \\
&\mathbf{E}_{2\mathbf{b}} = V_{2b}\frac{\mathbf{e}_2}{C_2} \quad \mathbf{H}_{2\mathbf{b}} = I_{2b}\frac{\mathbf{h}_2}{K_2}
\end{aligned}
\tag{5. 24}
$$

And the reciprocity theorem becomes :

$$
\begin{aligned}
&\left(V_{1a}I_{1b} - V_{1b}I_{1a}\right)\int_{s_1}\frac{1}{C_1K_1}\mathbf{e_1}\times\mathbf{h_1}\cdot\mathbf{ds} + \\
&\left(V_{2a}I_{2b} - V_{2b}I_{2a}\right)\int_{s_2}\frac{1}{C_2K_2}\mathbf{e}_2\times\mathbf{h}_2\cdot\mathbf{ds} = 0
\end{aligned}
\tag{5. 25}
$$

But, by definition

$$
\int_{s_1}\frac{1}{C_1K_1}\mathbf{e}_1\times\mathbf{h}_1\cdot\mathbf{ds} = \int_{s_1}\frac{1}{C_2K_2}\mathbf{e}_2\times\mathbf{h}_2\cdot\mathbf{ds} = 1 \tag{5. 26}
$$

Thus

$$V_{1a}I_{1b} - V_{1b}I_{1a} + V_{2a}I_{2b} - V_{2b}I_{2a} = 0 \tag{5. 27}$$

We have

$$\begin{aligned} I_1 &= Y_{11}V_1 + Y_{12}V_2 \\ I_2 &= Y_{21}V_1 + Y_{22}V_2 \end{aligned} \tag{5. 28}$$

Thus

$$\left(V_{1a}V_{2b} - V_{1b}V_{2a}\right)\left(Y_{12} - Y_{21}\right) = 0 \tag{5. 29}$$

This relation has to hold for any sources, thus for any voltage. This means that :

$$Y_{12} = Y_{21} \tag{5. 30}$$

This relation can be generalized to all the ports of the component. We can thus write in a general way, for a circuit or component having neither active elements, plasmas or ferrites that :

$$\begin{aligned} Y_{ij} &= Y_{ji} \\ Z_{ij} &= Z_{ji} \end{aligned} \tag{5. 31}$$

Thus the impedance and admittance matrices are symmetric for a reciprocal component.

5.3.4.2 Lossless circuit

Let us consider a lossless component with N ports. We can write that for this component the average power consumed by the circuit is zero

$$\mathrm{Re}\left\{P_{av}\right\} = 0 \tag{5. 32}$$

By definition of the voltages and the currents at the ports, the mean power delivered to the component is given by :

$$P_{av} = \left[V\right]^t \left[I\right]^* \tag{5. 33}$$

Which can be written in term of the impedance matrix as

$$\begin{aligned} P_{av} &= \left([Z][I]\right)^t [I]^* \\ &= [I]^t [Z][I]^* \\ &= \sum_{n=1}^{N} \sum_{m=1}^{N} I_m Z_{mn} I_n^* \end{aligned} \tag{5. 34}$$

The currents I_n are independent, thus the real part of each m=n term has to be zero :

$$\mathrm{Re}\left\{I_n Z_{nn} I_n^*\right\} = \left|I_n\right|^2 \mathrm{Re}\left\{Z_{nn}\right\} = 0 \tag{5. 35}$$

We deduce from this that the diagonal terms of the impedance matrix of a lossless circuit must be purely imaginary.

$$\mathrm{Re}\left\{Z_{nn}\right\} = 0 \tag{5. 36}$$

We suppose now that all the currents flowing into the circuit are equal to zero, excepted for I_n and I_m. We write

$$\mathrm{Re}\left\{\left(I_n I_m^* + I_m I_n^*\right) Z_{mn}\right\} = \left(I_n I_m^* + I_m I_n^*\right) \mathrm{Re}\left\{Z_{mn}\right\} = 0 \tag{5. 37}$$

From which we deduce that

$$\mathrm{Re}\left\{Z_{mn}\right\} = 0 \tag{5. 38}$$

We have thus shown that the impedance (and admittance) matrix of a lossless component has to be purely imaginary

5.3.5 Examples of impedance matrices

1) Transmission line

Consider the transmission line section depicted in figure 5.5

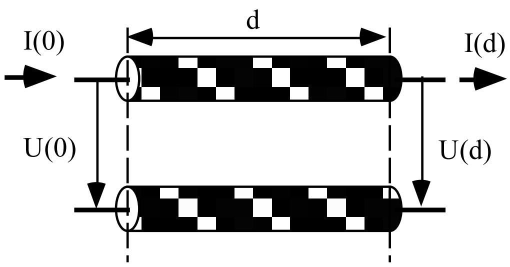

Fig. 5. 5 : Transmission line of length d

Its equivalent two-port is given by

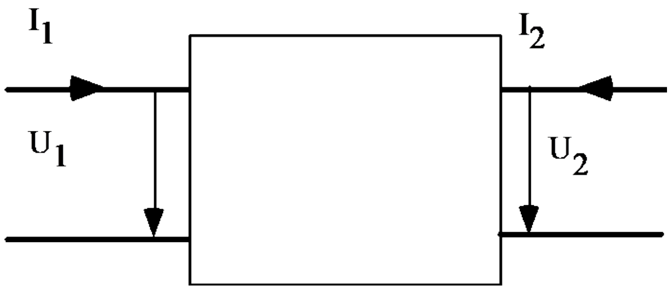

Fig. 5. 6 : Equivalent two-port

Where, by definition

$$U_1 = U(0), I_1 = I(0),\\ U_2 = U(d), I2 = -I(d) \tag{5. 39}$$

Knowing that

$$U(z) = U_+ \mathrm{e}^{-\gamma z} + U_- \mathrm{e}^{+\gamma z}\\ I(z) = I_+ \mathrm{e}^{-\gamma z} - I_- \mathrm{e}^{+\gamma z} \tag{5. 40}$$

We write

$$\begin{array}{ll} U(0) = U_+ + U_- & U(d) = U_+ \mathrm{e}^{-\gamma d} + U_- \mathrm{e}^{+\gamma d} \\ I(0) = I_+ - I_- & I(d) = I_+ \mathrm{e}^{-\gamma d} - I_- \mathrm{e}^{+\gamma d} \end{array} \tag{5. 41}$$

The impedance and admittance matrices are then written as

$$\begin{aligned} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} &= \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \\ &= Z_c \begin{bmatrix} \coth(\gamma d) & \dfrac{1}{\sinh(\gamma d)} \\ \dfrac{1}{\sinh(\gamma d)} & \coth(\gamma d) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \\ \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} &= \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \\ &= Y_c \begin{bmatrix} \coth(\gamma d) & \dfrac{-1}{\sinh(\gamma d)} \\ \dfrac{-1}{\sinh(\gamma d)} & \coth(\gamma d) \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \end{aligned} \tag{5. 42}$$

2) Equivalent T circuit of a reciprocal two-port

A reciprocal two-port has the following impedance matrix :

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12} & Z_{22} \end{bmatrix} \tag{5. 43}$$

It can be represented by an equivalent T circuit

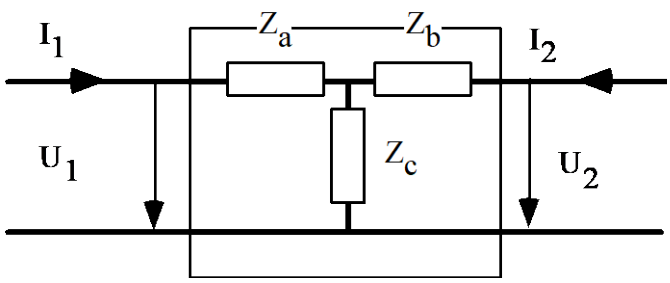

Fig. 5. 7 : Equivalent T circuit of a reciprocal two-port

where

$$\begin{aligned} Z_a &= Z_{11} - Z_{12} \\ Z_b &= Z_{22} - Z_{12} \\ Z_c &= Z_{12} \end{aligned} \tag{5. 44}$$

example : equivalent T circuit of a transmission line section

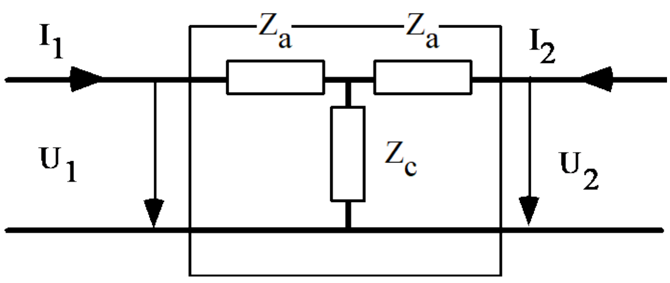

Fig. 5. 8 : Equivalent T circuit of a transmission line section

with

$$
\begin{aligned}
Z_a &= Z_{caract}\left(\coth(\gamma d) - \frac{1}{\sinh(\gamma d)}\right) \\
&= Z_{caract}\left(\frac{ch(\gamma d)-1}{\sinh(\gamma d)}\right) = Z_{caract}\tanh\left(\frac{\gamma d}{2}\right) \\
Z_c &= \frac{Z_{caract}}{\sinh(\gamma d)}
\end{aligned}
\tag{5. 45}
$$

3) Equivalent Π circuit of a reciprocal two-port

A reciprocal two-port has the following admittance matrix :

$$
\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12} & Y_{22} \end{bmatrix} \tag{5. 46}
$$

Such a two port can be represented by an equivalent Π circuit

.

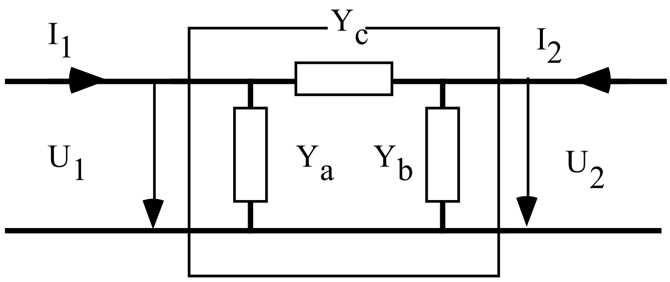

Fig. 5. 9 : Equivalent Π circuit of a reciprocal two-port

with

$$
\begin{aligned}
Y_a &= Y_{11} + Y_{12} \\
Y_b &= Y_{22} + Y_{12} \\
Y_c &= -Y_{12}
\end{aligned}
\tag{5. 47}
$$

Example : equivalent Π circuit of a transmission line section

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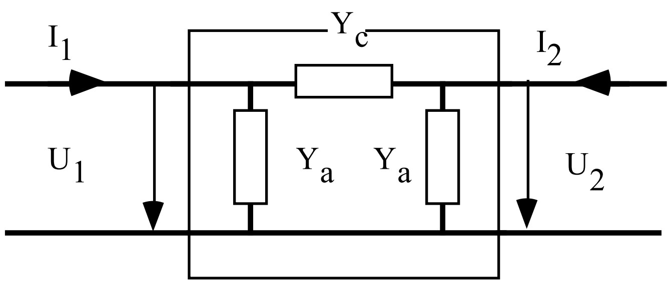


Fig. 5. 10 : Equivalent Π circuit of a transmission line section

With

$$Y_a = Y_{caract}\left(\coth(\gamma d) - \frac{1}{\sinh(\gamma d)}\right)$$

$$= Y_{caract}\left(\frac{ch(\gamma d)-1}{\sinh(\gamma d)}\right) = Y_{caract}\tanh\left(\frac{\gamma d}{2}\right) \qquad (5.\ 48)$$

$$Y_c = \frac{Y_{caract}}{\sinh(\gamma d)}$$

5.4 The scattering matrix

References:
R.E. Collin, "Foundations for Microwave Engineering", Mc Graw Hill, 1992.
F. E. Gardiol, "Hyperfréquences", volume XIII du Traité d'Electricité, Presses Polytechniques Romandes, chap. 6

We have seen in the preceding sections that voltages and currents are not really well suited for the microwave range. One of the direct consequences of the non uniqueness of these values are that they are often not measurable. They can thus be used for the theoretical characterization of circuits and components, as we have seen above, but these theoretical impedances and admittances cannot be corroborated by measured results. This is why we introduce normalized wave amplitudes, which are linked to power, in order to characterize microwave circuits.

5.4.1 Normalized wave amplitudes

We define the normalized waves amplitudes **a** and **b** as

$$a_i = \frac{v_i + Z_{ci} i_i}{2\sqrt{Z_{ci}}} \ , \ b_i = \frac{v_i - Z_{ci} i_i}{2\sqrt{Z_{ci}}} \qquad (5.\ 49)$$

Note : These normalized wave amplitudes have the dimension of the square root of the power, and power is easily measurable in microwaves.

The inverse relation is given by

$$v_i = \sqrt{Z_{ci}}\left(a_i + b_i\right) \ , \ i_i = \frac{\left(a_i - b_i\right)}{\sqrt{Z_{ci}}} \qquad (5.\ 50)$$

These normalizes amplitudes are defined on the transmission lines linking the ports of a component. But on these transmission lines, we have :

$$\begin{aligned} v_i &= v_i^+ \ \mathrm{e}^{-j\beta z} + v_i^- \ \mathrm{e}^{+j\beta z} \\ i_i &= i_i^+ \ \mathrm{e}^{-j\beta z} + i_i^- \ \mathrm{e}^{+j\beta z} \end{aligned} \qquad (5.\ 51)$$

From which we deduce

$$\begin{aligned} a_i &= \frac{v_i^+}{\sqrt{Z_{ci}}} \mathrm{e}^{-j\beta z} \\ b_i &= \frac{v_i^-}{\sqrt{Z_{ci}}} \mathrm{e}^{+j\beta z} \end{aligned} \qquad (5.\ 52)$$

Thus

- a_i : is a purely progressive (incident) wave giving the signal (square root of the power) flowing into the port i
- b_i : is a purely retrograde (reflected) wave, giving the signal flowing out of the port i.

5.4.2 Reference planes

A microwave component is defined between its ports, which are planes transverse to the transmission lines linking the component to the outside world. On these planes are located the origin of the longitudinal coordinate z_i related to the transmission line i (figure 5.11)

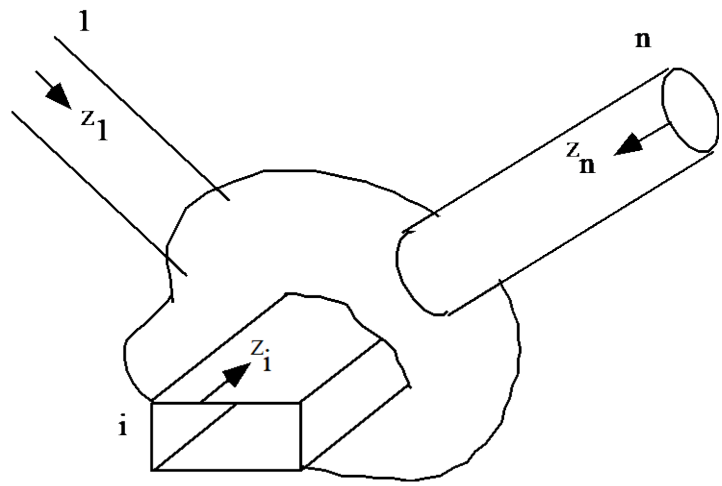


Fig. 5. 11 : Microwave component with its reference planes

By definition, the reference planes have to satisfy the following criteria :

- The reference planes are sufficiently far away from the component, to ensure that all evanescent modes have decayed.
- The transmission lines support only the dominant mode.
- The transmission lines are lossless

The active power at port i is given by

$$P_i = \mathrm{Re}\left[v_i i_i^*\right] = \mathrm{Re}\left[\left(a_i + b_i\right)\left(a_i^* - b_i^*\right)\right] = |a_i|^2 - |b_i|^2 \qquad (5.\ 53)$$

$|a_i|^2$ is thus the active power flowing into the component at port i, while $|b_i|^2$ is the active power flowing out of the component at port i.

5.4.3 Scattering matrix of a component

A microwave component is characterized as a function of the generalized wave amplitudes flowing on the transmission lines at the reference planes (figure 5.12). It is then characterized by its scattering matrix as :

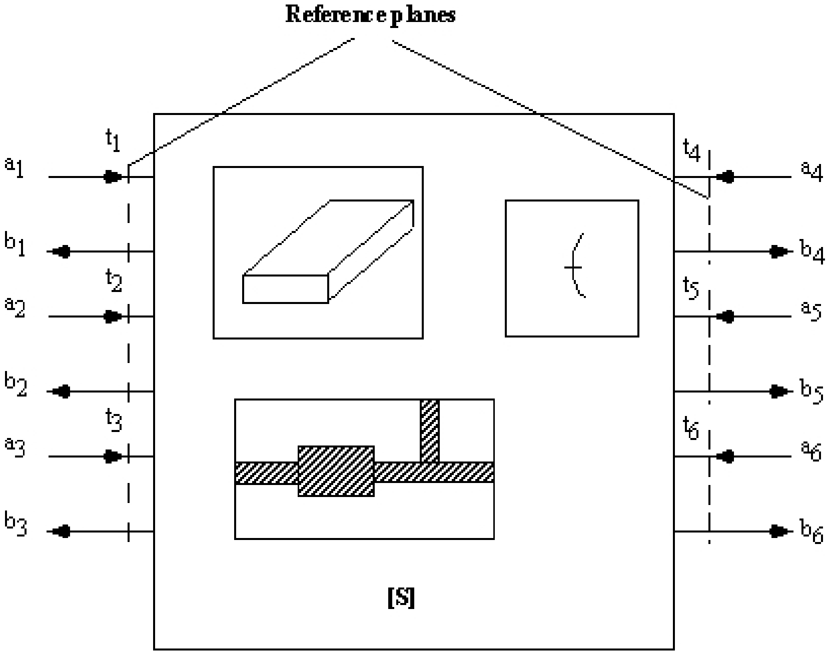


Fig. 5. 12 : Microwave component with its reference planes

$$[b]=[S][a] \tag{5. 54}$$

with

$$s_{ij}=\left.\frac{b_i}{a_j}\right|_{a_k=0\,,\,k\neq j} \tag{5. 55}$$

5.4.4 Properties of the scattering matrix

- The impedance matrix characterizes a component between open-circuits (Z_{ij} = v_i/i_j, i_k=0 for k≠j), while the admittance matrix characterizes a component between short-circuits (Y_{ij} = i_i/v_j, i_k=0 for k≠j). The scattering matrix characterizes a component between matched loads (S_{ij} = b_i/a_j, a_k=0 for k≠j).
- The term s_{ij} is the transfer function of the signal between port j and port i.
- The scattering matrix depends on the component itself, but also on the environment of the component through the transmission lines.
- Changing the characteristic impedance of the transmission lines means changing also the scattering matrix.

5.4.4.1 Reciprocity

In the case of a passive, linear and isotropic component, we have seen that

$$z_{ij} = z_{ji} \tag{5. 56}$$

It is easy to show through matrix transforms that for a reciprocal circuit

$$s_{ij} = s_{ji} \tag{5. 57}$$

Thus, a reciprocal circuit has a symmetrical scattering matrix.

5.4.4.2 Lossless circuit

A lossless circuit is circuit where no active power is dissipated. This means that for such a circuit, the sum of the active power flowing into the circuit must be equal to the active power flowing out of the circuit :

$$\sum |a_i|^2 = \sum |b_i|^2 \tag{5. 58}$$

In a matrix notation, this is equivalent to

$$[\tilde{a}][a] - [\tilde{b}][b] = 0 \tag{5. 59}$$

Where the tilde sign means the transpose complex conjugate of a matrix :

$$[\tilde{a}] = [a^*]^t \tag{5. 60}$$

Moreover, by definition,

$$\begin{aligned} [b] &= [S][a] \\ [\tilde{b}] &= [\tilde{a}][\tilde{S}] \end{aligned} \tag{5. 61}$$

Thus

$$\begin{aligned} &[\tilde{a}][a] - [\tilde{a}][\tilde{S}][S][a] = 0 \\ &[\tilde{a}]\left\{[1] - [\tilde{S}][S]\right\}[a] = 0 \\ &[\tilde{S}][S] = [1] \end{aligned} \tag{5. 62}$$

This can be written as

$$\sum_{i=1}^{N} s_{ij}^{*} s_{ik} = \delta_{jk} \qquad \delta_{jk} = \begin{cases} 1 & si \ \ j = k \\ 0 & si \ \ j \neq k \end{cases} \tag{5. 63}$$

5.4.4.3 Moving the reference plane

The origins of the axes zi, thus the position of the reference plane, are arbitrarily defined, as long as we are in single mode propagation. It can thus be interesting to study the effect of a translation of the reference plane along the axis on the scattering matrix (figure 5.13).

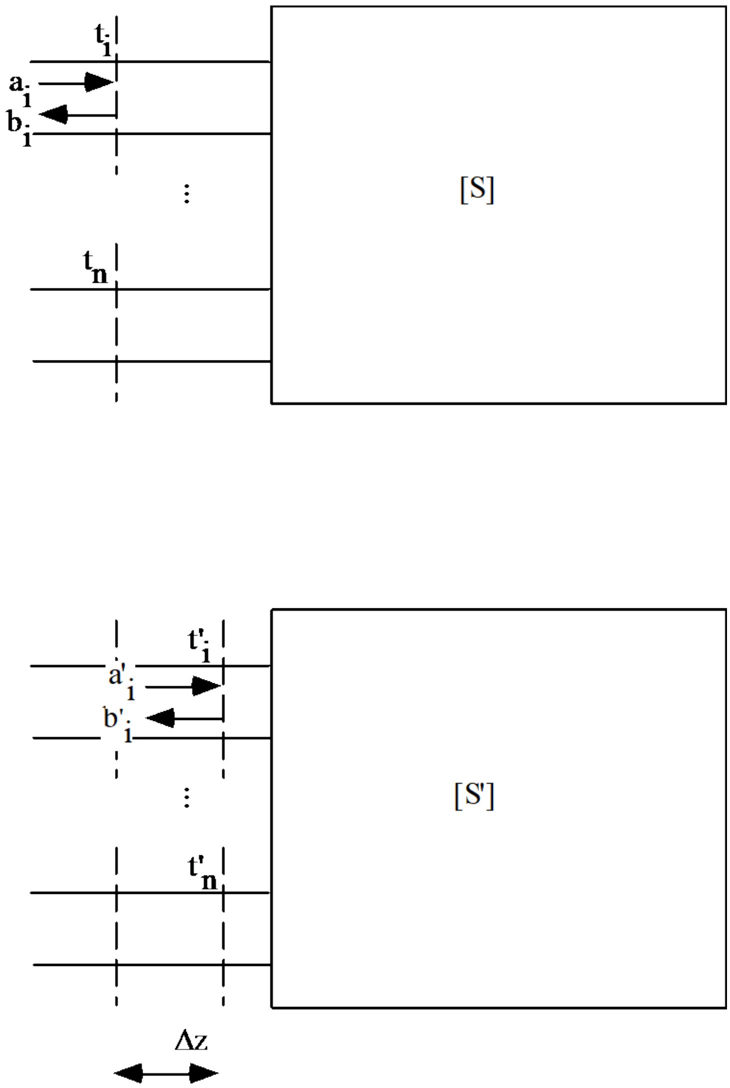


Fig. 5. 13 : Translation of the reference plane

The normalized wave amplitudes a'_i et b'_i linked to the translated coordinates system can be expressed in term of the normalized wave amplitudes a_i et b_i, linked to the original coordinate system, by

$$a'_i = a_i \, \mathrm{e}^{-j\varphi_i}$$
$$b'_i = b_i \, \mathrm{e}^{j\varphi_i} \qquad (5.\ 64)$$
$$\varphi_i = -\beta_i \Delta z_i$$

But we have also

$$[b'] = [S'][a'] \text{ et } [b] = [S][a] \qquad (5.\ 65)$$

We write

$$s'_{ii} = s_{ii}\,\mathrm{e}^{j2\varphi_i} \tag{5. 66}$$

And in general

$$\begin{aligned}
\left[a\right] &= \left[diag\,\mathrm{e}^{j\varphi}\right]\left[a'\right]\\
\left[a'\right] &= \left[diag\,\mathrm{e}^{-j\varphi}\right]\left[a\right]\\
\left[b\right] &= \left[diag\,\mathrm{e}^{-j\varphi}\right]\left[b'\right]\\
\left[b'\right] &= \left[diag\,\mathrm{e}^{j\varphi}\right]\left[b\right]
\end{aligned} \tag{5. 67}$$

With

$$\left[diag\,\mathrm{e}^{j\varphi}\right] = \begin{bmatrix} \mathrm{e}^{j\varphi_1} & 0 & \ldots & 0 \\ 0 & \mathrm{e}^{j\varphi_2} & & \vdots \\ \vdots & & & \\ 0 & \ldots & & \mathrm{e}^{j\varphi n} \end{bmatrix} \tag{5. 68}$$

Thus

$$\begin{aligned}
\left[b'\right] &= \left[diag\,\mathrm{e}^{j\varphi}\right]\left[S\right]\left[diag\,\mathrm{e}^{j\varphi}\right]\left[a'\right]\\
\left[S'\right] &= \left[diag\,\mathrm{e}^{j\varphi}\right]\left[S\right]\left[diag\,\mathrm{e}^{j\varphi}\right]\\
s'_{ij} &= s_{ij}\,\mathrm{e}^{j\left(\varphi_i+\varphi_j\right)}
\end{aligned} \tag{5. 69}$$

5.4.4.4 Relation between impedance matrix and scattering matrix

We define the two diagonal matrices

$$[G] = [diag\ Z_{ci}] = \begin{bmatrix} Z_{c1} & 0 & \ldots & 0 \\ 0 & Z_{c2} & & : \\ : & & & \\ 0 & \ldots & & Z_{cn} \end{bmatrix}$$

$$[F] = \left[diag\ \frac{1}{2\sqrt{Z_{ci}}} \right] = \begin{bmatrix} \frac{1}{2\sqrt{Z_{c1}}} & 0 & \ldots & 0 \\ 0 & \frac{1}{2\sqrt{Z_{c2}}} & & : \\ : & & & \\ 0 & \ldots & & \frac{1}{2\sqrt{Z_{cn}}} \end{bmatrix} \tag{5. 70}$$

and use the definition of the normalized wave amplitudes to write

$$\begin{aligned} [S] &= [F]\big[[Z]-[G]\big]\big[[F]\big[[Z]+[G]\big]\big]^{-1} \\ &= [F]\big[[Z]-[G]\big]\big[[Z]+[G]\big]^{-1}[F]^{-1} \end{aligned} \tag{5. 71}$$

and

$$[Z] = [F]^{-1}\big[[1]+[s]\big]\big[[1]-[s]\big]^{-1}[F][G] \tag{5. 72}$$

where [1] is the identity matrix.

5.4.5 Flow charts

The terms of the scattering matrix are transfer functions, linking an input port to an output port. They can be represented graphically by flow charts.

example 1 : two-port

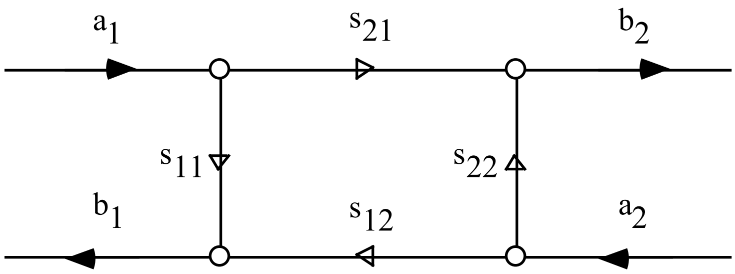


Fig. 5. 14 : Flow chart of a two-port

example 2 : three-port

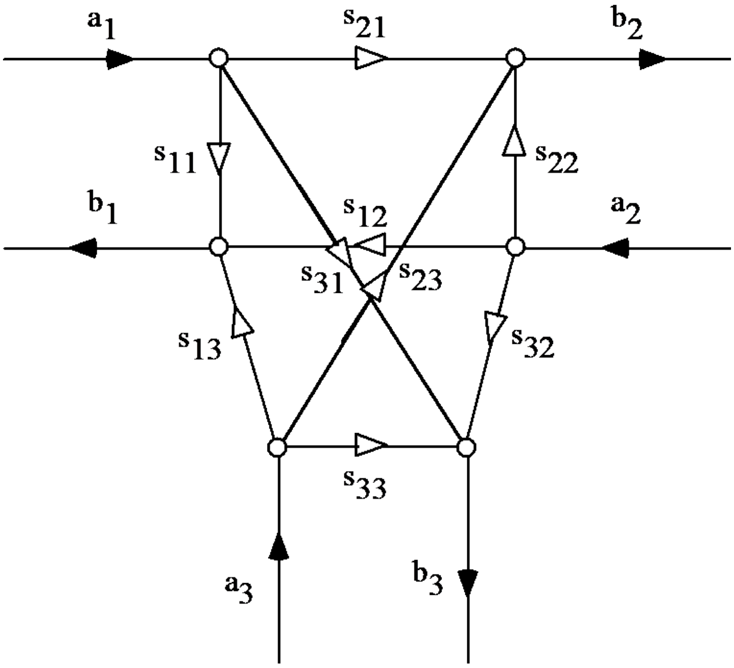


Fig. 5. 15 : Flow chart of a three-port

example 3 : two two-ports cascaded

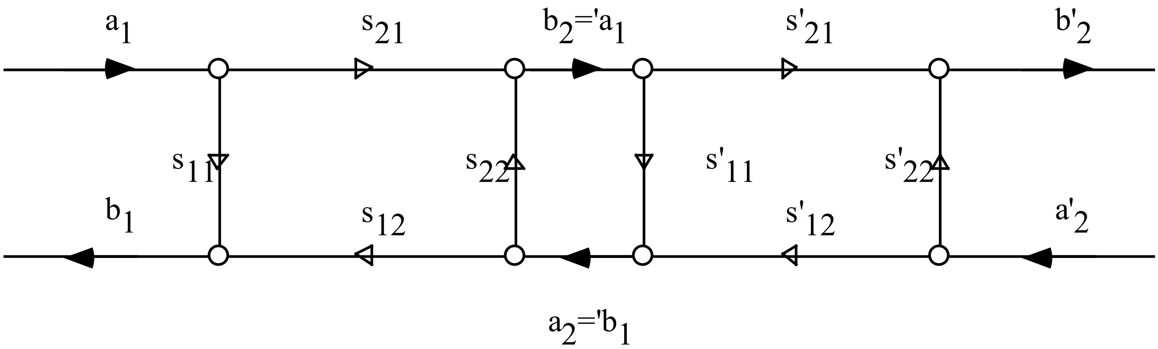


Fig. 5. 16 : Cascaded two-ports

5.4.5.1 Flow chart reduction rules

1) multiplication

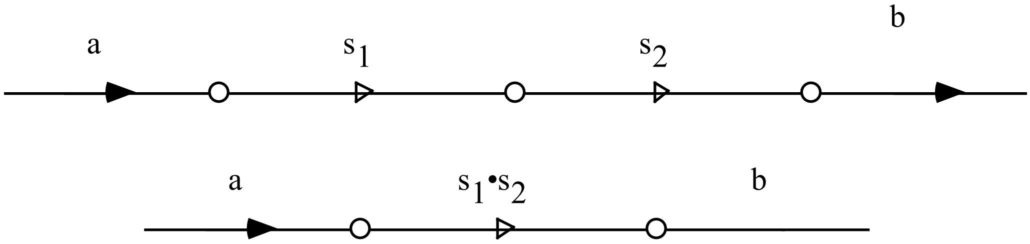


Fig. 5. 17 : Two flow chart in series

2) addition

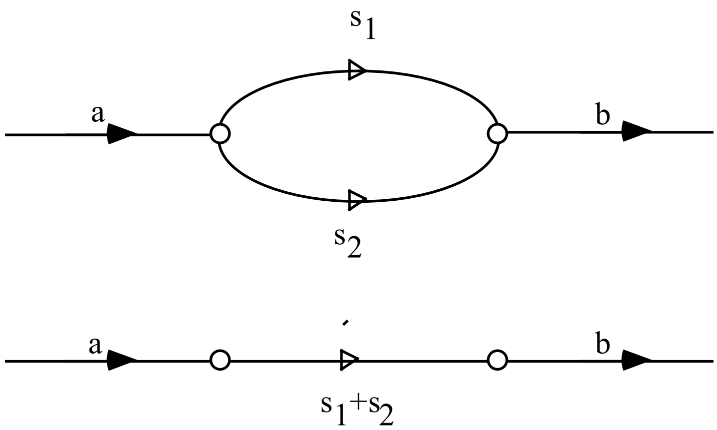


Fig. 5. 18: Two flow charts in parallel

3) retroaction

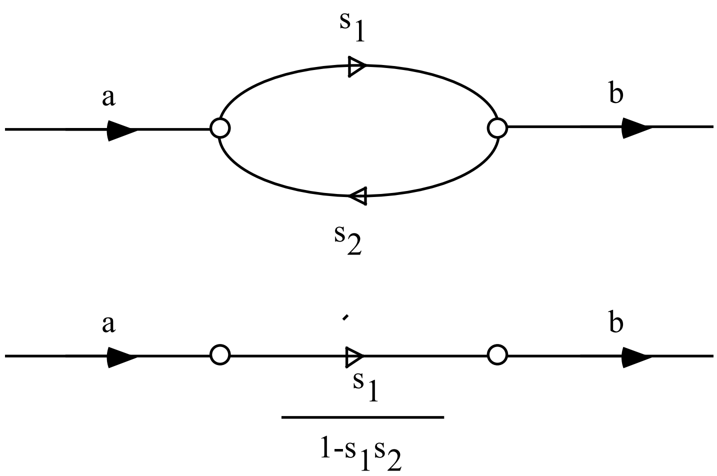


Fig. 5. 19 : Retroaction of two flow charts

5.4.5.2 Example

Find the reflection coefficient at the input of a reciprocal two-port terminated by a short circuit

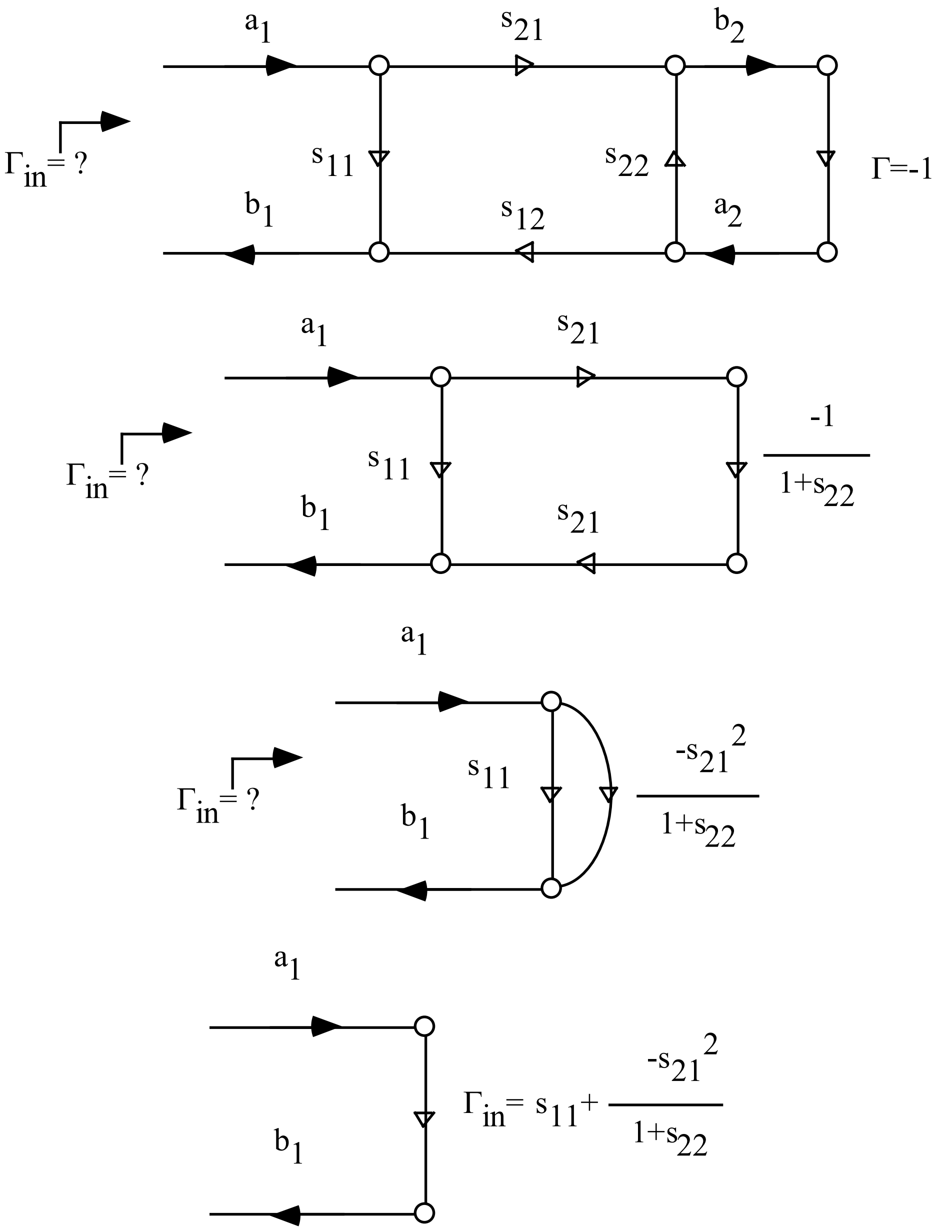
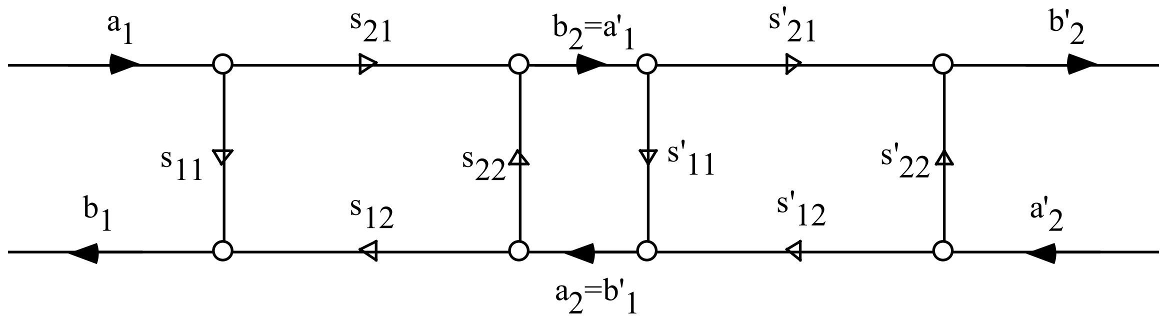
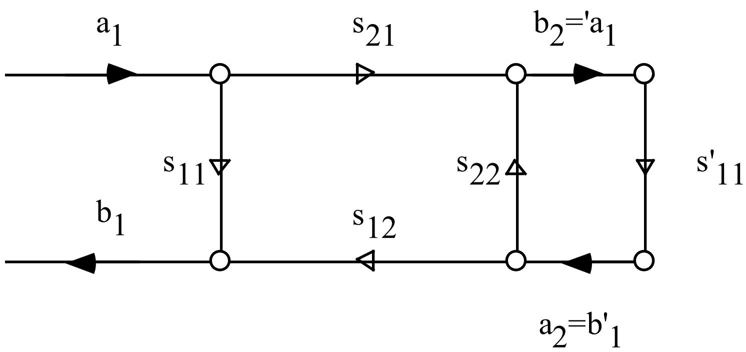


Fig. 5. 20 : Reduction of flow chart

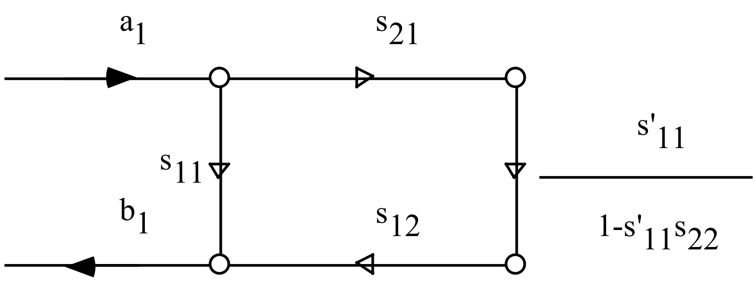
5.4.5.3 Example : two cascaded two-ports

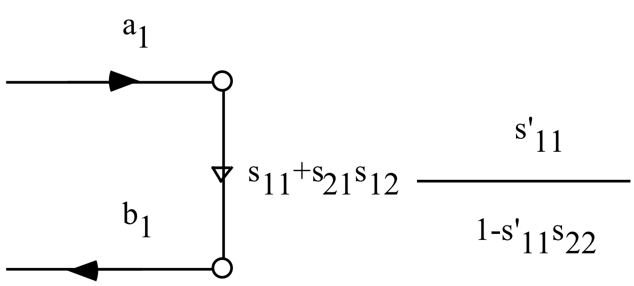


First stage : we look for the possible paths going from a_1 to b_1 :

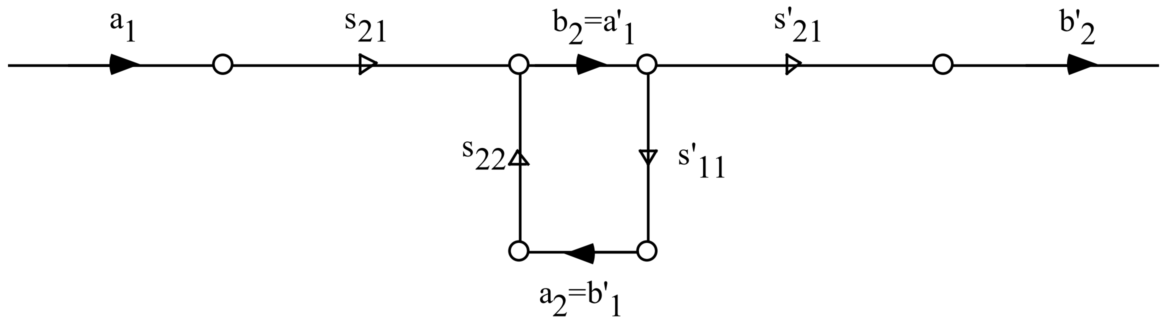


Stage two : we reduce

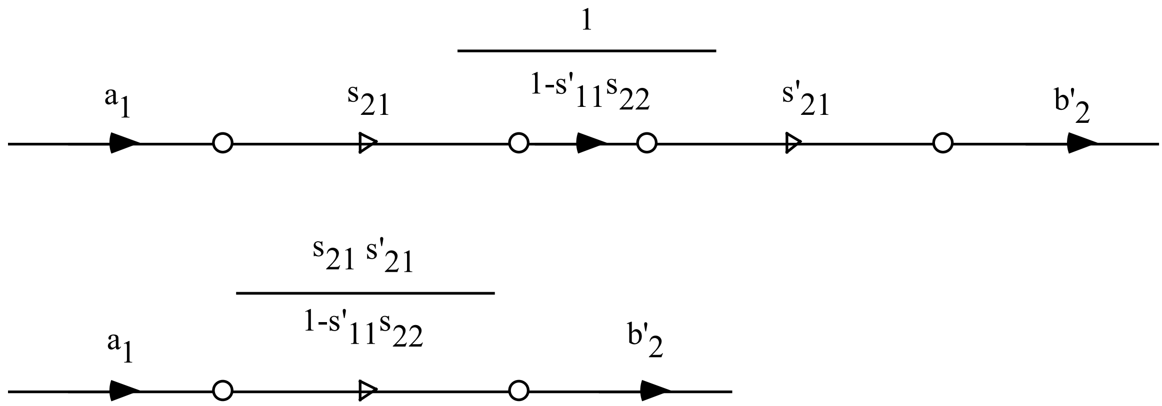




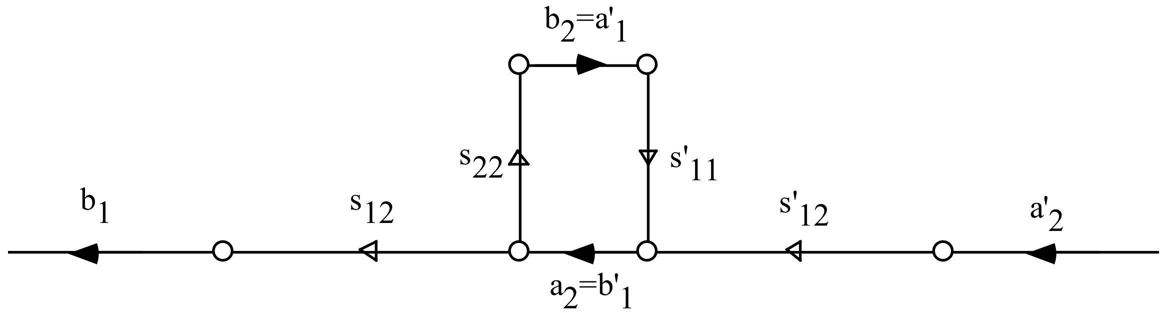
Stage three : we look for the possible paths going from a_1 to b'_2 :



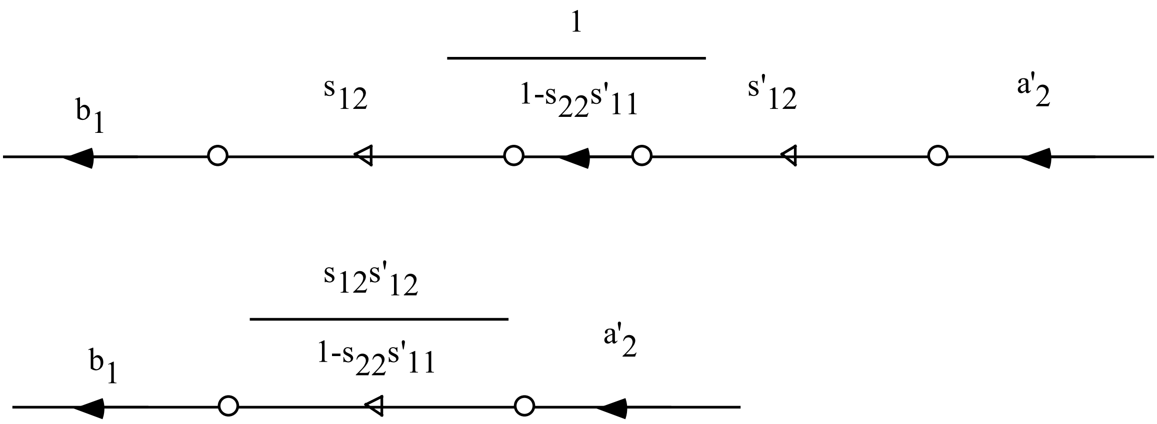
Stage 4 : we reduce



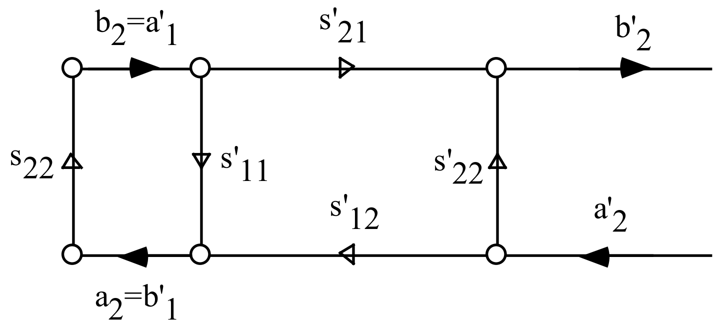
Stage 5 : we look for the possible paths going from a'_2 to b_1 :



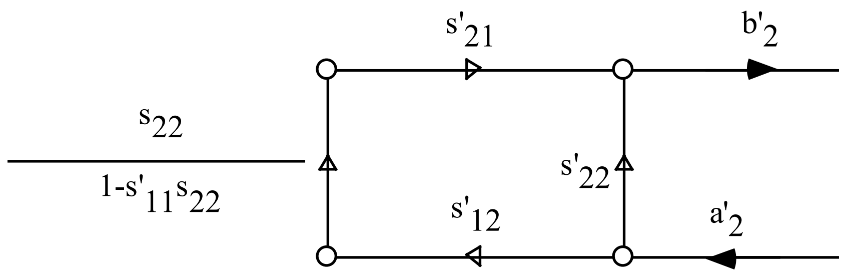
Stage 6: we reduce

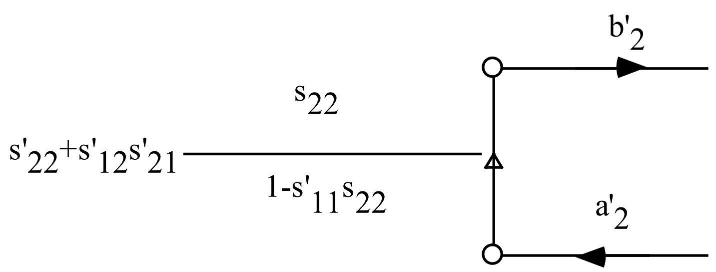


Stage 7 : we look for the possible paths going from a'_2 to b_2' :



Stage 8 : we reduce





And we get finally the scattering matrix of two cascaded two-ports :

$$\begin{bmatrix} b_1 \\ b'_2 \end{bmatrix} = \begin{bmatrix} s_{11} + \dfrac{s_{21}s_{12}s'_{11}}{1-s_{11}'s_{22}} & \dfrac{s_{12}s'_{12}}{1-s_{22}s'_{11}} \\ \dfrac{s_{21}s'_{21}}{1-s'_{11}\,s_{22}} & s'_{22} + \dfrac{s'_{12}\,s'_{21}\,s_{22}}{1-s'_{11}\,s_{22}} \end{bmatrix} \begin{bmatrix} a_1 \\ a'_2 \end{bmatrix} \tag{5. 73}$$

5.4.6 Summary of the general characteristics of the scattering matrix

- s_{ij} : transfer function between port j and i
- s_{ii} : reflection coefficient at port i
- $\left|s_{ij}\right|^2 = \dfrac{P_i}{P_j}$ normalized transferred power from j to i

- The scattering of a reciprocal network is symmetric
- The scattering matrix of a lossless network is of the type $[S][\tilde{S}]=[1]$
- The scattering matrix of a matched network has zeros on its main diagonal.

5.5 Voltage standing wave ratio

The voltage standing wave ration, or VSWR, is another useful mean to characterize the refflection coefficient at the ports of a device. It results from the fact that a reflection at a port will induced a reflected wave along the feeding line. This reflected wave will combine itself to the incident wave, in order to form a standing wave (fig. 5.21)

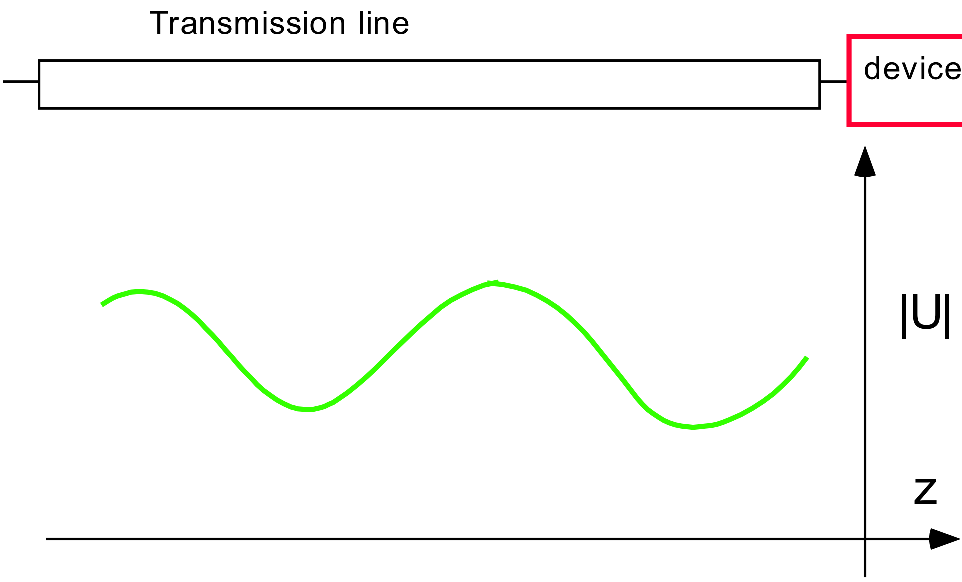


Fig. 5. 21 : standing wave

The voltage in the transmission line is given by :

$$U_i = \sqrt{Z_{ci}}\left(a_i + b_i\right) = \sqrt{Z_{ci}}\left(a_i + s_{ii} a_i\right) \tag{5. 74}$$

Thus

$$U_i(z) = \sqrt{Z_{ci}}\, a_i(z)\left[1 + s_{ii}\, \mathrm{e}^{2j\beta z}\right] \tag{5. 75}$$

The modulus of the voltage can be written as

$$\begin{aligned} \left|U_i(z)\right| &= \sqrt{Z_{ci}}\left|a_i(z)\right|\sqrt{\left[1+\left|s_{ii}\right|\cos(\varphi+2\beta z)\right]^2+\left|s_{ii}\right|^2 \sin^2(\varphi+2\beta z)} \\ \left|U_i(z)\right| &= \sqrt{Z_{ci}}\left|a_i(z)\right|\sqrt{1+\left|s_{ii}\right|^2+2\left|s_{ii}\right|\cos(\varphi+2\beta z)} \end{aligned} \tag{5. 76}$$

Let us now consiider th eminimum and the maximum values of the modulus of the voltage :

$$
\begin{aligned}
U_{\max} &= \sqrt{Z_{ci}}\,|a_i|\left(1+|s_{ii}|\right) \quad \text{en } \varphi+2\beta z_i = 2n\pi \\
U_{\min} &= \sqrt{Z_{ci}}\,|a_i|\left(1-|s_{ii}|\right) \quad \text{en } \varphi+2\beta z_i = (2n+1)\pi
\end{aligned}
\tag{5. 77}
$$

and take the ratio between these values, which is called the voltage standing wave ration :

$$
ROS = VSWR = \frac{|U_{\max}|}{|U_{\min}|} = \frac{1+|s_{ii}|}{1-|s_{ii}|} \tag{5. 78}
$$

For a matched load, the reflected vave is equal to zero, and thus the VSWR is equal to 1. For a total reflection the VSWR is infinite. Examples of standing ewaves for different reflection coefficients are depicted in figure 5.22.