A Mathematical Derivations

A.1 Frame Synchronization

In this section, we give some theoretical background on the frame synchronization problem and derive the decision rule (2.2).

A.1.1 Problem Statement

The frame synchronization works as follows: Each data frame starts with a known preamble of N_p symbols. Now, in every time step n, the receiver must decide if the most recently received N_p symbols are the preamble, distorted by noise, or if the symbols are noise only.

We denote the known preamble sequence as $p[i]$, $i = 0, \ldots, N_p - 1$, summarized in the vector $\mathbf{p}$. The preamble symbols are assumed to have unit energy, i.e. $|p[i]|^2 = 1$. The N_p most recently received symbols at time instant n are denoted as the vector $\mathbf{r}_n = (r[n-N_p+1], \ldots, r[n])^T$. In order to keep the following discussion general, we assume that the signal is attenuated by an unknown factor and also has an unknown phasing. These effects can be expressed by multiplying the transmitted signal with a complex channel coefficient h.

The problem is now to decide between the following two hypotheses:

$$\begin{aligned} \mathcal{H}_0 &: \mathbf{r}_n = \mathbf{w} \\ \mathcal{H}_1 &: \mathbf{r}_n = h\mathbf{p} + \mathbf{w}, \end{aligned} \tag{A.1}$$

where the entries of the vector $\mathbf{w}$ are complex Gaussian distributed random variables with zero mean and variance σ_w^2, denoted as $w[n] \sim \mathcal{CN}(0, \sigma_w^2)$. The PDF of the complex Gaussian distribution is

$$p(w) = \frac{1}{\pi\sigma_w^2} \exp\left(-\frac{|w|^2}{\sigma_w^2}\right). \tag{A.2}$$

Because the noise symbols are statistically independent, the distribution of the vector $\mathbf{w}$ is given as the product of the distributions of the individual random variables:

$$\begin{aligned} p(\mathbf{w}) &= \prod_{i=0}^{N_p-1} p(w[i]) \\ &= \frac{1}{(\pi\sigma_w^2)^{N_p}} \exp\left(-\frac{1}{\sigma_w^2}\sum_{i=0}^{N_p-1} |w[i]|^2\right) \\ &= \frac{1}{(\pi\sigma_w^2)^{N_p}} \exp\left(-\frac{\mathbf{w}^H\mathbf{w}}{\sigma_w^2}\right). \end{aligned} \tag{A.3}$$

A.1.2 The Generalized Likelihood Ratio Test

The problem (A.1) is a standard problem in detection theory. The approach is to calculate the *likelihood ratio* $L(\mathbf{r}_n)$, which is defined as the ratio of the likelihood that hypothesis $\mathcal{H}_1$ is true and the likelihood that hypothesis $\mathcal{H}_0$ is true, given the received data $\mathbf{r}_n$:

$$L(\mathbf{r}_n) = \frac{p(\mathbf{r}_n; \mathcal{H}_1)}{p(\mathbf{r}_n; \mathcal{H}_0)}. \tag{A.4}$$

The likelihood ratio is then compared to a threshold γ, and the following decision is made: If $L(\mathbf{r}_n) > \gamma$, decide that $\mathcal{H}_1$ is true, otherwise decide that $\mathcal{H}_0$ is true.

The problem in our case is that the PDF of $\mathbf{r}_n$ depends on the unknown noise variance σ_w^2, and in case of $\mathcal{H}_1$ also on the unknown channel coefficient h. This leads us to the *Generalized Likelihood Ratio Test* (GLRT), where the unknown parameters are replaced by their *Maximum Likelihood Estimates* (MLEs). The decision rule is as follows: Decide in favor of $\mathcal{H}_1$ if

$$L_G(\mathbf{r}_n) = \frac{p(\mathbf{r}_n; \hat{h}, \hat{\sigma}_1^2, \mathcal{H}_1)}{p(\mathbf{r}_n; \hat{\sigma}_0^2, \mathcal{H}_0)} > \gamma \tag{A.5}$$

where $\hat{h}$ and $\hat{\sigma}_i^2$ are the MLEs of h and σ_w^2 under the assumption of $\mathcal{H}_i$.

In the following, we will calculate the PDFs $p(\mathbf{r}_n; \hat{\sigma}_0^2, \mathcal{H}_0)$ and $p(\mathbf{r}_n; \hat{h}, \hat{\sigma}_1^2, \mathcal{H}_1)$ and finally the likelihood ratio L_G. For convenience, we will replace $\mathbf{r}_n$ by $\mathbf{r}$, because the time index is not important here.

A.1.2.1 Distribution of r assuming $\mathcal{H}_0$

Under the assumption of $\mathcal{H}_0$, i.e. $\mathbf{r} = \mathbf{w}$, the joint PDF of $\mathbf{r}$ is

$$p(\mathbf{r}; \hat{\sigma}_0^2, \mathcal{H}_0) = \frac{1}{(\pi\, \hat{\sigma}_0^2)^{N_p}} \exp\left(-\frac{1}{\hat{\sigma}_0^2}\, \mathbf{r}^H \mathbf{r}\right), \tag{A.6}$$

where $\hat{\sigma}_0^2$ is the estimate of σ_w^2 that maximizes (A.6). By differentiating (A.6) with respect to $\hat{\sigma}_0^2$ and setting it to zero, it can easily be shown that

$$\hat{\sigma}_0^2 = \tfrac{1}{N_p}\, \mathbf{r}^H \mathbf{r}. \tag{A.7}$$

Substituting (A.7) into (A.6) yields

$$p(\mathbf{r}; \hat{\sigma}_0^2, \mathcal{H}_0) = \frac{1}{\left(\frac{\pi}{N_p} \mathbf{r}^H \mathbf{r}\right)^{N_p}} \exp(-N_p). \tag{A.8}$$

A.1.2.2 Distribution of r assuming $\mathcal{H}_1$

In this case,

$$p(\mathbf{r}; \hat{h}, \hat{\sigma}_1^2, \mathcal{H}_1) = \frac{1}{(\pi\, \hat{\sigma}_1^2)^{N_p}} \exp\left(-\frac{1}{\hat{\sigma}_1^2}\, (\mathbf{r} - \hat{h}\mathbf{p})^H (\mathbf{r} - \hat{h}\mathbf{p})\right), \tag{A.9}$$

where $\hat{\sigma}_1^2$ and $\hat{h}$ are the estimates of σ_w^2 and h, respectively, that maximize (A.9). Since $\mathbf{p}$ and $\mathbf{w}$ are uncorrelated, the best estimate of the noise power is

$$\hat{\sigma}_1^2 = \tfrac{1}{N_p}\mathbf{r}^H\mathbf{r} - |h|^2, \tag{A.10}$$

i.e. the total received power minus the signal power.
The parameter h now has to be chosen in order to maximize

$$p(\mathbf{r}; h, \mathcal{H}_1) = \frac{1}{\left(\pi\left(\frac{\mathcal{E}}{N_p} - |h|^2\right)\right)^{N_p}} \exp\left(-\frac{\mathcal{E} - h^*c - hc^* + |h|^2N_p}{\frac{\mathcal{E}}{N_p} - |h|^2}\right), \tag{A.11}$$

where, for the sake of simplicity, the abbreviations

$$\mathcal{E} = \mathbf{r}^H\mathbf{r} \tag{A.12}$$

for the energy of the received sequence, and

$$c = \mathbf{p}^H\mathbf{r} \tag{A.13}$$

for the output of the correlator filter have been introduced. The derivative of (A.11) with respect to h is[1]

$$\begin{aligned}\tfrac{\partial}{\partial h}p(\mathbf{r}; h, \mathcal{H}_1) = {} & N_p\pi h^*\left(\pi\left(\frac{\mathcal{E}}{N_p} - |h|^2\right)\right)^{-N_p-1}\exp\left(-\frac{\mathcal{E} - h^*c - hc^* + |h|^2N_p}{\frac{\mathcal{E}}{N_p} - |h|^2}\right)\\ & - \left(\pi\left(\frac{\mathcal{E}}{N_p} - |h|^2\right)\right)^{-N_p}\exp\left(-\frac{\mathcal{E} - h^*c - hc^* + |h|^2N_p}{\frac{\mathcal{E}}{N_p} - |h|^2}\right)\\ & \times \frac{(-c^* + h^*N_p)(\frac{\mathcal{E}}{N_p} - |h|^2) + (\mathcal{E} - h^*c - hc^* + |h|^2N_p)\,h^*}{\left(\frac{\mathcal{E}}{N_p} - |h|^2\right)^2}.\end{aligned} \tag{A.14}$$

Setting (A.14) to zero provides

$$\frac{N_p\pi h^*}{\pi\left(\frac{\mathcal{E}}{N_p} - |h|^2\right)} - \frac{-\frac{\mathcal{E}}{N}c^* + 2\mathcal{E}h^* - h^{*2}c}{\left(\frac{\mathcal{E}}{N_p} - |h|^2\right)^2} = 0 \tag{A.15}$$

$$\Longleftrightarrow \qquad -N_p hh^{*2} + h^{*2}c - \mathcal{E}h^* + \tfrac{\mathcal{E}}{N_p}c^* = 0 \tag{A.16}$$

with the solution

$$\hat{h} = \frac{c}{N_p}, \tag{A.17}$$

as can easily be verified.
Substituting (A.10) and (A.17) into (A.9) results in

$$p(\mathbf{r}; \hat{h}, \hat{\sigma}_1^2, \mathcal{H}_1) = \frac{1}{\left(\frac{\pi}{N_p}\left(\mathcal{E} - \frac{|c|^2}{N_p}\right)\right)^{N_p}} \exp(-N_p). \tag{A.18}$$

[1] Note that $\frac{\partial h^*}{\partial h} = 0$ and therefore $\frac{\partial |h|^2}{\partial h} = h^*$.

A.1.2.3 Likelihood Ratio

Dividing (A.18) by (A.8) results in the likelihood ratio

$$\begin{aligned} L_G &= \left(\frac{\frac{\pi}{N_p}\mathcal{E}}{\frac{\pi}{N_p}\left(\mathcal{E} - \frac{|c|^2}{N_p}\right)} \right)^{N_p} \\ &= \left(\frac{1}{1 - \frac{|c|^2}{\mathcal{E}N_p}} \right)^{N_p}. \end{aligned} \tag{A.19}$$

We will now show that the denominator of L_G is nonnegative and equals zero if and only if $\sigma_w^2 = 0$, i.e. $\mathbf{w} = \mathbf{0}$: Using the Cauchy-Schwarz inequality, we find that

$$|c|^2 = |\mathbf{p}^H\mathbf{w}|^2 \le \|\mathbf{p}\|^2 \cdot \|\mathbf{w}\|^2 = N_p\,\mathbf{w}^H\mathbf{w}, \tag{A.20}$$

with equality if and only if $\mathbf{p}$ and $\mathbf{w}$ are linearly dependent, i.e. $\mathbf{w} = \lambda\mathbf{p}$ with a constant $\lambda \in \mathbb{C}$. But since $\mathbf{w}$ is a random vector if $\sigma_w^2 > 0$, $\Pr\{\mathbf{w} = \lambda\mathbf{p}; \lambda \in \mathbb{C}\} = 0$. Furthermore, from (A.20) follows

$$|\mathbf{p}^H\mathbf{w}|^2 \le N_p\,\mathbf{w}^H\mathbf{w} \tag{A.21}$$

$$\iff \quad (hN_p + \mathbf{p}^H\mathbf{w})(h^*N_p + \mathbf{w}^H\mathbf{p}) \le (h^*\mathbf{p}^H + \mathbf{w}^H)(h\mathbf{p} + \mathbf{w})\,N_p \tag{A.22}$$

$$\iff \quad \mathbf{p}^H(h\mathbf{p} + \mathbf{w})(h^*\mathbf{p}^H + \mathbf{w}^H)\mathbf{p} \le \mathbf{r}^H\mathbf{r}\,N_p \tag{A.23}$$

$$\iff \quad (\mathbf{p}^H\mathbf{r})(\mathbf{r}^H\mathbf{p}) \le \mathbf{r}^H\mathbf{r}\,N_p \tag{A.24}$$

$$\iff \quad \frac{|c|^2}{\mathcal{E}N_p} \le 1. \qquad \square \tag{A.25}$$

Since in practice $\sigma_w^2 > 0$, it can be safely assumed that the denominator of L_G is always positive. The GLRT decides in favor of $\mathcal{H}_1$ if L_G is larger than a threshold γ'. Rearranging (A.19) yields

$$\left(\frac{1}{1 - \frac{|c|^2}{\mathcal{E}N_p}} \right)^{N_p} > \gamma' \tag{A.26}$$

$$\iff \frac{|c|^2}{\mathcal{E}} > N_p\left(1 - \gamma'^{-\frac{1}{N_p}}\right) = \gamma. \tag{A.27}$$

The GLRT can therefore be stated as follows:

$$\frac{|\mathbf{p}^H\mathbf{r}|^2}{\mathbf{r}^H\mathbf{r}} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \gamma, \tag{A.28}$$

which finally leads us to Equation (2.2) if we reintroduce the time index n.