Fundamentals of Analog & Mixed Signal VLSI Design

Oscillators

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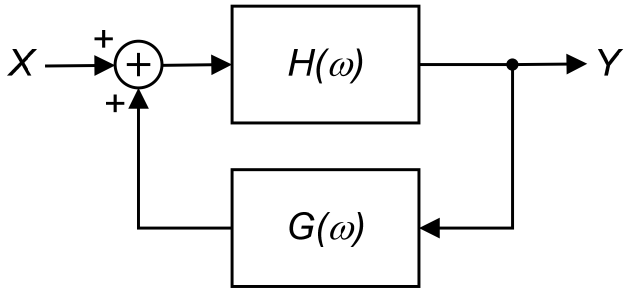
Outline

- General considerations
- The 3-points oscillator
- The cross-coupled pair oscillator

The Barkhausen Criteria



$$Y = \frac{H(\omega)}{1 - H(\omega) \cdot G(\omega)} \cdot X$$

- Most oscillators can be viewed as positive feedback systems with $H(\omega)$ being the feed forward gain and $G(\omega)$ the transfer function of the feedback circuit which is usually a frequency selective network (resonator)
- Oscillations occur at ω_0 if the loop gain $H(\omega_0)G(\omega_0)$ is exactly equal to unity, leading to the Barkhausen criteria

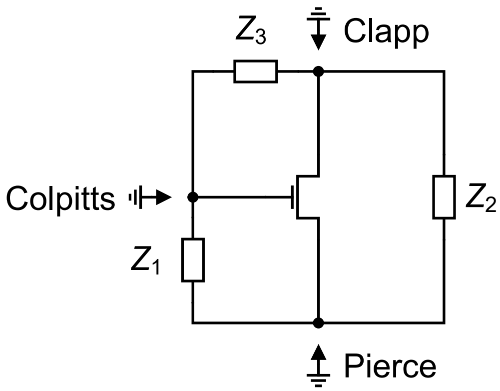
$$|H(\omega_0) \cdot G(\omega_0)| = 1 \text{ and } arg(H(\omega_0) \cdot G(\omega_0)) = 0$$

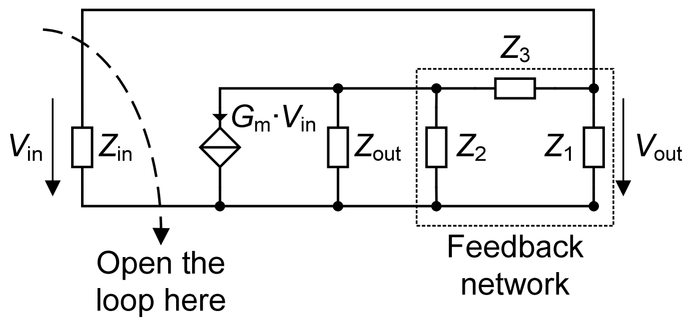
- The feedback network is usually frequency dependent and hence determines the oscillation frequency
- The Barkhausen criteria allows to derive the oscillation frequency, but does not say anything about the oscillation amplitude
- The latter is determined by the circuit nonlinearities

Outline

- General considerations
- The 3-points oscillator (or single transistor oscillator)
- The cross-coupled pair oscillator

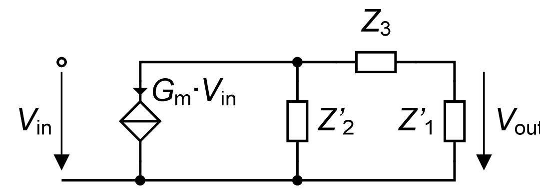
The 3-Points Oscillator – Barkhausen Criteria





Z_{in}: transistor input impedance

Z_{out}: transistor output impedance



$Z_1' = Z_1 \parallel Z_{in}$

$Z_2' = Z_2 \parallel Z_{out}$

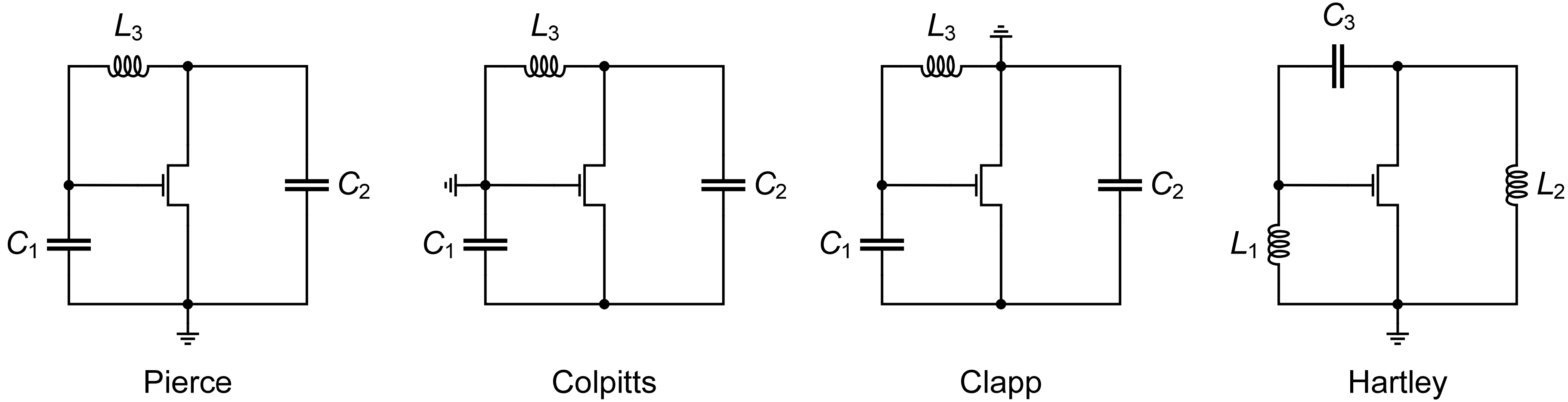
- Many basic (single transistor) oscillators can be described by the generic 3-points oscillator
- The transistor parasitic can be embedded into the impedances Z_k (like for example the transistor input and output impedances are included in Z_1 and Z_2 defining Z_1' and Z_2')
- Opening the loop at the gate allows to calculate the loop gain

$$G \cdot H = \frac{V_{out}}{V_{in}} = \frac{-G_m Z_1' Z_2'}{Z_1' + Z_2' + Z_3} = \frac{-G_m}{Y_1'(1 + Y_1' Z_3) + Y_2'}$$

- The loop gain has to be equal to unity to satisfy the Barkhausen criteria

$$G_m Z_1' Z_2' + Z_1' + Z_2' + Z_3 = 0 \text{ or } G_m + Y_1'(1 + Y_1' Z_3) + Y_2' = 0$$

The 3-Points Oscillator – Basic Oscillators



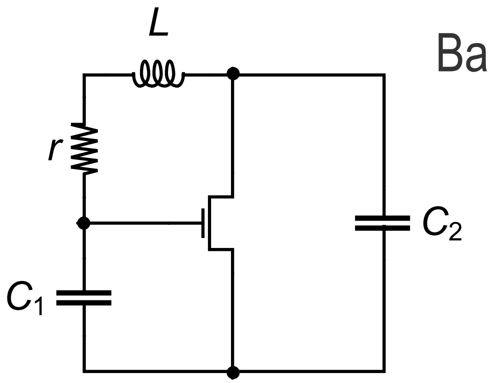
- In the case all the components of the feedback network are reactive $Z_k = jX_k$ $(k = 1,2,3)$, neglecting the input impedance Z_{in} but accounting for the output impedance $Z_{out} = 1/G_{ds}$

$$X_1 + X_2 + X_3 + j[(G_m + G_{ds})X_1X_2 + G_{ds}X_2X_3] = 0$$

$$X_2 = A_{dc} \cdot X_1 \text{ and } X_3 = -(A_{dc} + 1) \cdot X_1 \text{ with } A_{dc} = \frac{G_m}{G_{ds}}$$

- Since $A_{dc} > 0$, Z_2 should be of the same type of reactance than Z_1, whereas Z_3 should be of opposite sign leading to the following four basic single transistor oscillators depending on which node is the ground node

The 3-Points Oscillator – Critical Transconductance



Barkhausen criteria: $G_m + Y_1'\left(1 + Y_2' Z_3\right) + Y_2' = 0$

with: $Y_1' = Y_1 = j\omega C_1 \quad Y_2' = G_{ds} + j\omega C_2 \cong j\omega C_2 \quad Z_3 = r + j\omega L$

Leads to: $\begin{cases} G_m - \omega^2 r C_1 C_2 = 0 \\ C_1 + C_2 - \omega^2 L C_1 C_2 = 0 \end{cases}$

- The **resonant frequency** is then given by

$$\omega_0 = \frac{1}{\sqrt{L C_{12}}} \quad \text{with} \quad C_{12} = \frac{C_1 C_2}{C_1 + C_2}$$

- The **critical transconductance** required to maintain the oscillation is given by

$$G_{mcrit} = \omega_0^2 r C_1 C_2 = \frac{\left(C_1 + C_2\right) r}{L} = \frac{\omega_0 \left(C_1 + C_2\right)}{Q_L} \quad \text{with} \quad Q_L = \frac{\omega_0 L}{r}$$

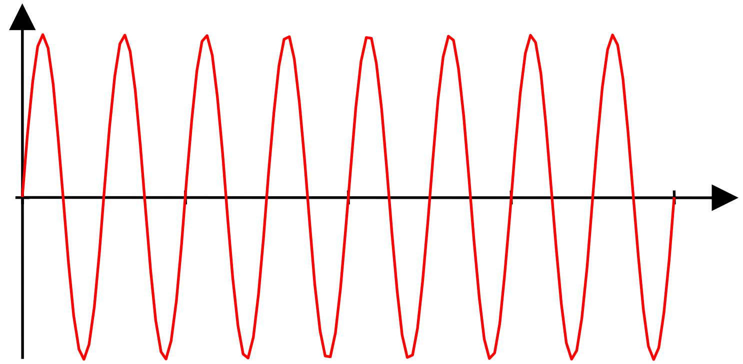
where Q_L is the unloaded Q of the inductor

- The larger the loss r (the smaller the Q_L), the larger the required G_{mcrit}
- G_{mcrit} also increases with frequency ω_0 and capacitances C_1 and C_2

The 3-Points Oscillator – Oscillation Conditions

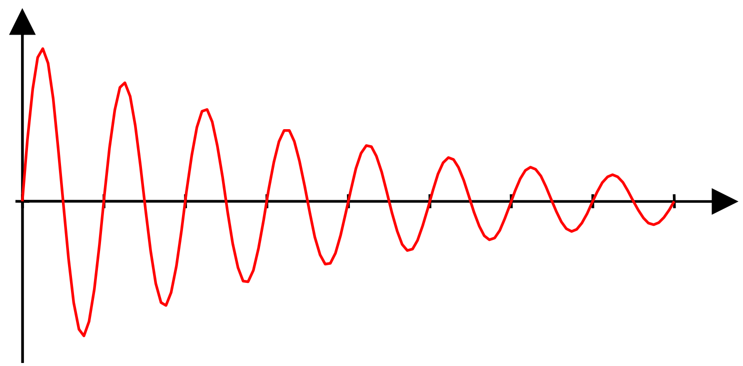
- Oscillations are maintained for

$$G_m = G_{mcrit} = \frac{\omega_0 \left(C_1 + C_2\right)}{Q_L}$$



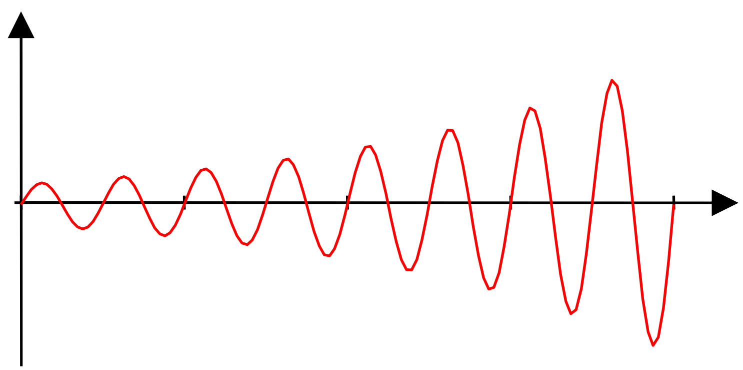
- Oscillations vanish if

$$G_m < G_{mcrit} = \frac{\omega_0 \left(C_1 + C_2\right)}{Q_L}$$



- Oscillations amplitude increase if

$$G_m > G_{mcrit} = \frac{\omega_0 \left(C_1 + C_2\right)}{Q_L}$$



Accounting for Loss in Output Conductance

- If the output conductance is accounted for, the Barkhausen criteria becomes

$$\begin{cases} G_m + G_{ds} - \omega^2 C_1 \left(G_{ds} L + r C_2 \right) = 0 \\ C_1 + C_2 + C_1 \left(G_{ds} r - \omega^2 L C_2 \right) = 0 \end{cases}$$

- The oscillation frequency is then slightly modified by the presence of the output conductance

$$\omega_0 = \frac{1}{\sqrt{L C_{eq}}} \quad \text{with} \quad C_{eq} \cong \frac{C_1 C_2'}{C_1 + C_2'} \quad \text{and} \quad C_2' \cong \frac{C_2}{1 + G_{ds} r}$$

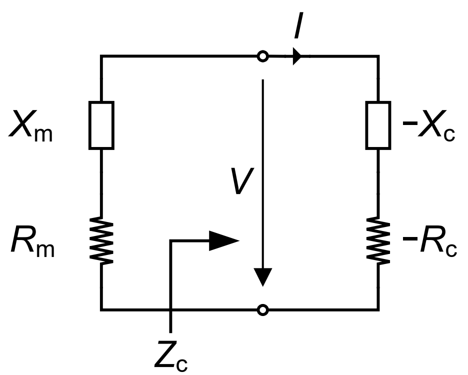
- The critical transconductance is then given by

$$G_{mcrit} = \alpha\, G_{ds} + (1+\alpha) \frac{\omega_0 C_2}{Q_L} \cong \alpha\, G_{ds} + \frac{\omega_0 \left(C_1 + C_2 \right)}{Q_L} \quad \text{with} \quad \alpha = \frac{C_1}{C_2'} = \frac{C_1 \left(1 + G_{ds} r \right)}{C_2} \cong \frac{C_1}{C_2}$$

- The critical transconductance has to be larger by $\alpha \cdot G_{ds}$ compared to the case where G_{ds} is negligible

Negative Resistance Analysis Method

- In a **linear analysis**, any oscillator can be viewed as a resonant circuit (X_m and X_c) in series with a negative resistance $-R_c$ that compensates for the loss R_m
- The impedance seen at the input of the circuit Z_c should hence have a negative real part $-R_c$ and a negative imaginary part $-X_c$

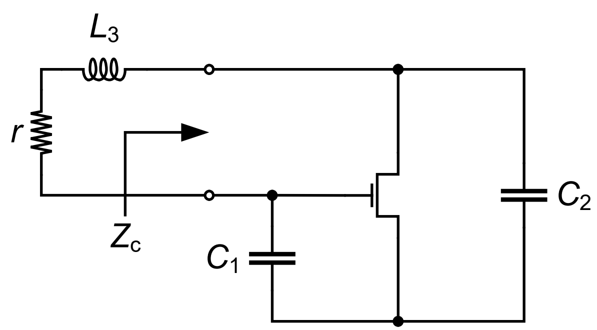


$$Z_m(\omega) = R_m + jX_m$$
$$Z_c(\omega, G_m) = -R_c(\omega, G_m) - jX_c(\omega)$$

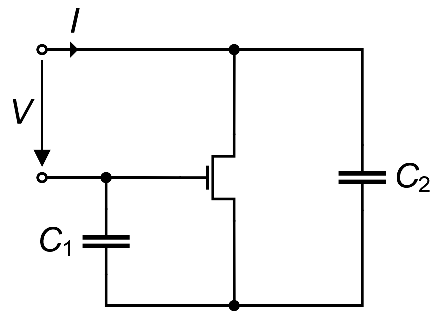
Such that their sum is equal to zero:

$$Z_m(\omega) + Z_c(\omega, G_m) = 0 \quad \rightarrow \quad \begin{cases} -\mathrm{Re}\{Z_c\} = R_c(\omega, G_m) = r \\ -\mathrm{Im}\{Z_c\} = X_c(\omega) = X_m(\omega) \end{cases}$$

- Can be applied to the Pierce oscillator

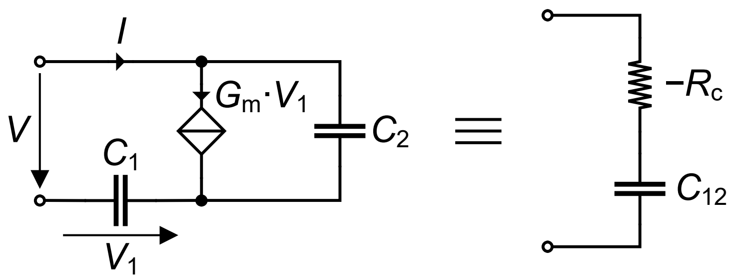


$$Z_c(\omega) = \frac{V}{I}$$



Negative Resistance Analysis Method

- The corresponding small-signal circuit is given by



$$Z_c = \frac{V}{I} = -\frac{G_m + j\omega(C_1 + C_2)}{\omega^2 C_1 C_2} = -\frac{G_m}{\omega^2 C_1 C_2} + \frac{1}{j\omega C_{12}}$$

$$R_c = \frac{G_m}{\omega^2 C_1 C_2}$$

$$X_c = \frac{1}{\omega C_{12}} \quad \text{with} \quad C_{12} = \frac{C_1 C_2}{C_1 + C_2}$$

- The oscillation frequency is then given by the condition on the imaginary part

$$-\operatorname{Im}\{Z_c\} = X_c = X_m \;\rightarrow\; \frac{1}{\omega C_{12}} = \omega L \;\rightarrow\; \omega_0 = \frac{1}{\sqrt{L C_{12}}}$$

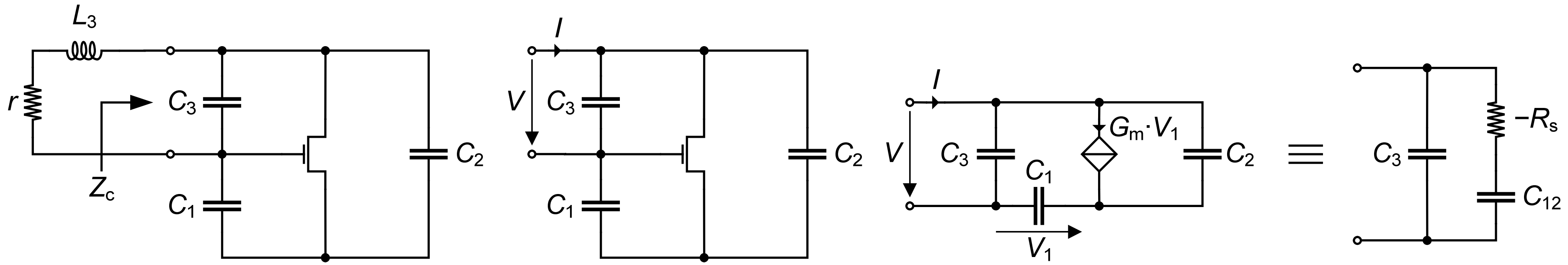
- The critical transconductance to insure oscillation is given by setting $R_c = r$

$$\frac{G_{mcrit}}{\omega^2 C_1 C_2} = r \;\rightarrow\; G_{mcrit} = r \cdot \omega^2 C_1 C_2 = \frac{r}{L} \cdot (C_1 + C_2) = \frac{\omega_0 \cdot (C_1 + C_2)}{Q_L}$$

which corresponds to the result obtained earlier using the Barkhausen criteria

Negative Resistance Analysis Method

- The same analysis can be conducted accounting for capacitance C_3 embedding the parasitic capacitances of the inductor and the transistor



$$R_s = \frac{G_m}{\omega^2 C_1 C_2}$$

$$C_{12} = \frac{C_1 C_2}{C_1 + C_2}$$

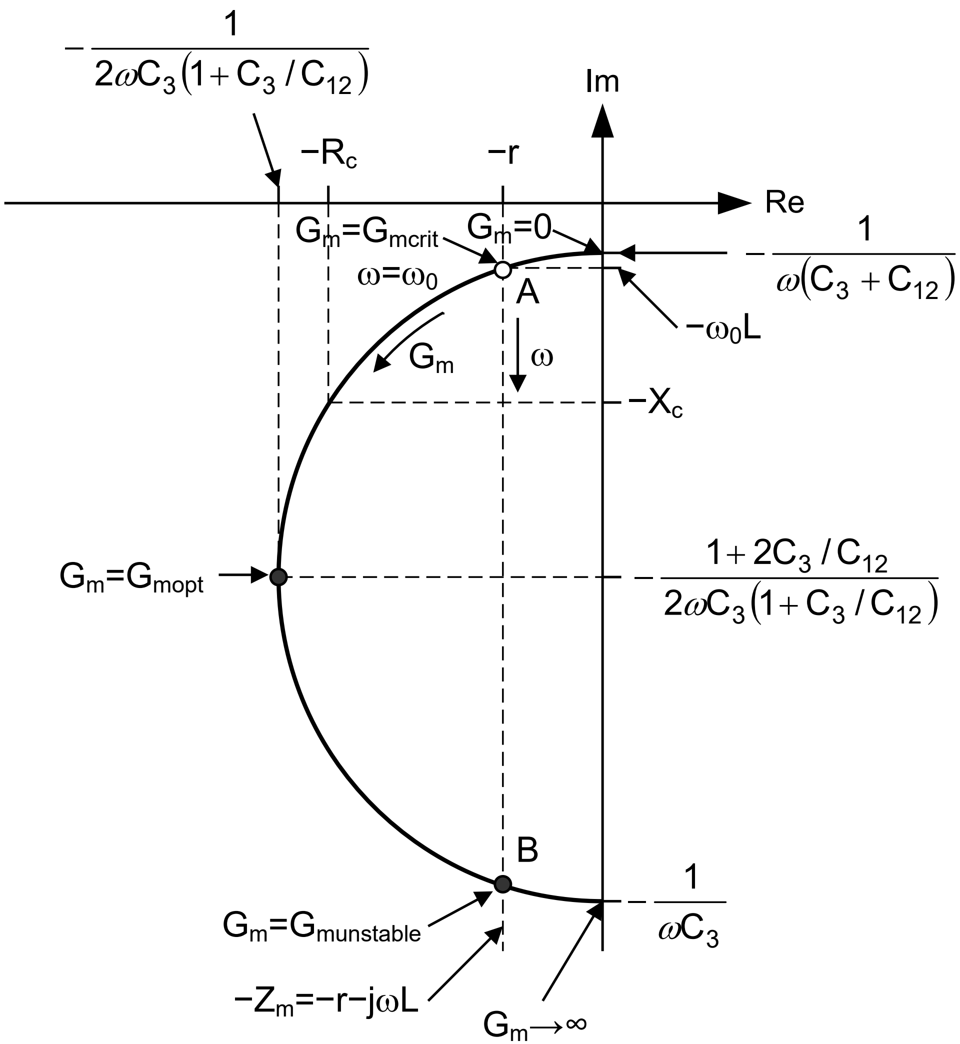
$$Z_c = -\frac{G_m + j\omega(C_1 + C_2)}{\omega^2 (C_1 C_2 + C_1 C_3 + C_2 C_3) - j\omega G_m C_3}$$

- Which leads to

$$R_c = \frac{G_m C_1 C_2}{(G_m C_3)^2 + \omega^2 (C_1 C_2 + C_1 C_3 + C_2 C_3)^2}$$

$$X_c = \frac{G_m^2 C_3 + \omega^2 (C_1 + C_2)(C_1 C_2 + C_1 C_3 + C_2 C_3)}{\omega\left[(G_m C_3)^2 + \omega^2 (C_1 C_2 + C_1 C_3 + C_2 C_3)^2\right]}$$

Impedance Locus



- When plotted in the complex plane for a given frequency (usually the resonance frequency ω_0), versus the parameter G_m, the circuit impedance $Z_c(G_m)$ describes a half-circle
- The impedance $-Z_m = -r - j\omega L$ of the lossy inductor can be plotted versus ω and describes a vertical line at $-r$
- The condition $Z_c = -Z_m$ corresponds to the intersections of the circle and the vertical line (points A and B)
- It can be shown that only point A corresponds to a stable point
- By definition, at point A we have:

$$G_m = G_{mcrit} \quad \text{and} \quad \omega = \omega_0$$

Impedance Locus

- G_{mcrit} and ω_0 can be found by solving

$$\begin{cases} -\mathrm{Re}\{Z_c\} = R_c(\omega, G_m) = r \\ -\mathrm{Im}\{Z_c\} = X_c(\omega) = \omega L \end{cases}$$

- R_c reaches a minimum (max in absolute value) given by

$$R_{c,\max} = \frac{1}{2\omega C_3\left(1+\frac{C_3}{C_{12}}\right)} \quad \text{for} \quad G_m = G_{mopt} = \omega\left(C_1 + C_2 + \frac{C_1 C_2}{C_3}\right) \quad \text{with} \quad C_{12} = \frac{C_1 C_2}{C_1 + C_2}$$

- If $r > R_{c,max}$ there are no intersections and no oscillations can take place
- The condition for a solution to exist is hence given by

$$r \le R_{c,\max} = \frac{1}{2\omega C_3\left(1+\frac{C_3}{C_{12}}\right)}$$

- If C_1 and/or C_2 decrease, point A moves downwards and ω_0 increases
- If $C_3 = 0$ the circle becomes a horizontal line independent of G_m

G_{mcrit} for Given ω_0 and Q_L

- In the case the oscillation frequency ω_0 and the quality factor of the inductor Q_L are set, G_{mcrit} can be found from

$$\frac{X_c(\omega_0, G_{mcrit})}{R_c(\omega_0, G_{mcrit})} = Q_L \quad \Rightarrow \quad \frac{G_{mcrit}^2 C_3 + \omega_0^2 (C_1 + C_2)(C_1 C_2 + C_1 C_3 + C_2 C_3)}{\omega_0 G_{mcrit} C_1 C_2} = Q_L$$

which leads to

$$G_{mcrit} = \omega_0 C_2 \frac{\alpha_1 Q_L}{2\alpha_3} \left[1 - \sqrt{1 - \left(\frac{2\alpha_3}{\alpha_1 Q_L} \right)^2 (\alpha_1 + 1) \left(1 + \alpha_1 + \frac{\alpha_1}{\alpha_3} \right)} \right] \quad \text{where} \quad \alpha_1 = \frac{C_1}{C_2} \quad \alpha_3 = \frac{C_3}{C_2}$$

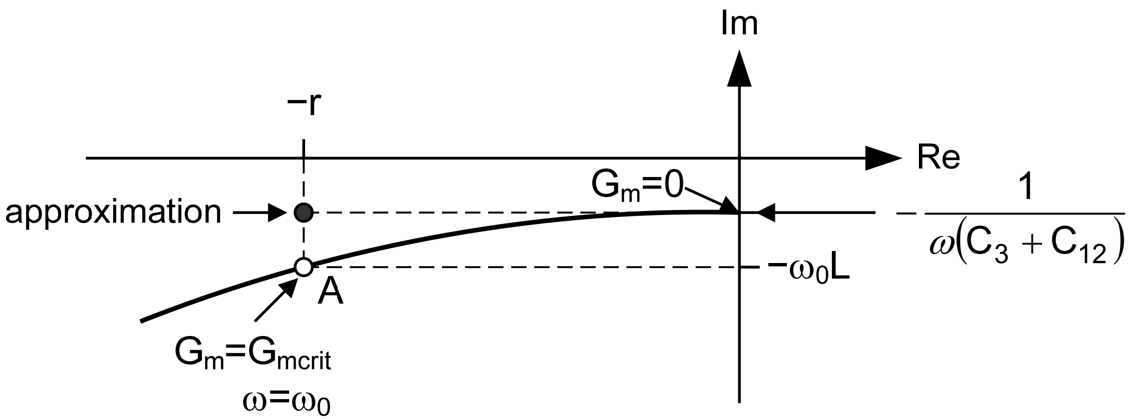
- The solution obviously only exists if

$$Q_L > \frac{2\alpha_3}{\alpha_1} \cdot \sqrt{(\alpha_1 + 1)\left(1 + \alpha_1 + \frac{\alpha_1}{\alpha_3} \right)}$$

- An approximate solution can be found for $Q_L \gg 1$

$$G_{mcrit} \cong \omega_0 C_2 \frac{\alpha_3}{\alpha_1 Q_L} (\alpha_1 + 1) \left(1 + \alpha_1 + \frac{\alpha_1}{\alpha_3} \right) = \frac{\omega_0}{Q_L} (C_1 + C_2) \left(1 + \frac{C_3}{C_{12}} \right) \quad \text{with} \quad C_{12} = \frac{C_1 C_2}{C_1 + C_2}$$

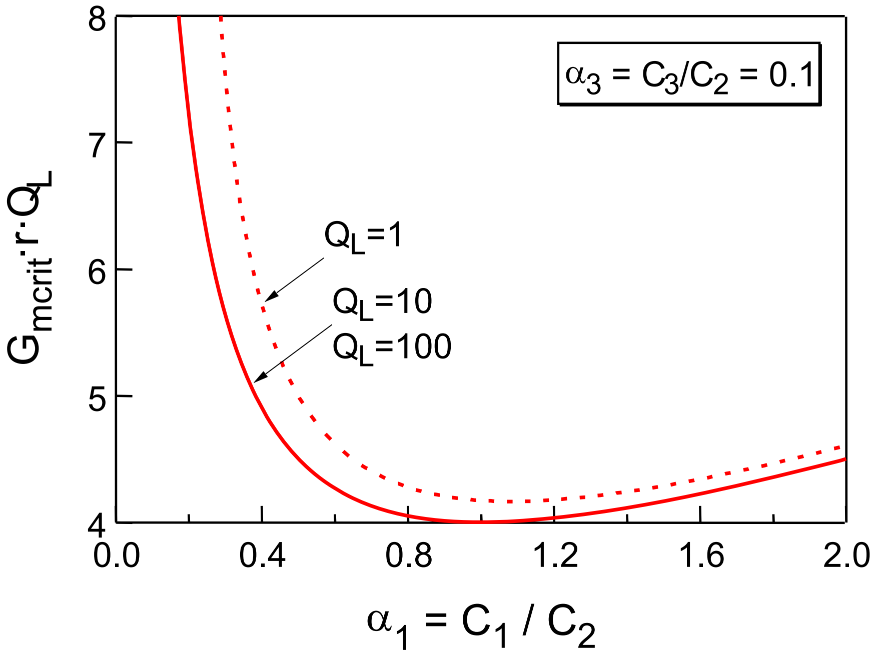
Approximation of G_{mcrit}



- As shown above, the oscillation frequency depends on r and therefore on the quality factor Q_L of the inductor which is not desirable since it may vary significantly
- When losses are small (r small) or Q_L becomes large, the vertical line gets closer to the imaginary axis and the sensitivity of ω_0 to Q_L becomes small
- In this condition, the oscillation frequency can be approximated by setting $G_m = 0$ in $X_c(\omega, G_m) = X_m(\omega)$ and solving for ω leads to

$$\omega_0 \cong \frac{1}{\sqrt{L(C_3 + C_{12})}}$$

Minimum Value of G_{mcrit}



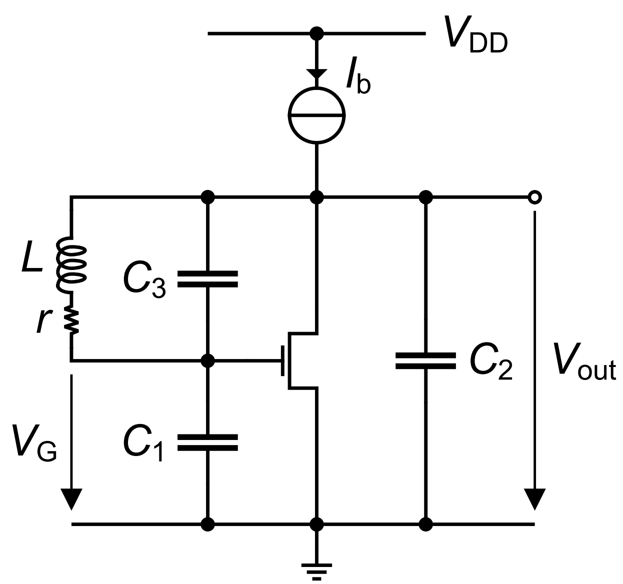
$$G_{mcrit} \cdot r \cdot Q_L \cong \frac{(\alpha_1 + 1)^2}{\alpha_1}$$

- As shown above, G_{mcrit} is minimum for $\alpha_1 = 1$ ($C_1 = C_2$)

$$G_{mcrit,\min} = \frac{1}{r}\left(\frac{2}{Q_L}\right)^2 = \frac{\omega_0}{Q_L} 2\left(C_1 + 2C_3\right) \quad \text{for} \quad C_1 = C_2$$

Sinusoidal Control Voltage

- For $G_m > G_{mcrit}$, the oscillation will start and amplitude will grow, generating harmonic components due to the nonlinearity of the active element
- The above analysis was linear assuming small-signal operation. It did not give any information about the oscillation amplitude. This can only be obtained from a nonlinear analysis which is not always possible to achieve in an analytical form

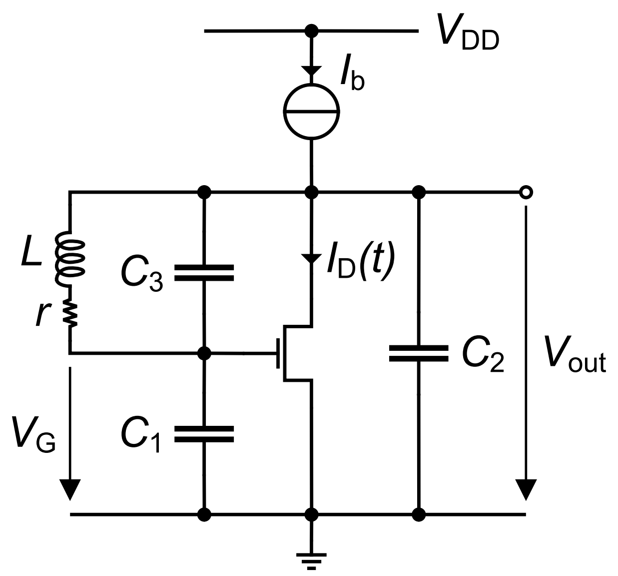


- If the quality factor of the resonator is assumed large (typically $Q_L > 10$), the current going through the LC tank is filtered from its harmonics and generates a voltage at the gate that can be considered as quasi-sinusoidal

$$V_G(t) \cong V_{G0} + A \cdot \cos(\omega_0 t)$$

where V_{G0} is the dc gate voltage when there are no oscillations ($A = 0$)

Nonlinear Analysis of the Pierce Oscillator (weak inv.)



- In the case of the Pierce oscillator the gate voltage can therefore be assumed to be sinusoidal

$$V_G(t) = V_{G0} + A \cdot \cos(\omega_0 t)$$

- If the transistor is biased in weak inversion, the drain current is then given by

$$I_D(t) = I_{D0} \cdot e^{\frac{V_G(t)}{nU_T}} = I_{D0} \cdot e^{\frac{V_{G0} + A \cdot \cos(\omega_0 t)}{nU_T}}$$

$$= I_0 \cdot e^{x \cdot \cos(\omega_0 t)}$$

$$A = \Delta V_G \cong -\Delta V_{out}$$

with $I_0 = I_{D0} \cdot e^{\frac{V_{G0}}{nU_T}} = I_{spec} \cdot e^{\frac{V_{G0} - V_{T0}}{nU_T}}$ and $x \triangleq \frac{A}{nU_T}$

- Notice that it is essentially capacitance C_3 that couples harmonic components directly to the gate. Therefore the assumption of the gate voltage being quasi-sinusoidal only holds if C_3 is much smaller than C_{12}

Nonlinear Analysis of the Pierce Oscillator (WI)

- Function $e^{x \cdot cos(\omega_0 t)}$ can be developed in a Fourier series given by

$$e^{x \cdot \cos(\omega_0 t)} = \mathrm{I}_{B0}(x) + 2 \cdot \sum_{n=1}^{+\infty} \mathrm{I}_{Bn}(x) \cdot \cos(n\omega_0 t)$$

where $\mathrm{I}_{B0}(x)$ and $\mathrm{I}_{Bn}(x)$ are the **modified Bessel functions of the first kind** of order 0 and n

- The drain current is then given by

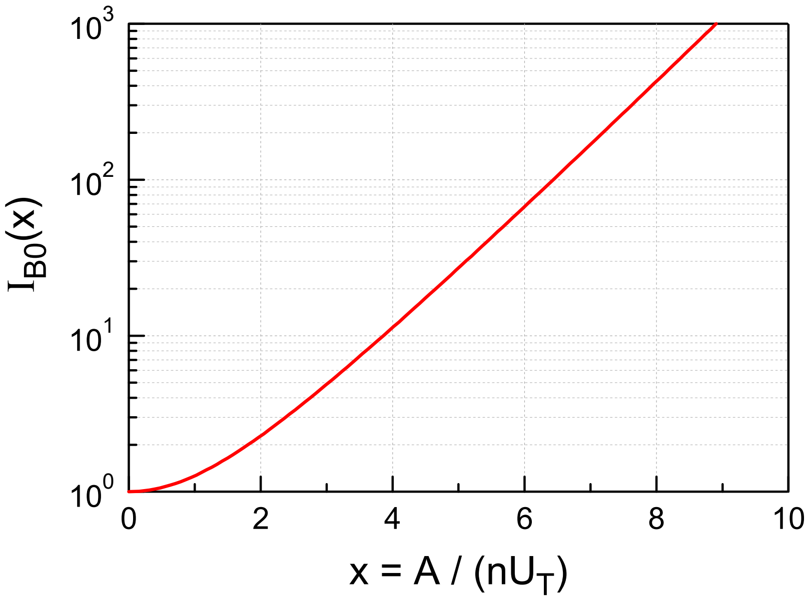
$$I_D(t) = I_0 \cdot e^{x \cdot \cos(\omega_0 t)} = I_{dc} + 2I_0 \cdot \sum_{n=1}^{+\infty} \mathrm{I}_{Bn}(x) \cdot \cos(n\omega_0 t)$$

where I_{dc} is the average current (dc current) given by

$$I_{dc} = I_0 \cdot \mathrm{I}_{B0}(x)$$

- Notice that in the case of the 3-points oscillators, the dc current I_{dc} is set by a constant bias current I_b, whereas I_0 is the quiescent current defined as the current that flows when there are no oscillations (or their amplitude is zero $x = 0$)

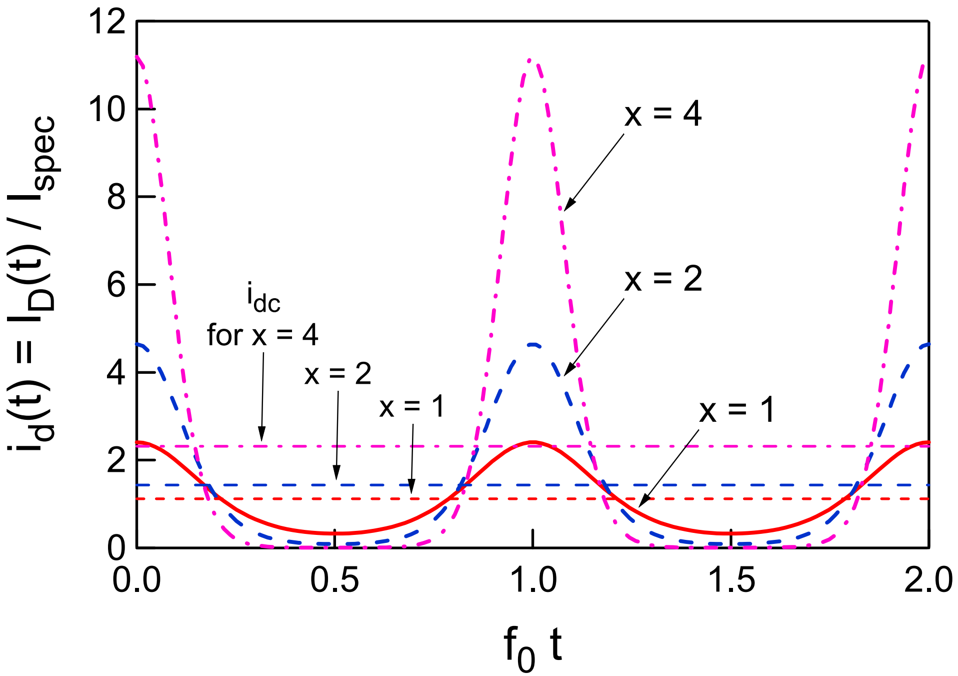
Quiescent Current I_0 and Voltage V_{G0}



$$I_{dc} = I_0 \cdot \mathrm{I}_{B0}(x)$$

- The average of $e^{x \cdot cos(\omega_0 t)}$ is given by $\mathrm{I}_{B0}(x)$ which increases exponentially
- For the 3-points oscillators, the dc current I_{dc} is maintained constant and equal to I_b
- The current I_0 and hence the gate bias voltage V_{G0} need therefore to decrease in order to compensate for the increase in $\mathrm{I}_{B0}(x)$ and maintain the dc current equal to I_b
- There is therefore a relation between the oscillation amplitude and the dc bias which will be derived later

Drain Current Waveforms

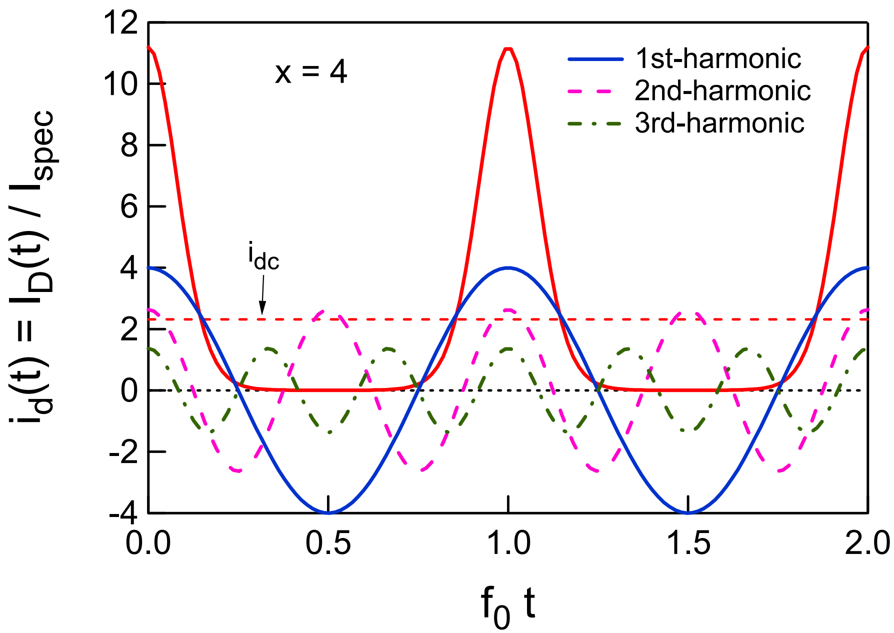


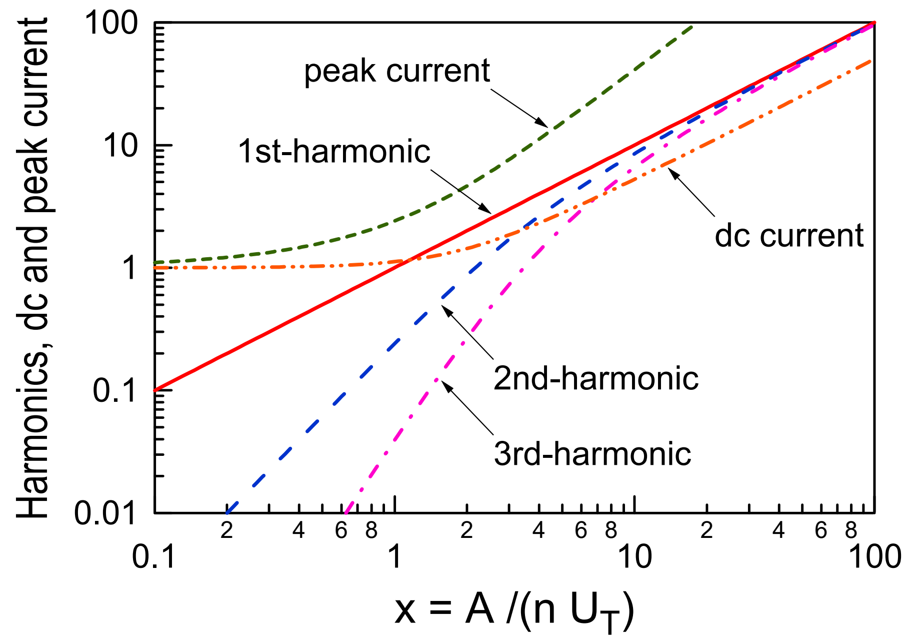
- The above plot shows the drain current normalized to I_{spec} for several oscillation amplitudes and accounting for the dependence of I_0 and I_{dc} (I_b) on x

$$i_d(t) \triangleq \frac{I_D(t)}{I_{spec}} = i_0(x) \cdot e^{x \cdot \cos(\omega_0 t)} = i_{dc}(x) + 2 i_0(x) \cdot \sum_{n=1}^{+\infty} \mathrm{I}_{Bn}(x) \cdot \cos(n \omega_0 t)$$

with $\quad i_0(x) \triangleq \dfrac{I_0}{I_{spec}} = \dfrac{x}{2 \mathrm{I}_{B1}(x)} \quad$ and $\quad i_{dc}(x) \triangleq \dfrac{I_b}{I_{spec}} = i_0(x) \cdot \mathrm{I}_{B0}(x)$

Drain Current Harmonics

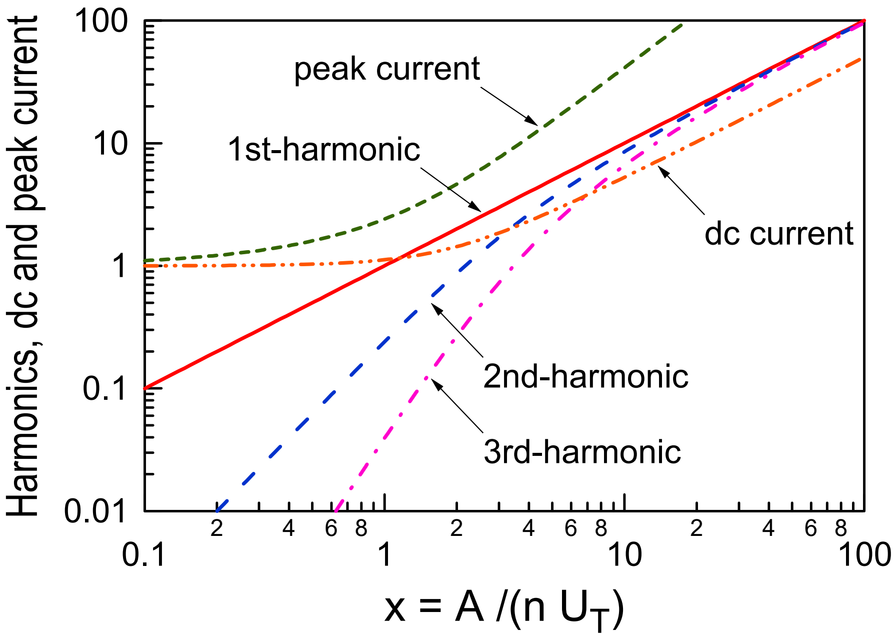


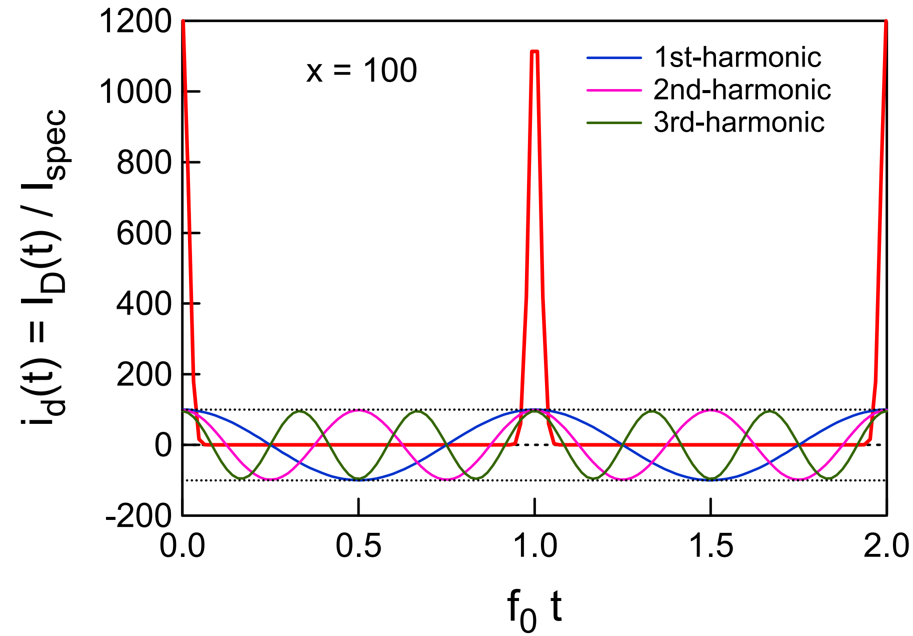


DC current: $i_{dc}(x) \square \dfrac{I_b(x)}{I_{spec}} = i_0(x) \cdot \mathrm{I}_{B0}(x)$ with $i_0(x) \square \dfrac{I_0}{I_{spec}} = \dfrac{x}{2\mathrm{I}_{B1}(x)}$

n^th-harmonic: $i_{d(n)} \square \dfrac{I_{D(n)}}{I_{spec}} = 2 i_0(x) \cdot \mathrm{I}_{Bn}(x)$ with $i_{d(1)} = \dfrac{2x}{2\mathrm{I}_{B1}(x)} \cdot \mathrm{I}_{B1}(x) = x$

Harmonics for Large Amplitudes





- It is interesting to note that for large values of x, all harmonics tend to the same value, since

$$\mathrm{I}_{Bn}(x) \cong \frac{e^x}{\sqrt{2\pi x}} \quad \text{for} \quad x \gg 1 \quad \rightarrow \quad i_{d(n)} = 2 i_0(x) \cdot \mathrm{I}_{Bn}(x) = x \cdot \frac{\mathrm{I}_{Bn}(x)}{\mathrm{I}_{B1}(x)} \cong x \quad \text{for} \quad x \gg 1$$

Equivalent Impedance for the Fundamental Component

- The active element is usually nonlinear and generates harmonic components in the drain current
- The latter are filtered out by the resonator even though the current across it can be strongly distorted
- The energy exchange between the active element and the resonator occurs therefore mostly at the fundamental frequency
- The active circuit can therefore be replaced by the impedance for the fundamental defined as

$$Z_{c(1)} = -\frac{V}{I_{(1)}}$$

where $I_{(1)}$ is the complex current at the fundamental frequency which depends on the amplitude of the sinusoidal voltage V

Transconductance for the Fundamental Component

- At low frequency the variation of the fundamental component of the drain current $\Delta I_{D(1)}(t)$ and of the gate voltage $\Delta V_G(t)$ are in-phase
- The small-signal transconductance can be replaced by the transconductance for the fundamental $G_{m(1)}$ given by

$$G_{m(1)} = \frac{\Delta I_{D(1)}}{A} = \frac{2I_0 \mathrm{I}_{B1}(x)}{A} = \frac{I_0}{nU_T} \cdot \frac{2\mathrm{I}_{B1}(x)}{x} \quad \text{where} \quad x \triangleq \frac{A}{nU_T}$$

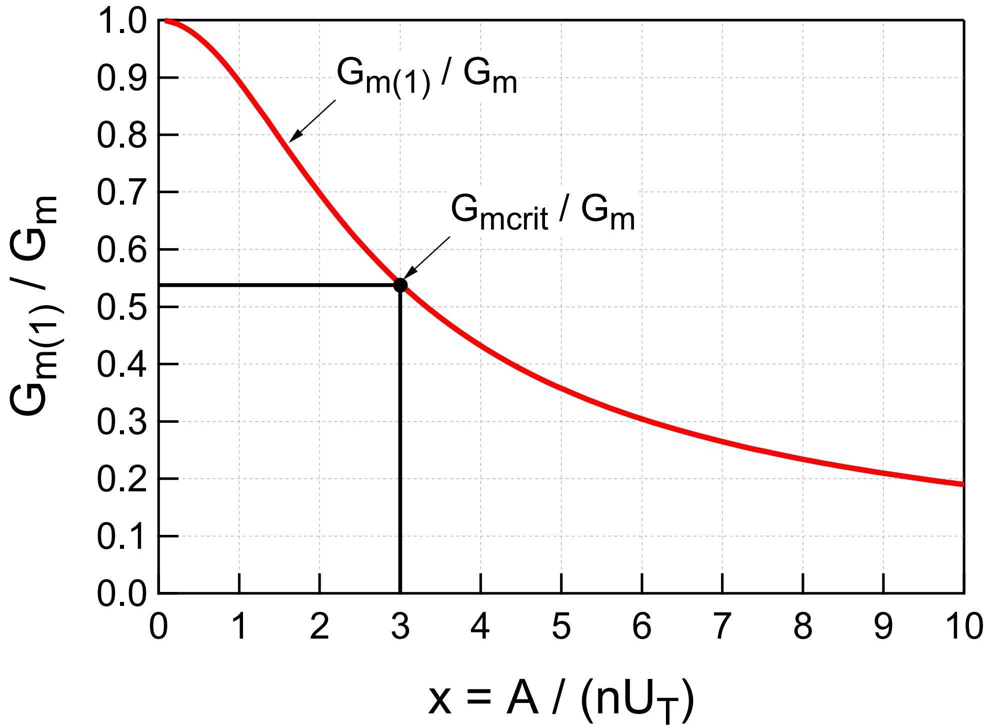
- The transconductance for the fundamental can be rewritten by introducing the dc current I_b

$$I_b = I_0 \cdot \mathrm{I}_{B0}(x) \quad \rightarrow \quad I_0 = \frac{I_b}{\mathrm{I}_{B0}(x)} \quad \rightarrow \quad G_{m(1)} = \frac{I_b}{nU_T} \cdot \frac{2\mathrm{I}_{B1}(x)}{x \cdot \mathrm{I}_{B0}(x)} = G_m \cdot \frac{2\mathrm{I}_{B1}(x)}{x \cdot \mathrm{I}_{B0}(x)}$$

where $G_m = I_b/(nU_T)$ is the small-signal transconductance set by the bias current I_b

$$G_m = \frac{I_{dc}}{nU_T} = \frac{I_b}{nU_T}$$

Transconductance for the Fundamental Component



$$\frac{G_{m(1)}}{G_m} = \frac{2 I_{B1}(x)}{x \cdot I_{B0}(x)}$$

with $G_m = \dfrac{I_b}{nU_T}$

and $x \triangleq \dfrac{A}{nU_T}$

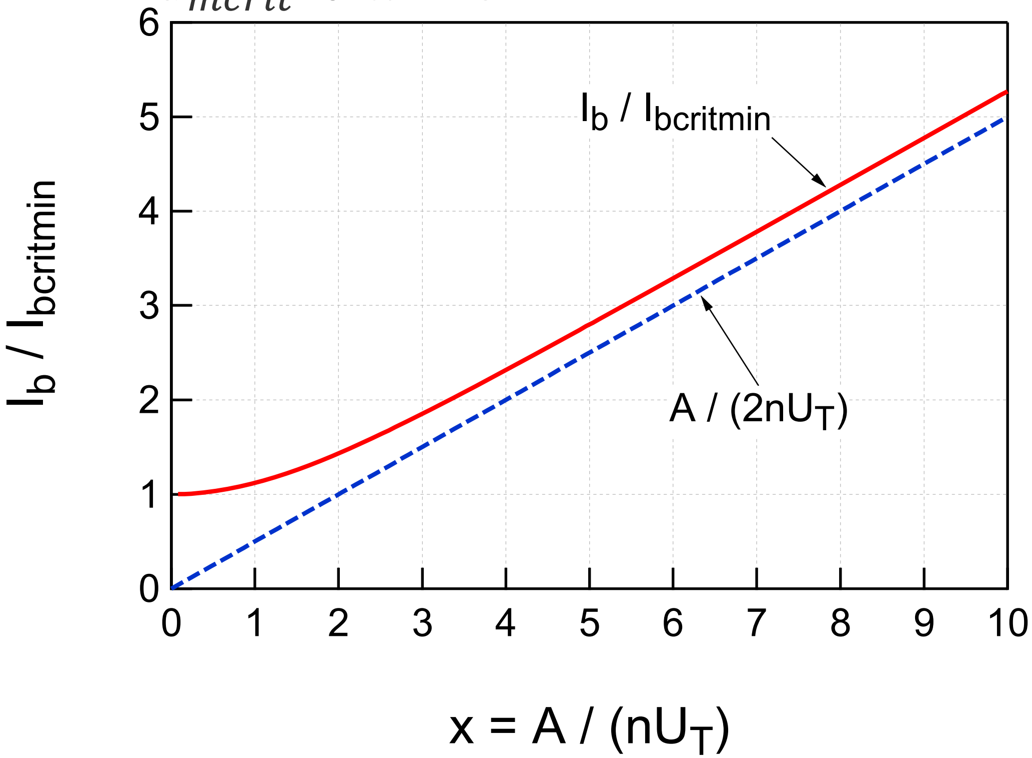
- The above plot shows the transconductance for the fundamental normalized to the small-signal transconductance versus the normalized oscillation amplitude
- The amplitude will stabilize for $G_{m(1)} = G_{mcrit}$ which is the condition that finally determines the oscillation amplitude

Bias Current versus Amplitude

- In weak inversion the condition $G_{m(1)} = G_{mcrit}$ translates into

$$G_{m(1)} = G_{mcrit} \quad \rightarrow \quad \frac{I_b}{nU_T} \cdot \frac{2\mathrm{I}_{B1}(x)}{x \cdot \mathrm{I}_{B0}(x)} = \frac{I_{bcritmin}}{nU_T} \quad \rightarrow \quad \frac{I_b}{I_{bcritmin}} = \frac{x \cdot \mathrm{I}_{B0}(x)}{2\mathrm{I}_{B1}(x)}$$

- $I_{bcritmin} \triangleq nU_T \cdot G_{mcrit}$ is the minimum current (reached in WI) to achieve G_{mcrit} for $x = 0$



- Since for $x \gg 1$

$$\mathrm{I}_{Bn}(x) \cong \frac{e^x}{\sqrt{2\pi x}} \quad \text{for} \quad x \gg 1$$

- we have

$$\frac{I_b}{I_{bcritmin}} = \frac{x \cdot \mathrm{I}_{B0}(x)}{2\mathrm{I}_{B1}(x)} \cong \frac{x}{2} \quad \text{for} \quad x \gg 1$$

- or

$$I_b \cong I_{bcritmin} \cdot \frac{A}{2nU_T} = \frac{G_{mcrit}}{2} \cdot A$$

for $A \gg nU_T$

DC Gate Voltage Bias Shift

- In case the bias current is set to the quiescent current $I_b = I_0$, by definition of I_0, the oscillation amplitude is zero ($x = 0$)

$$x = 0 \quad \rightarrow \quad I_b = I_0 \cdot \mathrm{I}_{B0}(x=0) = I_0 = I_{D0} \cdot e^{\frac{V_{G0}}{nU_T}} = I_{spec} \cdot e^{\frac{V_{G0}-V_{T0}}{nU_T}}$$

- For $I_b > I_0$, oscillations will start to grow until the condition $G_{m(1)} = G_{mcrit}$ is reached, at which the oscillations will stabilize with an amplitude set by $I_b / I_{bcritmin}$
- As shown in the previous plot, the dc drain current would increase wrt x, but it is actually constant and set to I_b by the current source. Since the current cannot grow when the oscillations are growing, the dc gate voltage has to adjust so that $I_{dc} = I_b$
- V_{G0} and I_0 therefore decrease compared to the condition $V_{G0} = V_{Gcrit}$ and $I_0 = I_b = I_{bcritmin}$ for which $x = 0$
- The quiescent voltage V_{G0} and the quiescent current I_0 are therefore indirectly also functions of the oscillation amplitude and hence of the I_b / I_{bcrit} ratio

DC Gate Voltage Bias Shift

- For a given bias current I_b and minimum critical bias current $I_{bcritmin}$, the relation between the quiescent current I_0 and the oscillation amplitude x can be found from the oscillation condition

$$G_{m(1)} = G_{mcrit} \quad \rightarrow \quad \frac{I_b}{I_{bcritmin}} = \frac{I_0 \cdot \mathrm{I}_{B0}(x)}{I_{bcritmin}} = \frac{x \cdot \mathrm{I}_{B0}(x)}{2\mathrm{I}_{B1}(x)} \quad \rightarrow \quad \frac{I_0}{I_{bcritmin}} = \frac{x}{2\mathrm{I}_{B1}(x)}$$

- Introducing the definition of the quiescent current I_0, we get

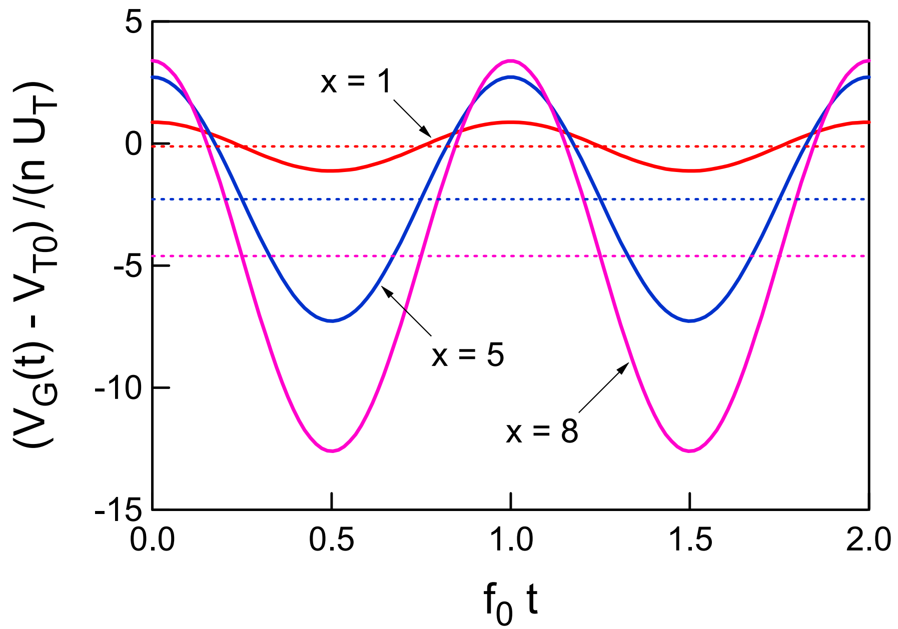
$$\frac{I_0}{I_{bcritmin}} = \frac{I_{spec}}{I_{bcritmin}} \cdot e^{\frac{V_{G0}-V_{T0}}{nU_T}} \quad \rightarrow \quad e^{\frac{V_{G0}-V_{T0}}{nU_T}} = \frac{I_{bcritmin}}{I_{spec}} \cdot \frac{x}{2\mathrm{I}_{B1}(x)}$$

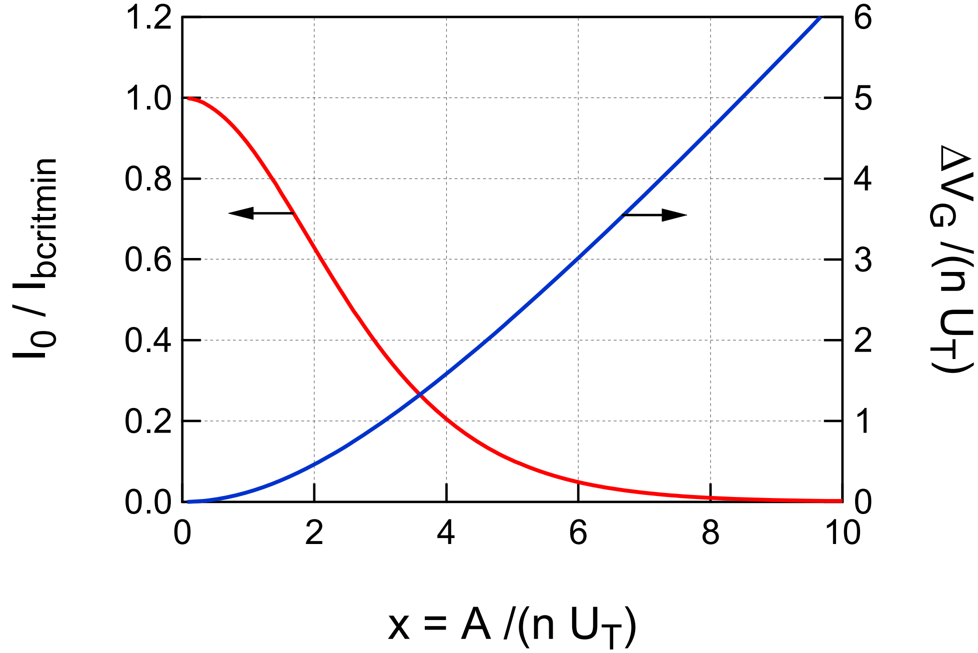
- We see that for a given $I_{bcritmin}$ and bias current I_b, as the amplitude grows, at the same time the overdrive voltage decreases according to

$$\frac{V_{G0} - V_{T0}}{nU_T} = \ln\left(\frac{I_0}{I_{spec}}\right) = \underbrace{\ln\left(\frac{I_{bcritmin}}{I_{spec}}\right)}_{\cong \frac{V_{Gcrit}-V_{T0}}{nU_T}} - \underbrace{\ln\left(\frac{2\mathrm{I}_{B1}(x)}{x}\right)}_{\cong \frac{\Delta V_G(x)}{nU_T}} = \frac{V_{Gcritmin} - V_{T0}}{nU_T} - \frac{\Delta V_G(x)}{nU_T}$$

where $V_{Gcritmin}$ is the gate voltage for a bias current $I_b = I_{bcritmin}$, i.e. $x = 0$

DC Gate Bias Voltage Shift





- As mentioned earlier, the gate bias has to decrease when the oscillations are growing to maintain the dc drain current equal to the bias current

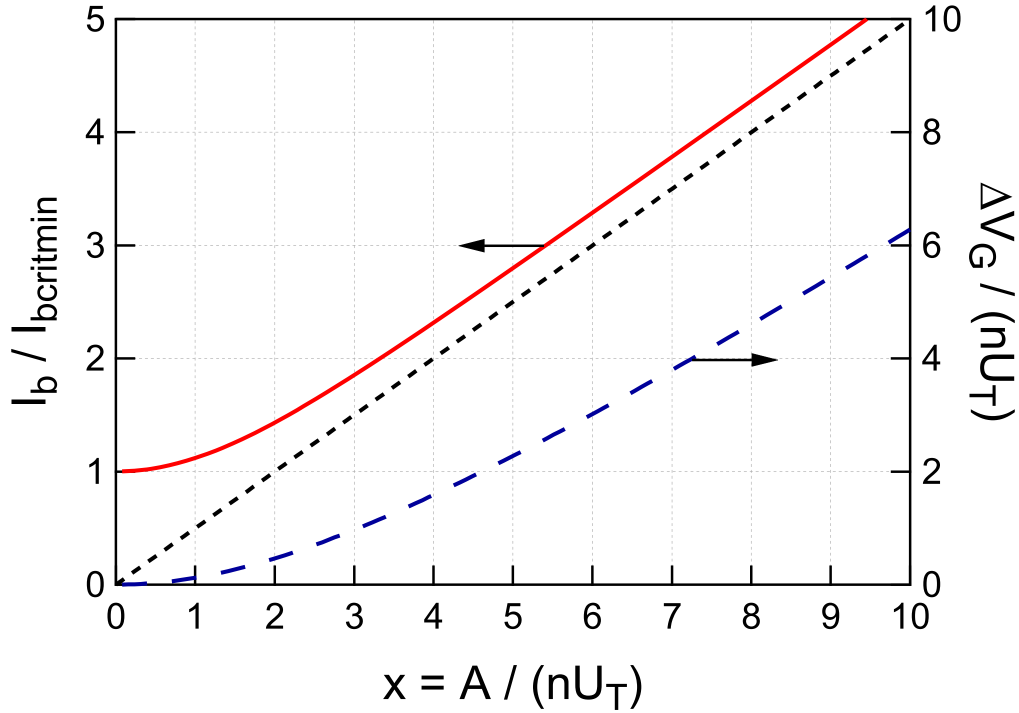
$$\frac{V_G(t)-V_{T0}}{nU_T}=\frac{V_{G0}(x)-V_{T0}}{nU_T}+x\cdot\cos(\omega_0\cdot t)$$

with $$\frac{V_{G0}(x)-V_{T0}}{nU_T}=\ln\left(\frac{I_{bcritmin}}{I_{spec}}\right)-\ln\left(\frac{2\mathrm{I}_{B1}(x)}{x}\right)$$

$$\frac{V_{G0}-V_{T0}}{nU_T}=\frac{V_{Gcritmin}-V_{T0}}{nU_T}-\frac{\Delta V_G(x)}{nU_T}$$

with $$\frac{\Delta V_G(x)}{nU_T}\cong\ln\left(\frac{2\mathrm{I}_{B1}(x)}{x}\right)$$

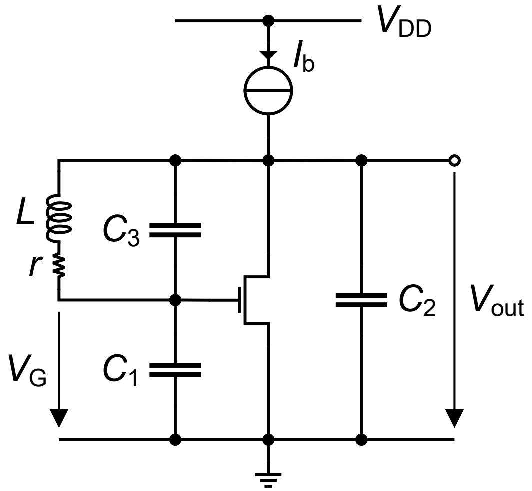
Amplitude and Gate Voltage Bias Shift vs Bias Current



- For a given resonator and hence a given $I_{bcritmin}$, this plot shows the bias current I_b that is required for achieving a given amplitude A and the resulting gate bias shift decrease

$$\frac{I_b}{I_{bcritmin}} = \frac{x \cdot \mathrm{I}_{B0}(x)}{2\mathrm{I}_{B1}(x)} \quad \text{and} \quad \frac{\Delta V_G}{nU_T} = \ln\left(\frac{2\mathrm{I}_{B1}(x)}{x}\right)$$

Example – The Pierce Oscillator



$$f_0 = 1\,GHz, Q_L = 10, C_1 = C_2 = 1\,pF, C_3 = 1\,pF$$

$$\omega_0 \cong \frac{1}{\sqrt{L(C_3 + C_{12})}} \Rightarrow L \cong \frac{1}{\omega_0^2 (C_3 + C_{12})} = 16.9\,nH$$

$$r = \frac{\omega_0 L}{Q_L} = 10.6\Omega$$

$$G_{mcrit} \cong \frac{\omega_0}{Q_L}(C_1 + C_2)\left(1 + \frac{C_3}{C_{12}}\right) = 3.8\frac{mA}{V}$$

- Since the inductance Q_L is not very high, the above approximation is not very accurate. The exact solution is then given by

$$G_{mcrit} = \omega_0 C_2 \frac{\alpha_1 Q_L}{2\alpha_3}\left[1 - \sqrt{1 - \left(\frac{2\alpha_3}{\alpha_1 Q_L}\right)^2 (\alpha_1 + 1)\left(1 + \alpha_1 + \frac{\alpha_1}{\alpha_3}\right)}\right] = 4\frac{mA}{V}$$

- The inductance value is then found from

$$L = \frac{X_c(\omega_0, G_{mcrit})}{\omega_0}$$

- This leads to $L = 17.256\,nH$ and $r = 10.8\,\Omega$

Pierce Oscillator Example – Bias Current in WI

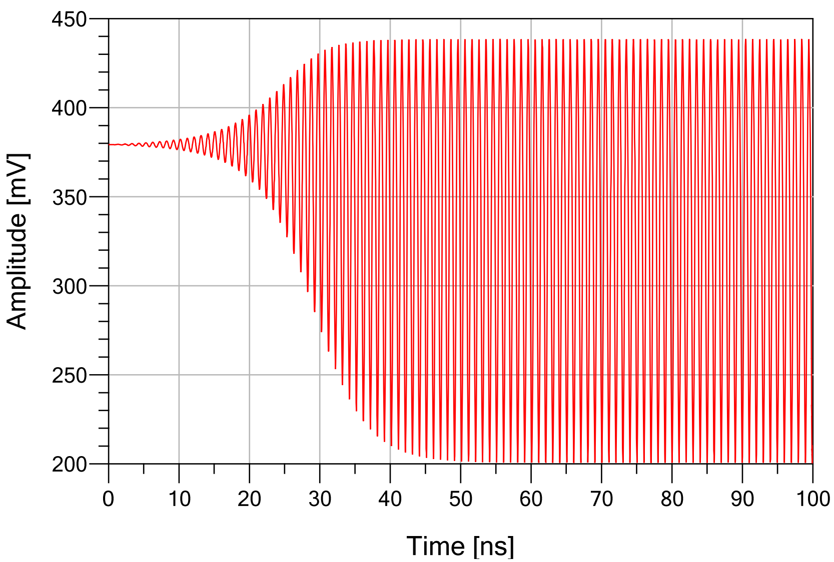
- If we assume that the transistor operates in weak inversion (with $n = 1.3$), the critical current is given by

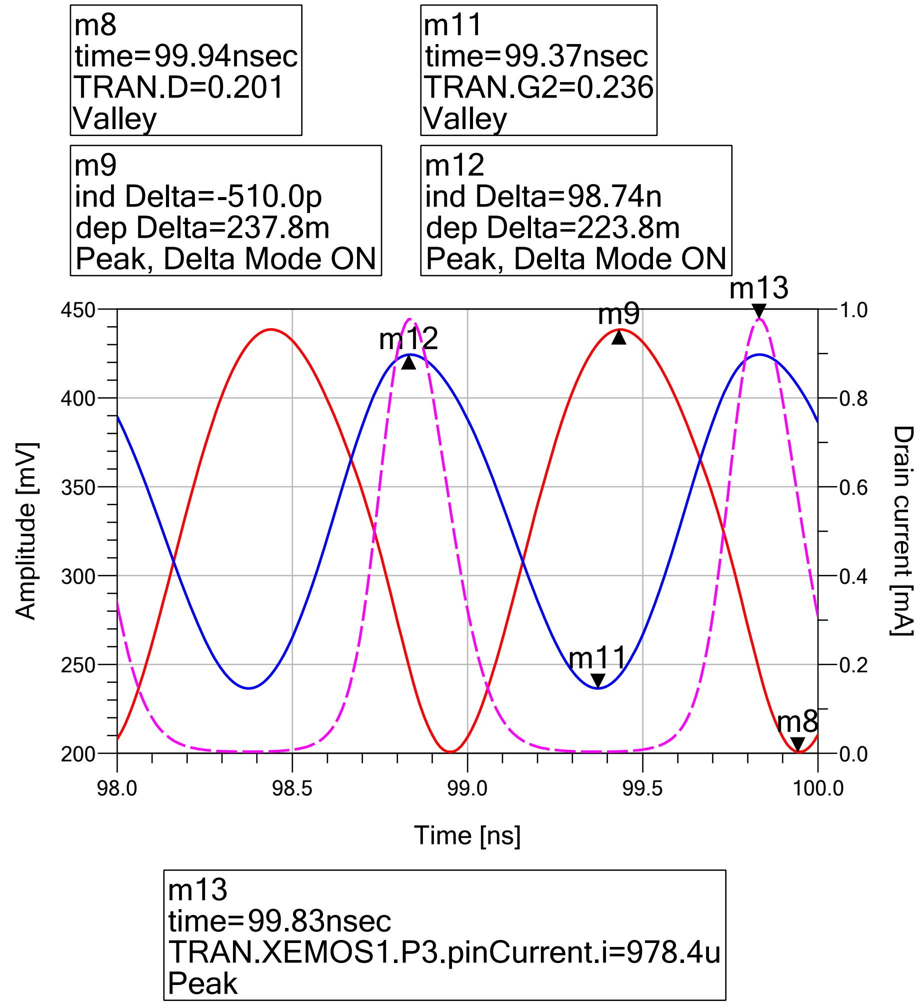
$$I_{bcritmin} = G_{mcrit} \cdot nU_T \cong 132\,\mu A$$

- Setting the oscillation amplitude to $A = 100\ mV$, we get

$$x = \frac{A}{nU_T} = 3 \quad \Rightarrow \quad \frac{x \cdot \mathrm{I}_{B0}(x)}{2\mathrm{I}_{B1}(x)} = 1.87 \quad \Rightarrow \quad I_b = I_{bcritmin} \cdot 1.87 = 247.7\,\mu A$$

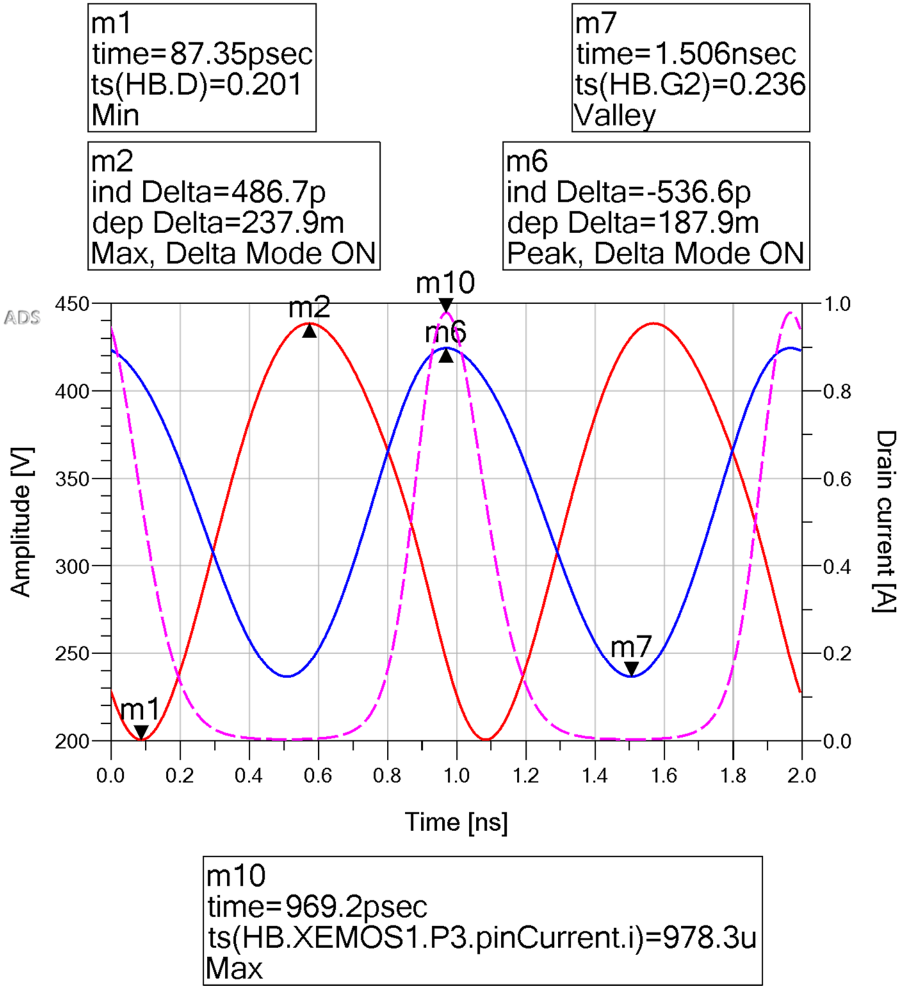
Pierce Oscillator Example in WI – Transient Simulations

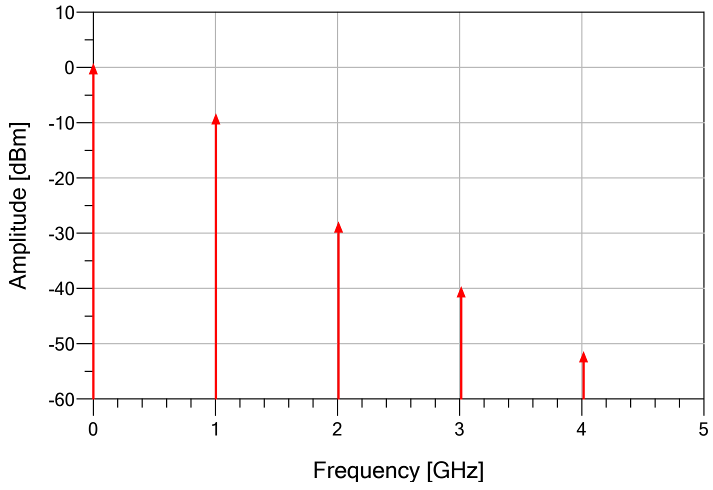




- Transient simulations performed with an ideal exponential transconductor
- The amplitude is slightly larger than 100mV (119 mV). This comes from the fact that Q_L is not that large generating harmonics which is in contradiction with the assumption of a sinusoidal gate voltage
- The above theory is based on the fundamental component only assuming a large Q and hence that the harmonics are negligible

Pierce Oscillator Example in WI – HB Simulations

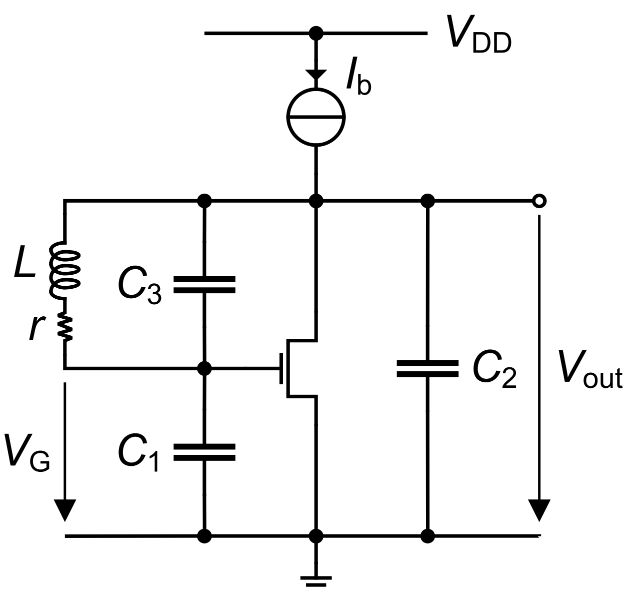




- Harmonic balance SS simulations performed with an ideal exponential transconductor
- Consistent with transient simulations
- The amplitude is slightly larger than 100mV (119 mV)

Pierce Oscillator in Strong Inversion

- The same analysis can be carried out assuming the transistor is operating in **strong inversion**
- It can be handled analytically as long as the oscillation amplitude A is assumed to be smaller than the overdrive voltage $V_G - V_{T0}$ in order for the current to remain positive **avoiding any current clipping**
- In this case the gate voltage and the drain current are given by



$$V_G(t) = V_{G0} + A \cdot \cos(\omega_0 t)$$

$$I_D(t) = I_{spec} \cdot \left(\frac{V_G(t) - V_{T0}}{2nU_T} \right)^2 = I_{spec} \cdot \left(\frac{V_{G0} - V_{T0} + A \cdot \cos(\omega_0 t)}{2nU_T} \right)^2$$

$$= \frac{I_{spec}}{4} \cdot \left(v_{gt0} + x \cdot \cos(\omega_0 t) \right)^2 \quad \text{for} \quad x \le v_{gt0}$$

with $\quad I_{spec} \triangleq 2n\beta U_T^2, \quad v_{gt0} \triangleq \dfrac{V_{G0} - V_{T0}}{nU_T}, \quad x \triangleq \dfrac{A}{nU_T}$

Pierce Oscillator in Strong Inversion

- Normalizing and developing the quadratic function leads to

$$i_d(t) = \frac{I_D(t)}{I_{spec}} = \frac{1}{4}\cdot\left(v_{gt0} + x\cdot\cos(\omega_0 t)\right)^2 = \frac{1}{4}\cdot\left(v_{gt0}^2 + 2v_{gt0}\cdot x\cdot\cos(\omega_0 t) + x^2\cdot\cos^2(\omega_0 t)\right)$$

$$= \underbrace{\frac{1}{4}\cdot\left(v_{gt0}^2 + \frac{x^2}{2}\right)}_{i_{dc}} + \underbrace{\frac{v_{gt0}\cdot x}{2}}_{i_{d(1)}}\cdot\cos(\omega_0 t) + \frac{x^2}{8}\cdot\cos(2\omega_0 t)$$

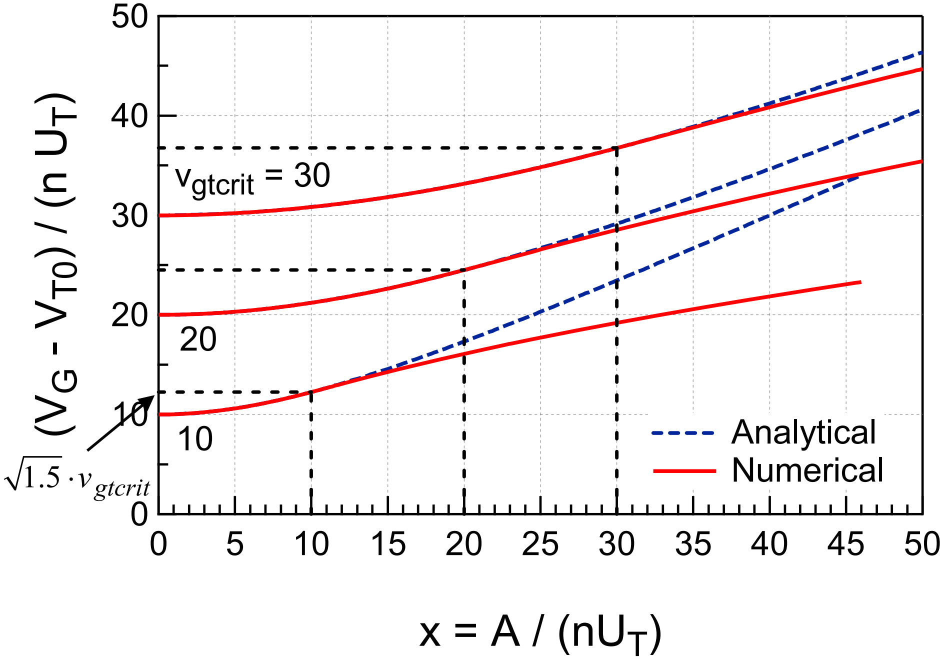
- The normalized **dc current** for $x < v_{gt0}$ is given by

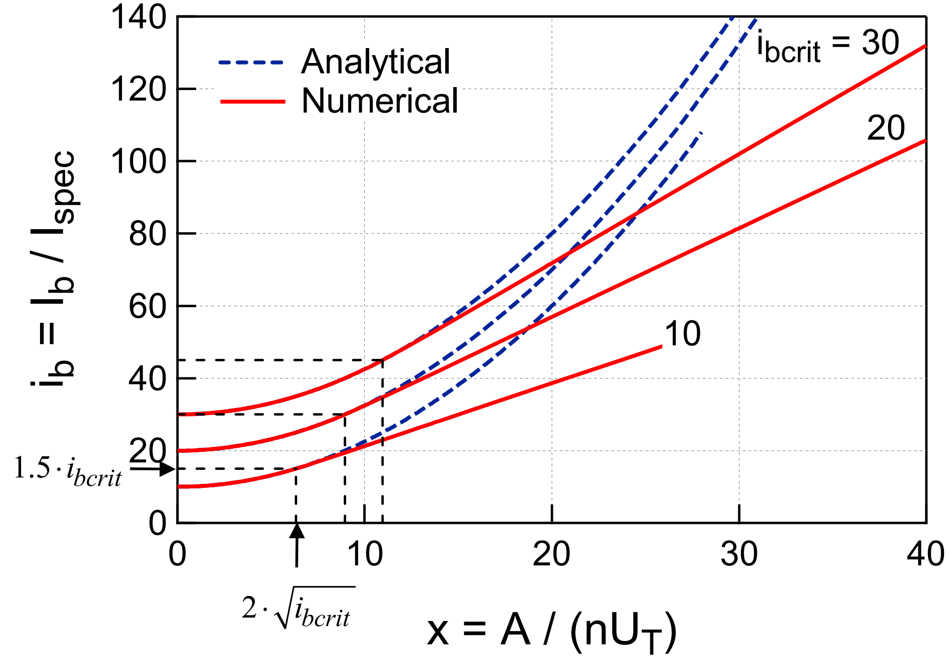
$$i_{dc} \triangleq \frac{I_{dc}}{I_{spec}} = \frac{I_b}{I_{spec}} = \frac{1}{4}\cdot\left(v_{gt0}^2 + \frac{x^2}{2}\right) = i_0 + \frac{x^2}{8} \quad \text{with} \quad i_0 \triangleq \frac{I_0}{I_{spec}} = \left(\frac{v_{gt0}}{2}\right)^2$$

- Where i_0 is the dc current for zero amplitude

$$i_0 \triangleq \frac{I_0}{I_{spec}} = \left(\frac{v_{gt0}}{2}\right)^2 \quad \text{with} \quad I_0 \triangleq I_D\big|_{x=0} = I_{spec}\cdot\left(\frac{V_{G0} - V_{T0}}{2nU_T}\right)^2 = I_{spec}\cdot\left(\frac{v_{gt0}}{2}\right)^2$$

Bias Voltage and Current





- The required normalized bias overdrive voltage v_{gt} (normalized bias current i_b or inversion factor) for a given critical overdrive voltage v_{gtcrit} (critical current i_{bcrit}) assuming $x < v_{gtcrit}$ is given by

$$v_{gt} \triangleq \frac{V_G - V_{T0}}{nU_T} = \sqrt{v_{gtcrit}^2 + \frac{x^2}{2}} \quad \text{or} \quad i_b \triangleq \frac{I_b}{I_{spec}} = \left(\frac{v_{gt}}{2}\right)^2 = i_{bcrit} + \frac{x^2}{8}$$

Transconductance for the Fundamental

- The fundamental component for $x < v_{gt0}$ is given by

$$I_{D(1)} = I_{spec} \cdot \frac{v_{gt0}}{2} \cdot x \quad \text{for} \quad x \le v_{gt0}$$

- The transconductance for the fundamental component is then given by

$$G_{m(1)} = \frac{I_{D(1)}}{A} = \frac{I_{spec}}{A} \cdot \frac{v_{gt0}}{2} \cdot x = \frac{I_{spec}}{nU_T} \cdot \frac{v_{gt0}}{2} = \frac{I_{spec}}{nU_T} \cdot \sqrt{i_0} \quad \text{for} \quad x \le v_{gt0}$$

- From the dc constraint, we get the relation between i_0 and v_{gt0} which should both decrease with the amplitude x

$$i_b = \left(\frac{v_{gt}}{2}\right)^2 = \left(\frac{v_{gt0}}{2}\right)^2 + \frac{x^2}{8} = i_0 + \frac{x^2}{8} \quad \rightarrow \quad i_0 = i_b - \frac{x^2}{8} \quad \text{or} \quad v_{gt0} = \sqrt{v_{gt}^2 - \frac{x^2}{2}}$$

- The transconductance for the fundamental then becomes

$$G_{m(1)} = \frac{I_{spec}}{nU_T} \cdot \sqrt{i_b - \frac{x^2}{8}} = \frac{I_{spec}}{nU_T} \cdot \sqrt{\left(\frac{v_{gt}}{2}\right)^2 - \frac{x^2}{8}} = \frac{I_{spec}}{nU_T} \cdot \frac{1}{2} \cdot \sqrt{v_{gt}^2 - \frac{x^2}{2}}$$

Transconductance for the Fundamental

- The small-signal transconductance is given by

$$G_m = \frac{I_{spec}}{nU_T} \cdot \frac{v_{gt}}{2} = \frac{I_{spec}}{nU_T} \cdot \sqrt{i_b} \quad \text{with} \quad v_{gt} \cong \frac{V_G - V_{T0}}{nU_T} \quad \text{and} \quad i_b = \left(\frac{v_{gt}}{2}\right)^2$$

- The transconductance for the fundamental normalized to the small-signal transconductance is then given by

$$\frac{G_{m(1)}}{G_m} = \sqrt{1 - \frac{x^2}{8i_b}} = \sqrt{1 - \frac{1}{2}\left(\frac{x}{v_{gt}}\right)^2} = \sqrt{1 - \frac{1}{2}\left(\frac{A}{V_G - V_{T0}}\right)^2} \quad \text{for} \quad x \le v_{gt}$$

- The critical condition is then given by

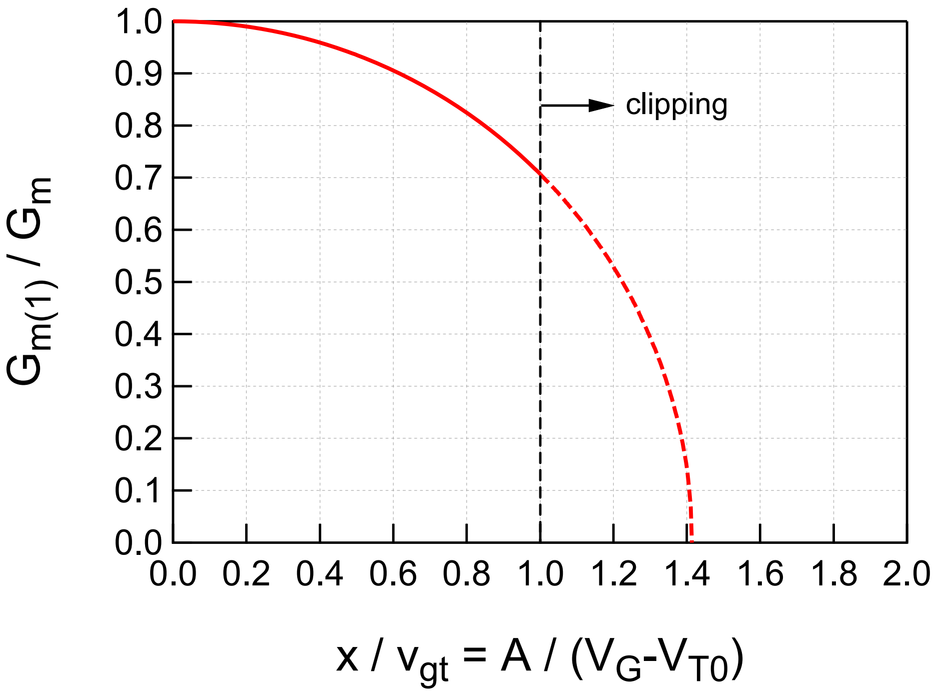
$$G_{m(1)} = G_{mcrit} \quad \rightarrow \quad \frac{I_{spec}}{nU_T} \cdot \frac{1}{2} \cdot \sqrt{v_{gt}^2 - \frac{x^2}{2}} = \frac{I_{spec}}{nU_T} \cdot \frac{v_{gtcrit}}{2} \quad \rightarrow \quad \sqrt{v_{gt}^2 - \frac{x^2}{2}} = v_{gtcrit}$$

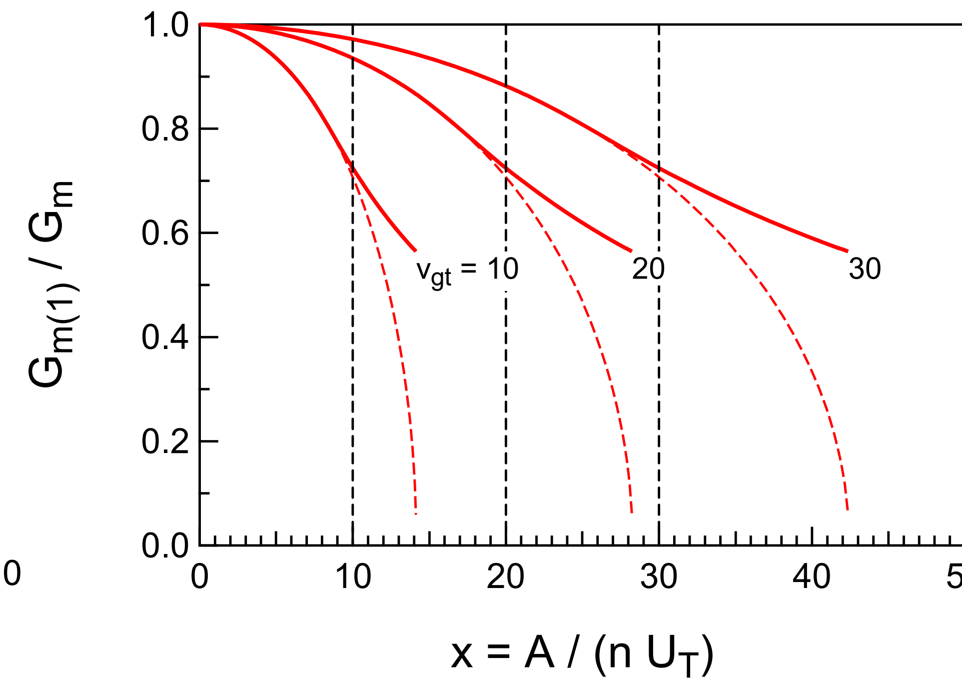
or in terms of currents $\quad \frac{I_{spec}}{nU_T} \cdot \sqrt{i_b - \frac{x^2}{8}} = \frac{I_{spec}}{nU_T} \cdot \sqrt{i_{bcrit}} \quad \rightarrow \quad i_b - \frac{x^2}{8} = i_{bcrit}$

- Since there is no current clipping and no bias shift, it is not surprising to find that

$$v_{gtcrit} = v_{gt0} = \sqrt{v_{gt}^2 - \frac{x^2}{2}} \quad \text{and} \quad i_{bcrit} = i_0 = i_b - \frac{x^2}{8}$$

Transconductance for the Fundamental





$$\frac{G_{m(1)}}{G_m} = \sqrt{1 - \frac{x^2}{8 i_b}} = \sqrt{1 - \frac{1}{2}\left(\frac{x}{v_{gt}}\right)^2} = \sqrt{1 - \frac{1}{2}\left(\frac{A}{V_G - V_{T0}}\right)^2} \quad \text{for} \quad x \le v_{gt}$$

with $v_{gt} \triangleq \dfrac{V_G - V_{T0}}{n U_T}$ and $x \triangleq \dfrac{A}{n U_T}$

Pierce Oscillator Example – Bias in SI

- If we assume the transistor operates in strong inversion (with $n = 1.3$) and choose the operating overdrive voltage as $V_G - V_{T0} = 300\ mV$ and the oscillation amplitude as $A = 200\ mV$, we can then calculate the normalized amplitude as

$$v_{gt} \triangleq \frac{V_G - V_{T0}}{nU_T} = 9.12 \quad i_b \triangleq \frac{I_b}{I_{spec}} = \left(\frac{v_{gt}}{2}\right)^2 = 20.8 \quad x \triangleq \frac{A}{nU_T} = 6.08$$

- The critical overdrive and critical current can then be calculated as

$$v_{gtcrit} = 2\sqrt{i_{bcrit}} = \sqrt{v_{gt}^2 - \frac{x^2}{2}} = 8.045 \quad \text{or} \quad i_{bcrit} = \left(\frac{v_{gtcrit}}{2}\right)^2 = i_b - \frac{x^2}{8} = 16.18$$

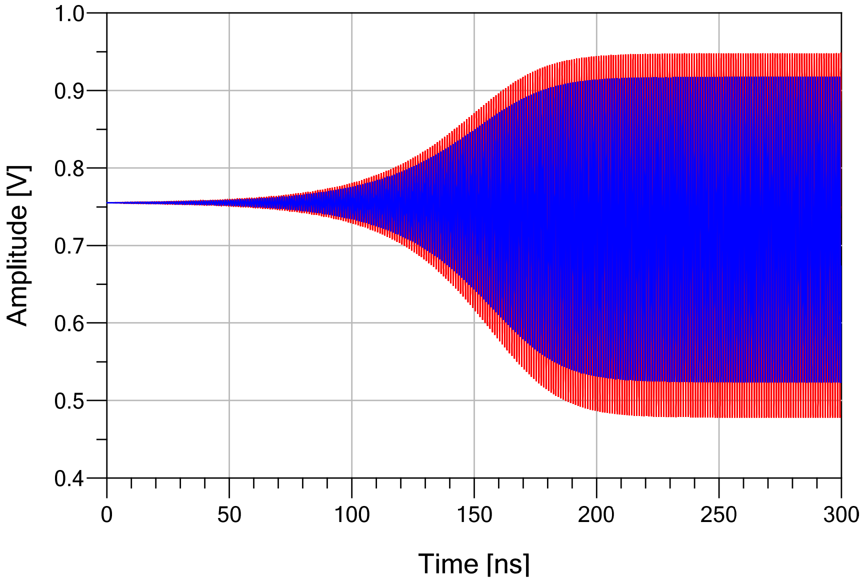
- The specific current is then obtained as

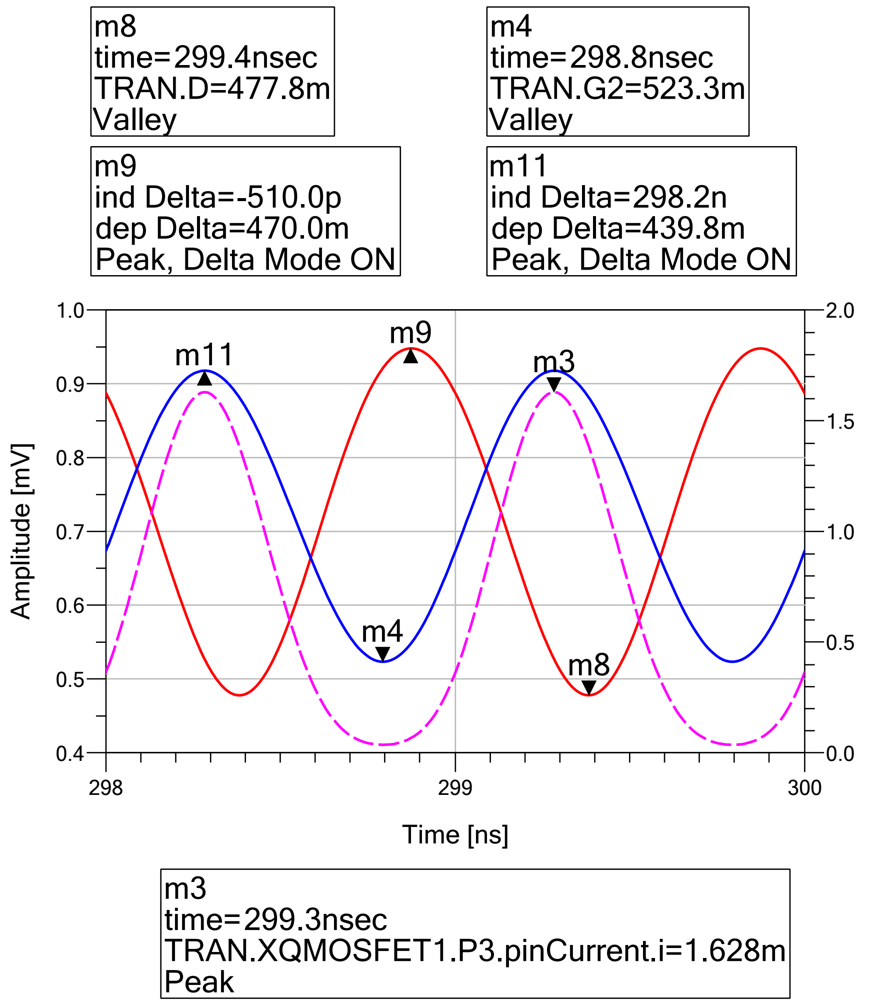
$$I_{spec} = \frac{2nU_T \cdot G_{mcrit}}{v_{gtcrit}} = \frac{nU_T \cdot G_{mcrit}}{\sqrt{i_{bcrit}}} = 32.9\ \mu A \quad \rightarrow \quad \frac{\beta}{2n} = \frac{I_b}{(V_G - V_{T0})^2}$$

- The critical current and the required bias current are given by

$$I_{crit} = I_{spec} \cdot i_{bcrit} = 532.875\ \mu A \quad I_b = I_{spec} \cdot i_b = I_{spec} \cdot \left(\frac{v_{gt}}{2}\right)^2 = 685.125\ \mu A$$

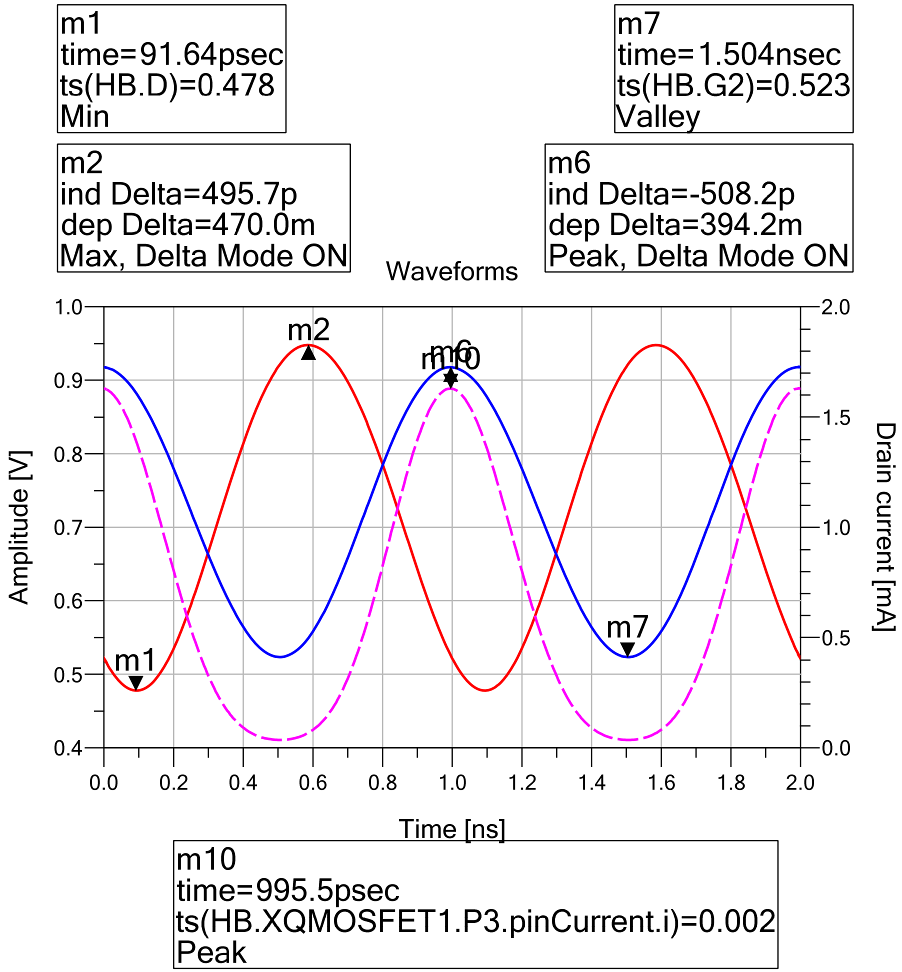
Pierce Oscillator Example in SI – Transient Simulations

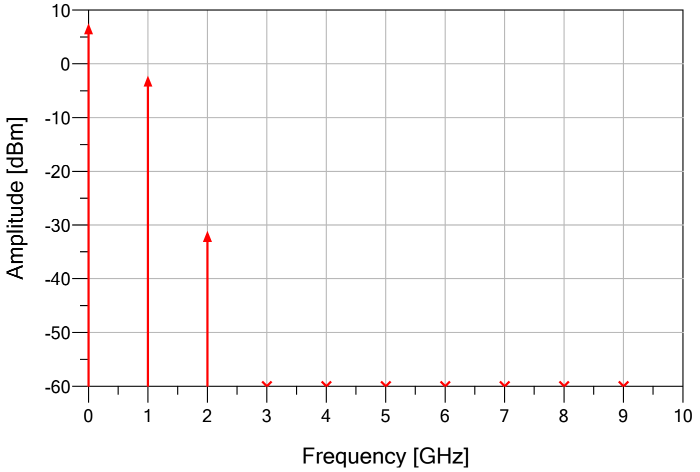




- Transient simulations performed with an ideal quadratic transconductor
- The amplitude at the drain (235 mV) and at the gate (220 mV) are slightly higher than 200mV
- The above theory is based on the fundamental component only assuming a large Q and hence that the harmonics are negligible

Pierce Oscillator Example in SI – HB Simulations





- Harmonic balance SS simulations performed with an ideal quadratic transconductor
- The amplitude at the gate is exactly 200mV, whereas the amplitude at the drain is slightly larger (235mV)

Design Procedure (Strong Inversion)

- From the Q of the tank and the capacitances, deduce the required G_{mcrit}
- Choose an appropriate critical overdrive voltage $V_{Gcrit} - V_{T0}$ (normalized form v_{gtcrit}) or critical inversion factor $i_{bcrit} = \left(v_{gtcrit}/2\right)^2$ which correspond to the average overdrive and bias current at the operating point
- We can now find the specific current according to

$$I_{spec} = \frac{2nU_T \cdot G_{mcrit}}{v_{gtcrit}} = \frac{nU_T \cdot G_{mcrit}}{\sqrt{i_{bcrit}}}$$

- Calculate the desired normalized amplitude $x = A/(nU_T)$
- For the chosen normalized critical overdrive voltage (or inversion factor) and normalized amplitude, deduce the normalized overdrive voltage or bias current from

$$v_{gt} \triangleq \frac{V_G - V_{T0}}{nU_T} = \sqrt{v_{gtcrit}^2 + \frac{x^2}{2}} \quad \text{or} \quad i_b \triangleq \frac{I_b}{I_{spec}} = \left(\frac{v_{gt}}{2}\right)^2 = i_{bcrit} + \frac{x^2}{8}$$

- The bias current I_b required for the desired amplitude is then given by

$$I_b = I_{spec} \cdot i_b = I_{spec} \cdot \left(\frac{v_{gt}}{2}\right)^2$$

Design Procedure (from Weak to Strong Inversion)

- From the Q of the tank and the capacitances, deduce the required G_{mcrit}
- Choose an appropriate inversion factor i_{bcrit}
- Calculate the minimum critical bias current $I_{bcritmin}$

$$I_{bcritmin} = n \cdot U_T \cdot G_{mcrit}$$

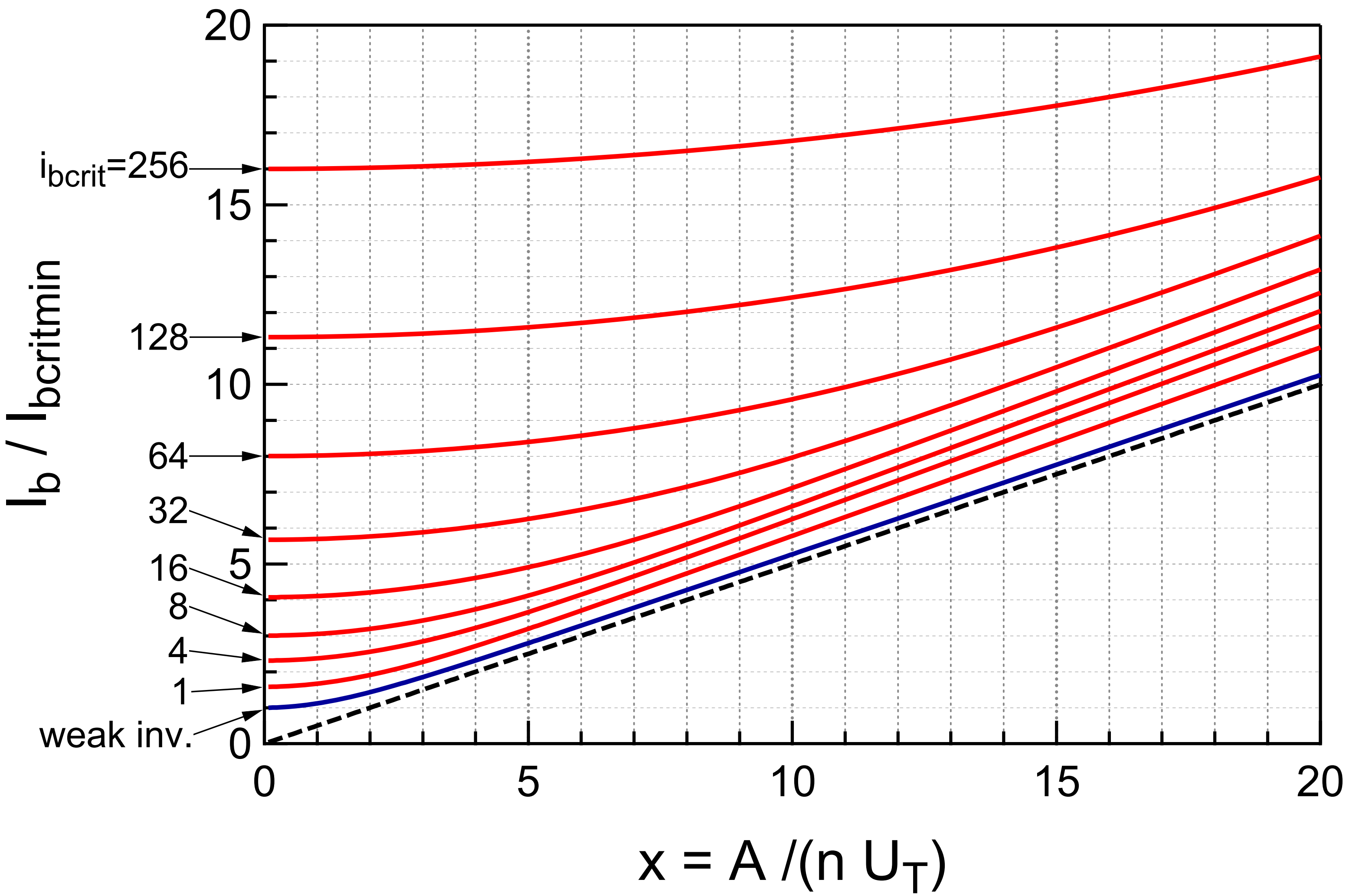
- Calculate the corresponding specific current

$$I_{spec} = \frac{I_{bcritmin}}{\sqrt{i_{bcrit}} \cdot \left(1 - \exp\left[-\sqrt{i_{bcrit}}\right]\right)}$$

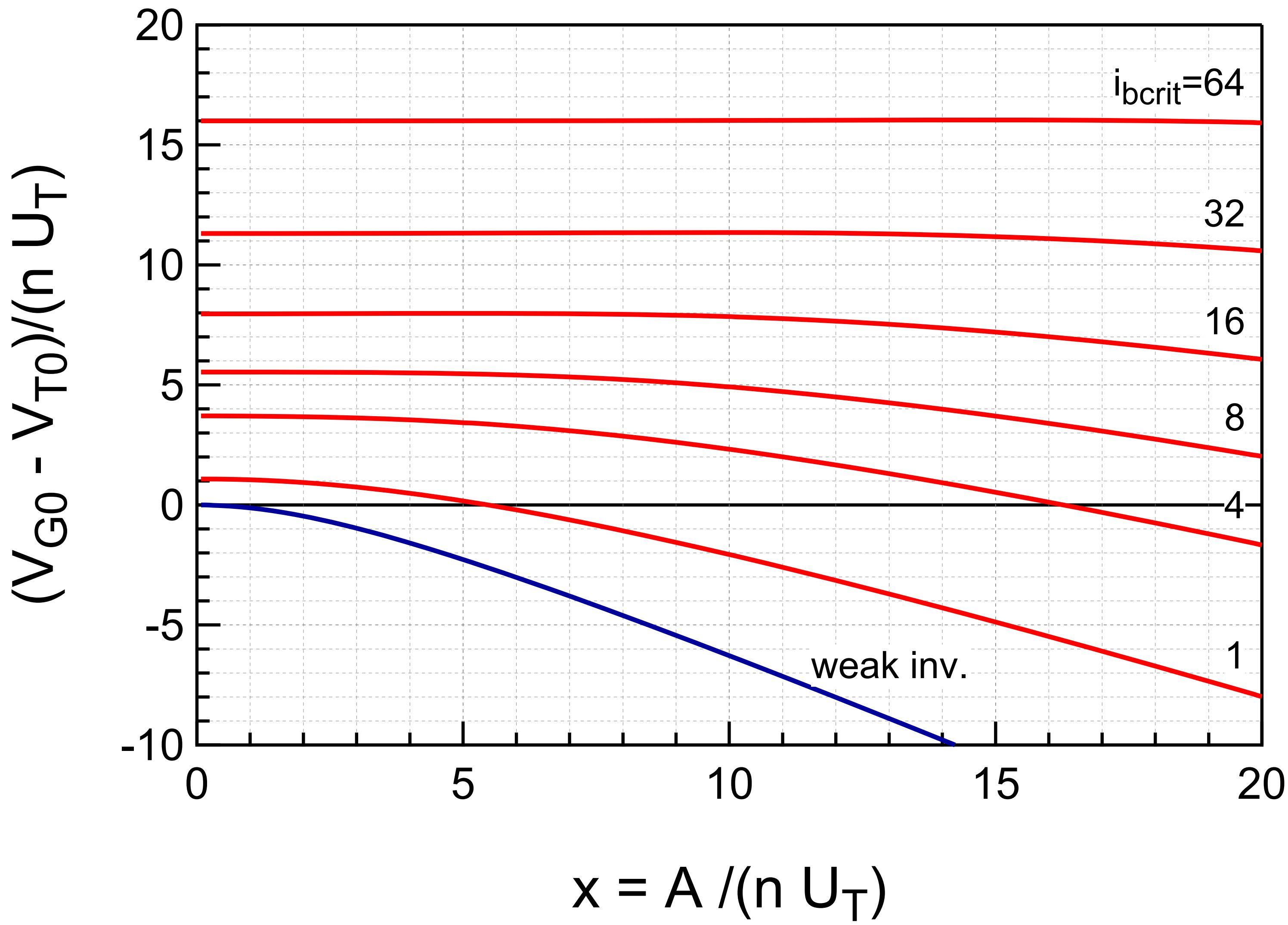
- Calculate the desired normalized amplitude $x = A/(nU_T)$
- For the chosen inversion factor i_{bcrit} and normalized amplitude x, deduce the normalized bias current $I_b/I_{bcritmin}$ from the abacus (next slide)
- Deduce the actual bias current

$$I_b = I_{bcritmin} \cdot \frac{I_b}{I_{bcritmin}}$$

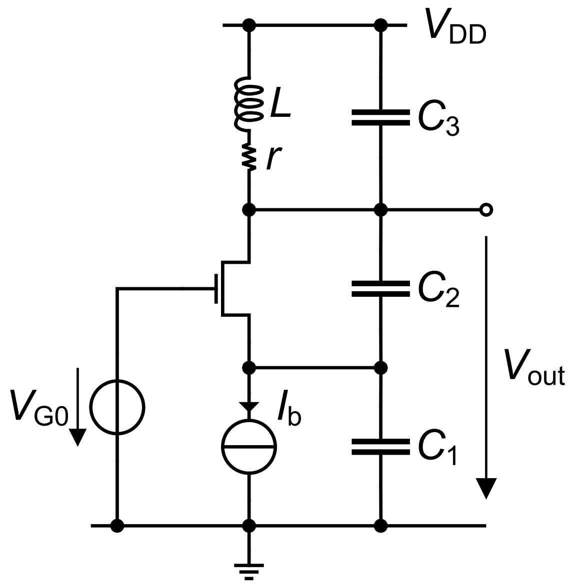
Bias Current from Weak to Strong Inversion

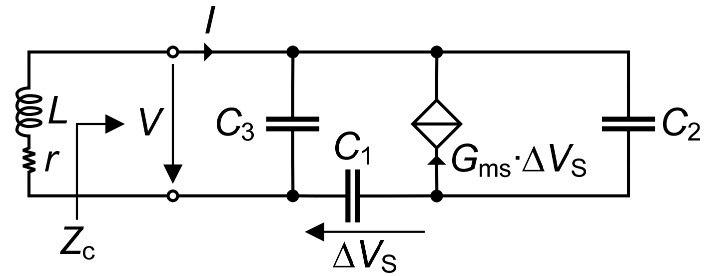


Bias Shift from Weak to Strong Inversion



The Colpitts Oscillator – Circuit Impedance





$$Z_c = -\frac{G_{ms} + j\omega(C_1 + C_2)}{\omega^2 (C_1 C_2 + C_1 C_3 + C_2 C_3) - j\omega G_{ms} C_3}$$

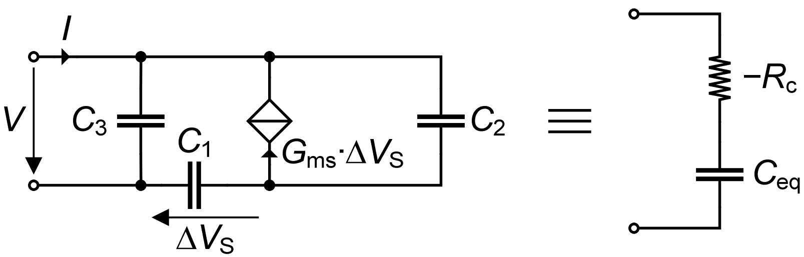
- Analysis almost identical to the Pierce except that G_m is replaced with G_{ms}

$$R_c = \frac{G_{ms} C_1 C_2}{(G_{ms} C_3)^2 + \omega^2 (C_1 C_2 + C_1 C_3 + C_2 C_3)^2}$$

$$X_c = \frac{G_{ms}^2 C_3 + \omega^2 (C_1 + C_2)(C_1 C_2 + C_1 C_3 + C_2 C_3)}{\omega\left[(G_{ms} C_3)^2 + \omega^2 (C_1 C_2 + C_1 C_3 + C_2 C_3)^2\right]}$$

The Colpitts Oscillator – Critical Transconductance

- For $G_{ms} \ll (\omega_0/C_3)(C_1C_2 + C_1C_3 + C_2C_3)$, R_c and X_c simplify to



$$R_c \cong \frac{G_{ms}C_1C_2}{\omega^2\left(C_1C_2 + C_1C_3 + C_2C_3\right)^2} \qquad X_c \cong \frac{1}{\omega C_{eq}}$$

with $\quad C_{eq} = C_3 + C_{12} = \dfrac{C_1C_2 + C_1C_3 + C_2C_3}{C_1 + C_2}$

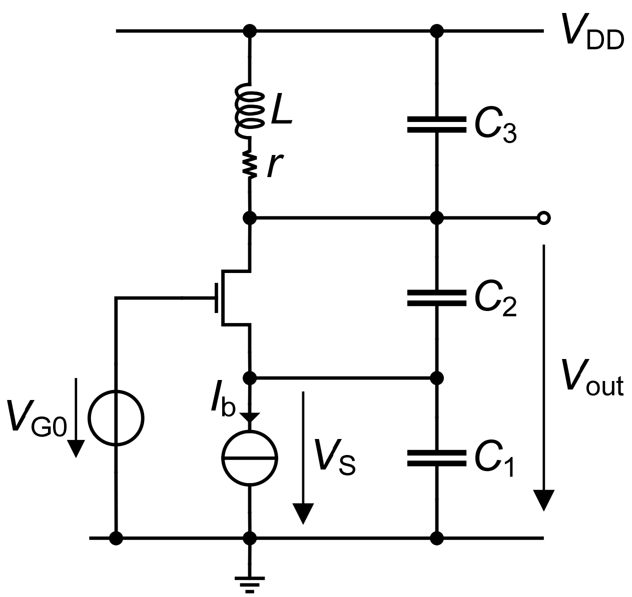
- The oscillation frequency is approximated by $\quad \omega_0 \cong \dfrac{1}{\sqrt{L \cdot C_{eq}}}$
- And the critical (source) transconductance is given by

$$G_{mscrit} = \omega_0 C_2 \frac{\alpha_1 Q_L}{2\alpha_3}\left[1 - \sqrt{1 - \left(\frac{2\alpha_3}{\alpha_1 Q_L}\right)^2 (\alpha_1 + 1)\left(1 + \alpha_1 + \frac{\alpha_1}{\alpha_3}\right)}\right]$$

$$\cong \frac{1}{r}\frac{\alpha_1}{2\alpha_3^2}\left[1 - \sqrt{1 - \left(\frac{2\alpha_3(\alpha_1 + 1)}{\alpha_1 Q_L}\right)^2}\right] \cong \frac{1}{r}\frac{(\alpha_1 + 1)^2}{\alpha_1 Q_L} = \frac{\omega_0}{Q_L}(C_1 + C_2)\left(1 + \frac{C_3}{C_{12}}\right)$$

where $\quad \alpha_1 \triangleq \dfrac{C_1}{C_2} \quad \alpha_3 \triangleq \dfrac{C_3}{C_2}$

Nonlinear Analysis of the Collpits Oscillator (weak inv.)



$$A = \Delta V_S = \frac{C_2}{C_1 + C_2} \cdot \Delta V_{out}$$

- In the case of the Collpits oscillator the source voltage can be assumed to be sinusoidal

$$V_S(t) = V_{S0} - A \cdot \cos(\omega_0 t)$$

- If the transistor is biased in weak inversion, the drain current is then given by

$$I_D(t) = I_{D0} \cdot e^{\frac{V_{G0} - nV_S(t)}{nU_T}} = I_{D0} \cdot e^{\frac{V_{G0} - nV_{S0} + nA \cdot \cos(\omega_0 t)}{nU_T}}$$

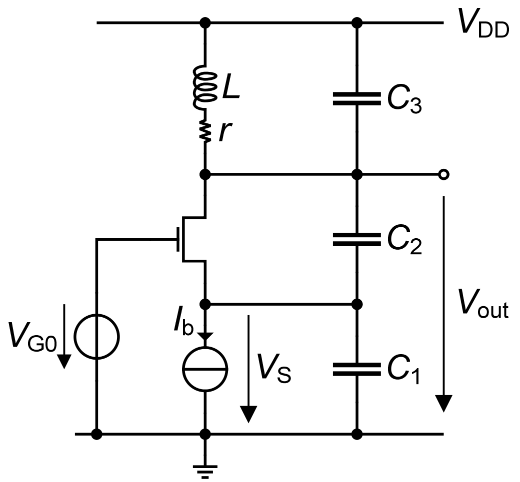
$$= I_0 \cdot e^{\frac{A \cdot \cos(\omega_0 t)}{U_T}} = I_0 \cdot e^{x \cdot \cos(\omega_0 t)}$$

with $I_0 \triangleq I_{D0} \cdot e^{\frac{V_{G0} - nV_{S0}}{nU_T}} = I_{spec} \cdot e^{\frac{V_{G0} - V_{T0} - nV_{S0}}{nU_T}}$

and $x \triangleq \dfrac{A}{U_T}$

- **Same analysis than the Pierce oscillator** and hence the results and normalized plots of the Pierce oscillator also apply to the Colpitts oscillator

Example – The Colpitts Oscillator



$$f_0 = 1\,GHz, Q_L = 10, C_1 = C_2 = 1\,pF, C_3 = 1\,pF$$

$$\omega_0 \cong \frac{1}{\sqrt{L(C_3 + C_{12})}} \quad \Rightarrow \quad L \cong \frac{1}{\omega_0^2 (C_3 + C_{12})} = 16.9\,nH$$

$$r = \frac{\omega_0 L}{Q_L} = 10.6\,\Omega$$

$$G_{mscrit} \cong \frac{\omega_0}{Q_L}(C_1 + C_2)\left(1 + \frac{C_3}{C_{12}}\right) = 3.8\,\frac{mA}{V}$$

- Since the inductance Q_L is not very high, the above approximation is not very accurate. The exact solution is then given by

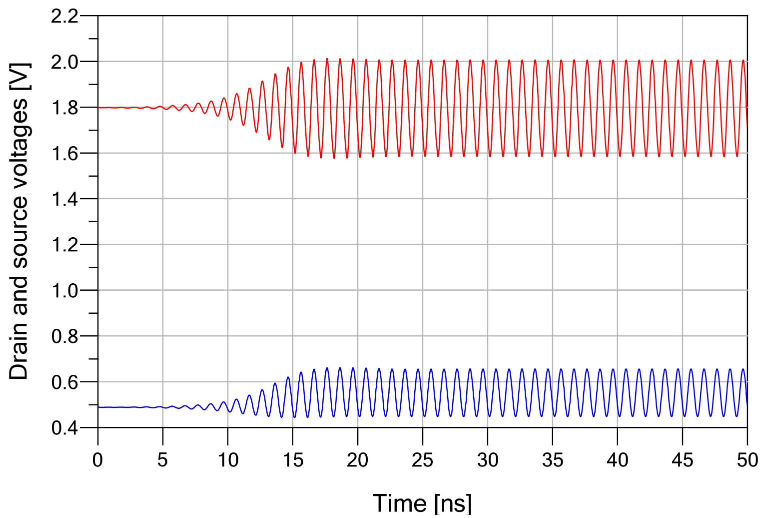
$$G_{mscrit} = \omega_0 C_2 \frac{\alpha_1 Q_L}{2\alpha_3}\left[1 - \sqrt{1 - \left(\frac{2\alpha_3}{\alpha_1 Q_L}\right)^2 (\alpha_1 + 1)\left(1 + \alpha_1 + \frac{\alpha_1}{\alpha_3}\right)}\right] = 4\,\frac{mA}{V}$$

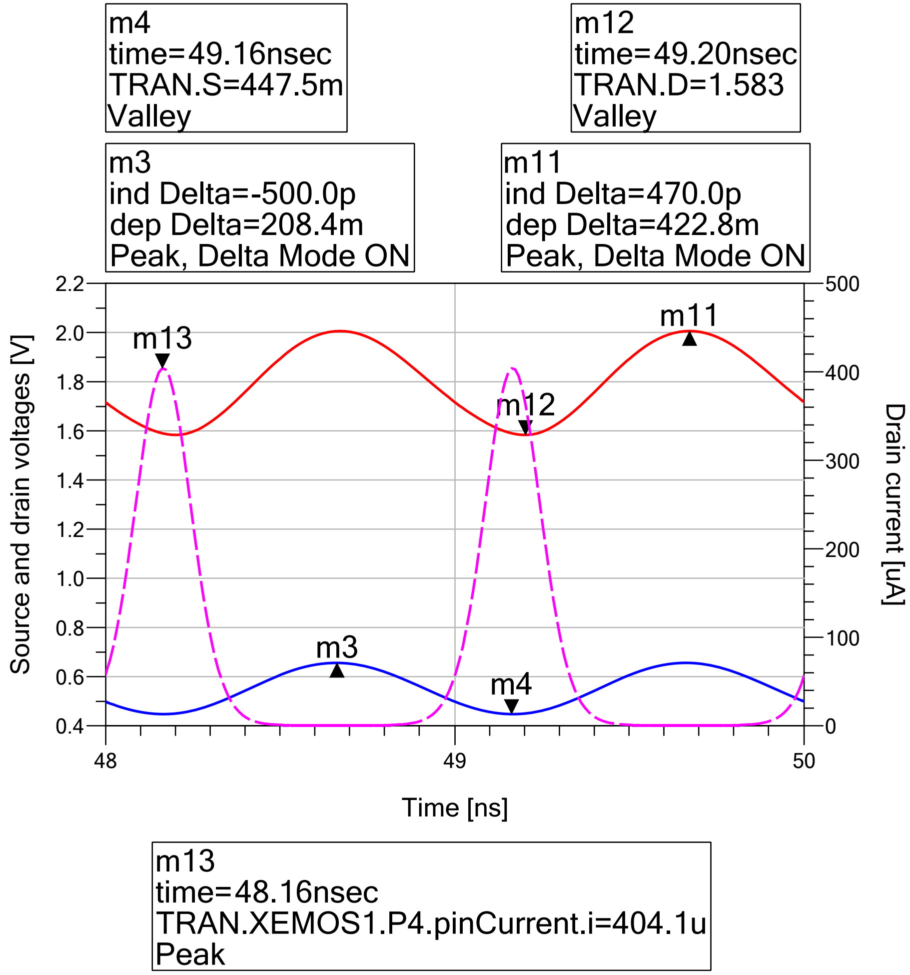
- The inductance value is then found from

$$L = \frac{X_c(\omega_0, G_{mscrit})}{\omega_0}$$

- This leads to $L = 17.256\,nH$ and $r = 10.8\,\Omega$

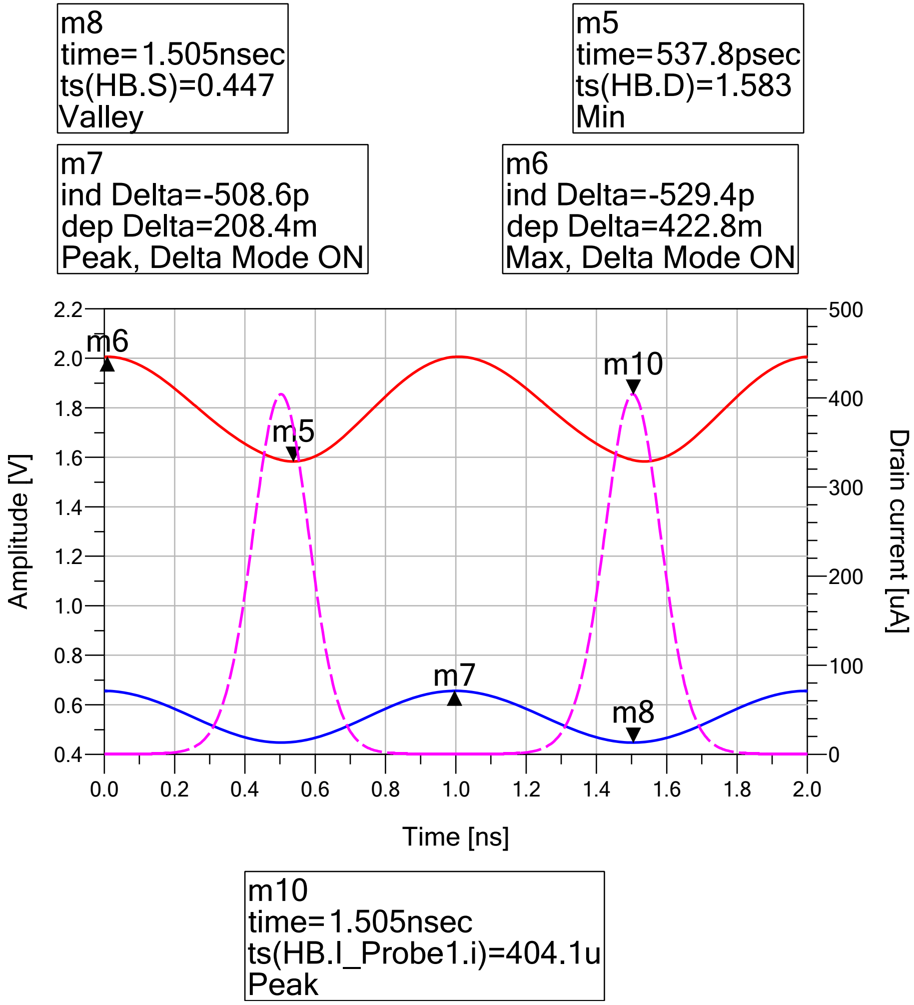
Colpitts Oscillator Example in WI – Transient Simulations

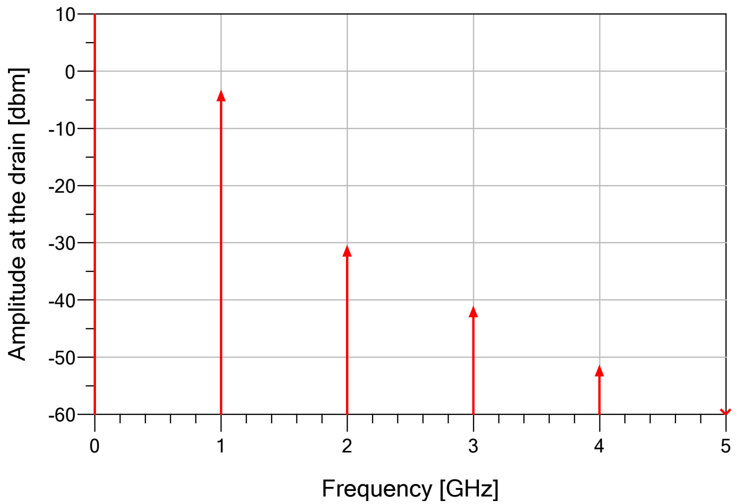




- Transient simulations performed with an ideal exponential transconductor
- The amplitude at the source is almost exactly 100 mV (104 mV), whereas the amplitude at the drain is slightly larger (211 mV)
- The above theory is based on the fundamental component only assuming a large Q and hence that the harmonics are negligible

Colpitts Oscillator Example in WI – HB Simulations

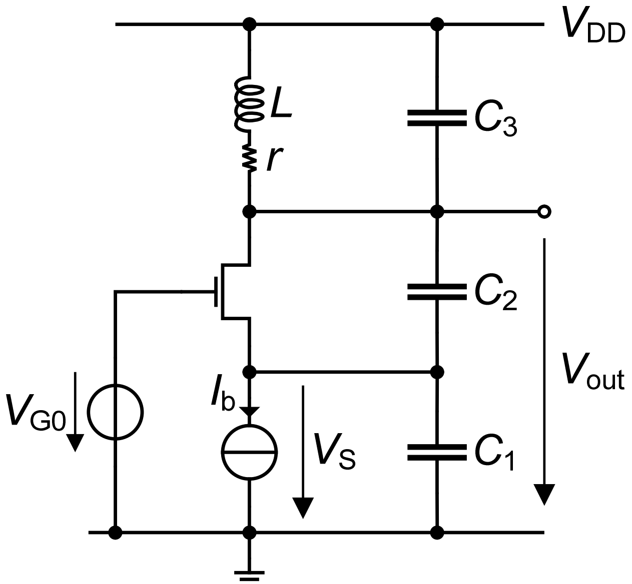




- Harmonic balance steady-state (HB-SS) simulations performed with an ideal quadratic transconductor
- The HB simulations are consistent with the transient simulations presented in the previous slide
- The amplitude at the source is almost exactly 100 mV (104 mV), whereas the amplitude at the drain is slightly larger (211 mV)

Colpitts Oscillator in Strong Inversion

- Similar analysis than for the Pierce oscillator. In this case we have



$$A = \Delta V_S = \frac{C_2}{C_1 + C_2} \cdot \Delta V_{out}$$

$$V_S(t) = V_{S0} - A \cdot \cos(\omega_0 t)$$

$$I_D(t) = I_{spec} \cdot \left(\frac{V_{P0} - V_S(t)}{2U_T} \right)^2$$

$$= I_{spec} \cdot \left(\frac{V_{P0} - V_{S0} - A \cdot \cos(\omega_0 t)}{2U_T} \right)^2$$

$$= \frac{I_{spec}}{4} \cdot \left(v_{ps0} - x \cdot \cos(\omega_0 t) \right)^2 \quad \text{for} \quad x \le v_{ps0}$$

- The results and plots obtained for the Pierce oscillator in strong inversion can be used for the Colpitts oscillator accounting for the different normalization given by

$$v_{ps0} \triangleq \frac{V_{P0} - V_{S0}}{U_T} = \frac{V_{G0} - V_{T0} - nV_{S0}}{nU_T}, \quad x \triangleq \frac{A}{U_T}$$

Colpitts Oscillator in Strong Inversion

- Normalizing and developing the quadratic function leads to

$$i_d(t) = \frac{I_D(t)}{I_{spec}} = \frac{1}{4} \cdot \left(v_{ps0} - x \cdot \cos(\omega_0 t)\right)^2 = \frac{1}{4} \cdot \left(v_{ps0}^2 - 2 v_{ps0} \cdot x \cdot \cos(\omega_0 t) + x^2 \cdot \cos^2(\omega_0 t)\right)$$

$$= \underbrace{\frac{1}{4} \cdot \left(v_{ps0}^2 + \frac{x^2}{2}\right)}_{i_{dc}} - \underbrace{\frac{v_{ps0} \cdot x}{2}}_{-i_{d(1)}} \cdot \cos(\omega_0 t) + \frac{x^2}{8} \cdot \cos(2\omega_0 t)$$

- The normalized dc current for $x < v_{ps0}$ is given by

$$i_{dc} \triangleq \frac{I_{dc}}{I_{spec}} = \frac{I_b}{I_{spec}} = \frac{1}{4} \cdot \left(v_{ps0}^2 + \frac{x^2}{2}\right) = i_0 + \frac{x^2}{8} \quad \text{with} \quad i_0 \triangleq \frac{I_0}{I_{spec}} = \left(\frac{v_{ps0}}{2}\right)^2$$

- The normalized fundamental component for $x < v_{ps0}$ is given by

$$i_{d(1)} \triangleq \frac{I_{D(1)}}{I_{spec}} = \frac{v_{ps0} \cdot x}{2} \quad \text{for} \quad x \le v_{ps0}$$

Transconductance for the Fundamental (Colpitts)

- The fundamental component for $x < v_{ps0}$ is given by

$$I_{D(1)} = I_{spec} \cdot \frac{v_{ps0}}{2} \cdot x \quad \text{for} \quad x \le v_{ps0}$$

- The transconductance for the fundamental component is given by

$$G_{m(1)} = \frac{I_{D(1)}}{A} = \frac{I_{spec}}{A} \cdot \frac{v_{ps0}}{2} \cdot x = \frac{I_{spec}}{U_T} \cdot \frac{v_{ps0}}{2} = \frac{I_{spec}}{U_T} \cdot \sqrt{i_0} \quad \text{for} \quad x \le v_{ps0}$$

- From the dc constraint we get

$$i_b = \left(\frac{v_{ps}}{2}\right)^2 = \left(\frac{v_{ps0}}{2}\right)^2 + \frac{x^2}{8} = i_0 + \frac{x^2}{8} \quad \rightarrow \quad i_0 = i_b - \frac{x^2}{8} \quad \text{or} \quad v_{ps0} = \sqrt{v_{ps}^2 - \frac{x^2}{2}}$$

- The transconductance for the fundamental then becomes

$$G_{ms(1)} = \frac{I_{spec}}{U_T} \cdot \sqrt{i_b - \frac{x^2}{8}} = \frac{I_{spec}}{U_T} \cdot \frac{1}{2} \cdot \sqrt{v_{ps}^2 - \frac{x^2}{2}}$$

Transconductance for the Fundamental (Colpitts)

- The small-signal source transconductance is given by

$$G_{ms} = \frac{I_{spec}}{U_T} \cdot \frac{v_{ps}}{2} = \frac{I_{spec}}{U_T} \cdot \sqrt{i_b} \quad \text{with} \quad v_{ps} \square \frac{V_P - V_S}{U_T} \quad \text{and} \quad i_b = \left(\frac{v_{ps}}{2}\right)^2$$

- The transconductance for the fundamental normalized to the small-signal transconductance is then given by

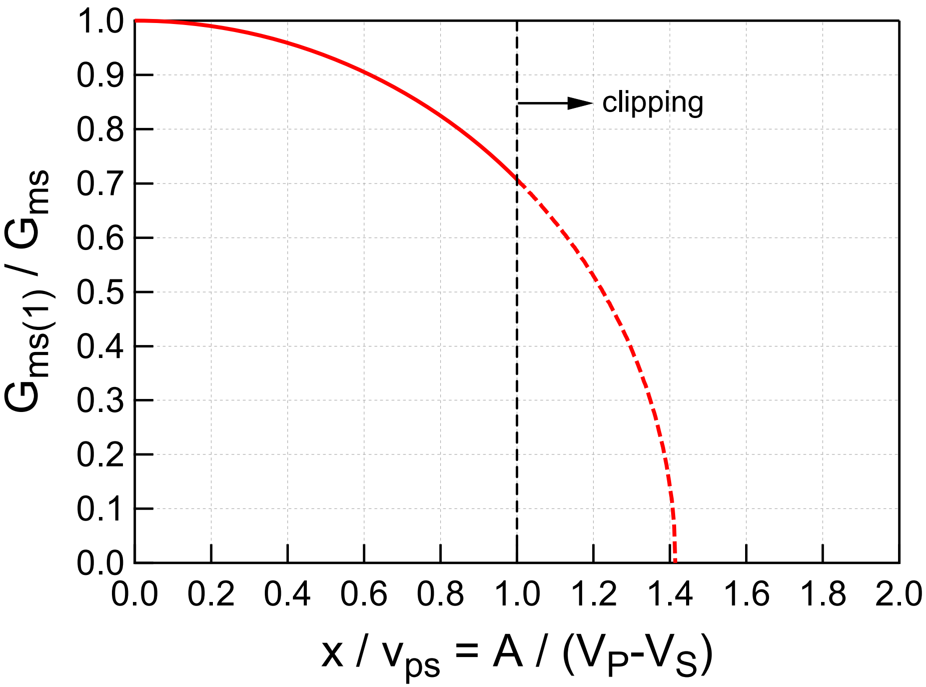
$$\frac{G_{ms(1)}}{G_{ms}} = \sqrt{1 - \frac{x^2}{8 i_b}} = \sqrt{1 - \frac{1}{2}\left(\frac{x}{v_{ps}}\right)^2} = \sqrt{1 - \frac{1}{2}\left(\frac{A}{V_P - V_S}\right)^2} \quad \text{for} \quad x \le v_{ps}$$

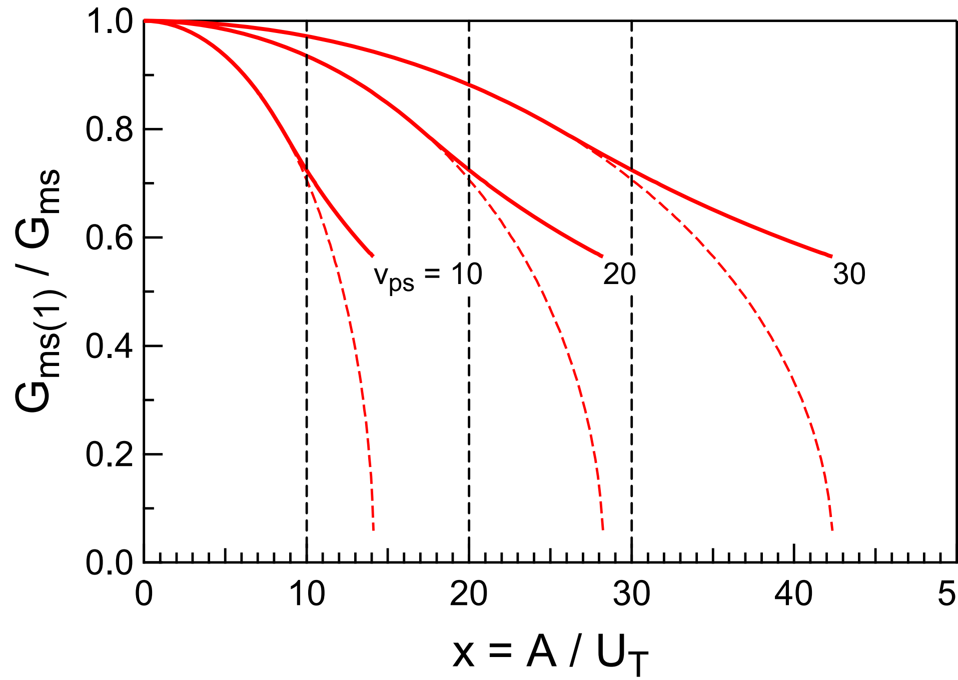
- The critical condition is then given by

$$G_{ms(1)} = G_{mscrit} \quad \rightarrow \quad \frac{I_{spec}}{U_T} \cdot \frac{1}{2} \cdot \sqrt{v_{ps}^2 - \frac{x^2}{2}} = \frac{I_{spec}}{U_T} \cdot \frac{v_{pscrit}}{2} \quad \rightarrow \quad \sqrt{v_{ps}^2 - \frac{x^2}{2}} = v_{pscrit}$$

or in terms of currents $\quad \frac{I_{spec}}{U_T} \cdot \sqrt{i_b - \frac{x^2}{8}} = \frac{I_{spec}}{U_T} \cdot \sqrt{i_{bcrit}} \quad \rightarrow \quad i_b - \frac{x^2}{8} = i_{bcrit}$

Transconductance for the Fundamental (Colpitts)

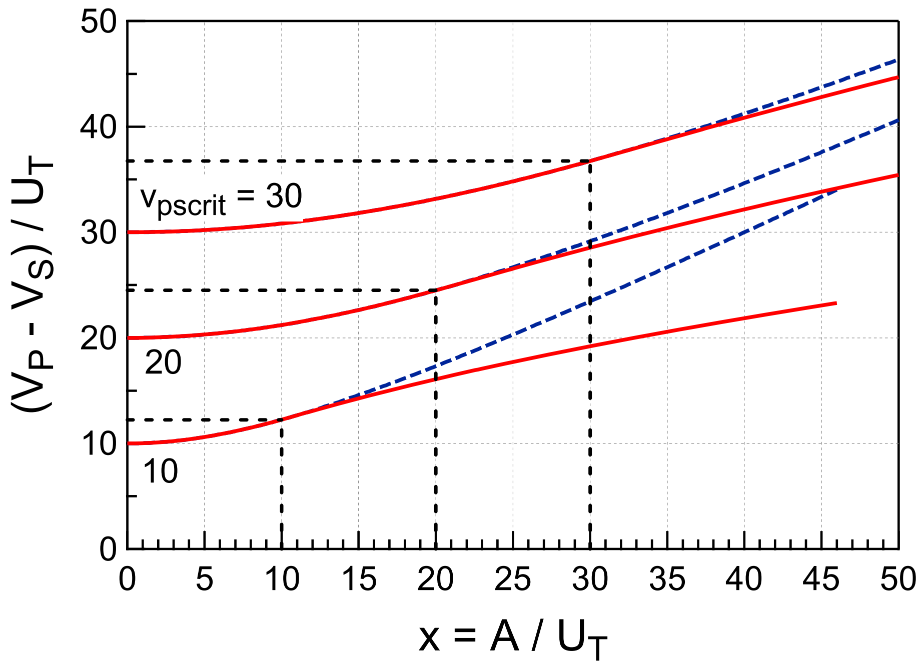


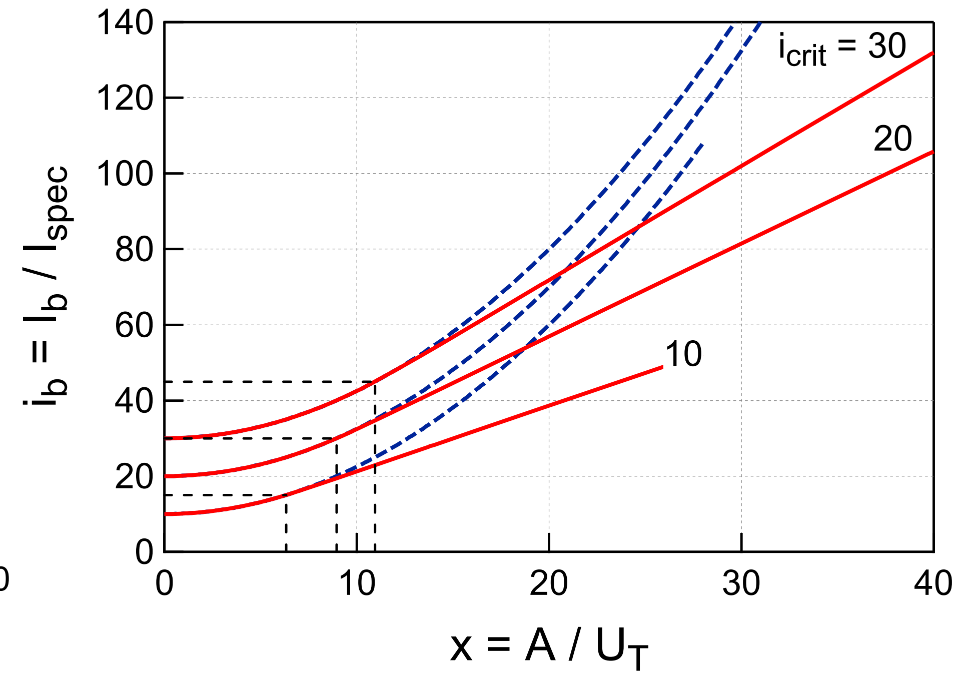


$$\frac{G_{ms(1)}}{G_{ms}} = \sqrt{1-\frac{x^2}{8i_b}} = \sqrt{1-\frac{1}{2}\left(\frac{x}{v_{ps}}\right)^2} = \sqrt{1-\frac{1}{2}\left(\frac{n\cdot A}{V_G - V_{T0} - n\cdot V_S}\right)^2} \quad \text{for} \quad x \le v_{ps}$$

with $v_{ps} \triangleq \dfrac{V_P - V_S}{U_T} \cong \dfrac{V_G - V_{T0} - n\cdot V_S}{n\cdot U_T}$ and $x \triangleq \dfrac{A}{U_T}$

Bias Voltage and Current (Colpitts)





- The required bias overdrive voltage v_{ps} (bias current i_b) for a given critical saturation voltage v_{pscrit} (critical current i_{bcrit}) assuming $x < v_{ps0}$ is given by

$$v_{ps} \triangleq \frac{V_P - V_S}{U_T} \cong \frac{V_G - V_{T0} - n \cdot V_S}{n U_T} = \sqrt{v_{pscrit}^2 + \frac{x^2}{2}} \quad \text{or} \quad i_b \triangleq \frac{I_b}{I_{spec}} = \left(\frac{v_{ps}}{2}\right)^2 = i_{bcrit} + \frac{x^2}{8}$$

Colpitts Oscillator Example – Bias in SI

- If we assume the transistor operates in strong inversion (with $n = 1.3$) and choose the operating saturation voltage as $V_P - V_S = 300\ mV$ and the output oscillation amplitude as $\Delta V_{out} = 200\ mV$ corresponding to an amplitude of the source voltage of $A = \Delta V_S = 100\ mV$, we can then calculate the normalized amplitude as

$$v_{ps} \triangleq \frac{V_P - V_S}{U_T} = 11.6 \quad i_b \triangleq \frac{I_b}{I_{spec}} = 33.6 \quad x \triangleq \frac{A}{U_T} = 3.86$$

- The critical overdrive and critical current can then be calculated as

$$v_{pscrit} = \sqrt{v_{ps}^2 - \frac{x^2}{2}} = 11.27 \quad \text{or} \quad i_{bcrit} = i_b - \frac{x^2}{8} = 31.7$$

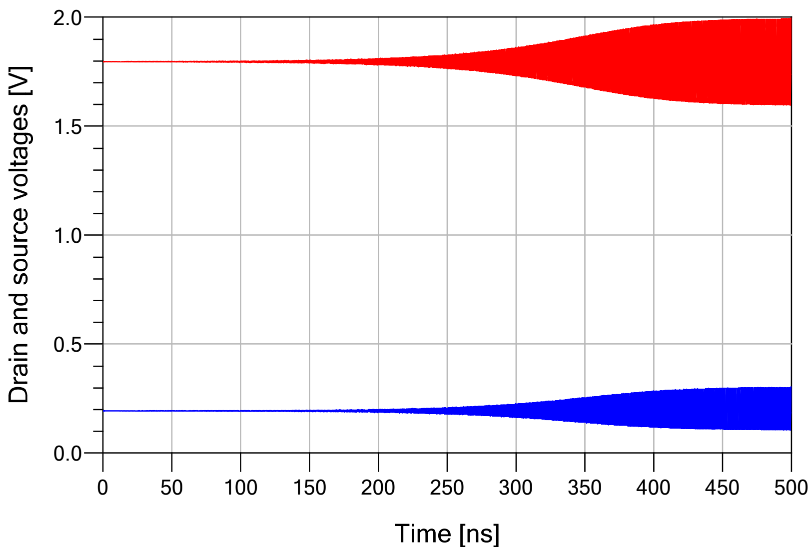
- The specific current is then obtained as

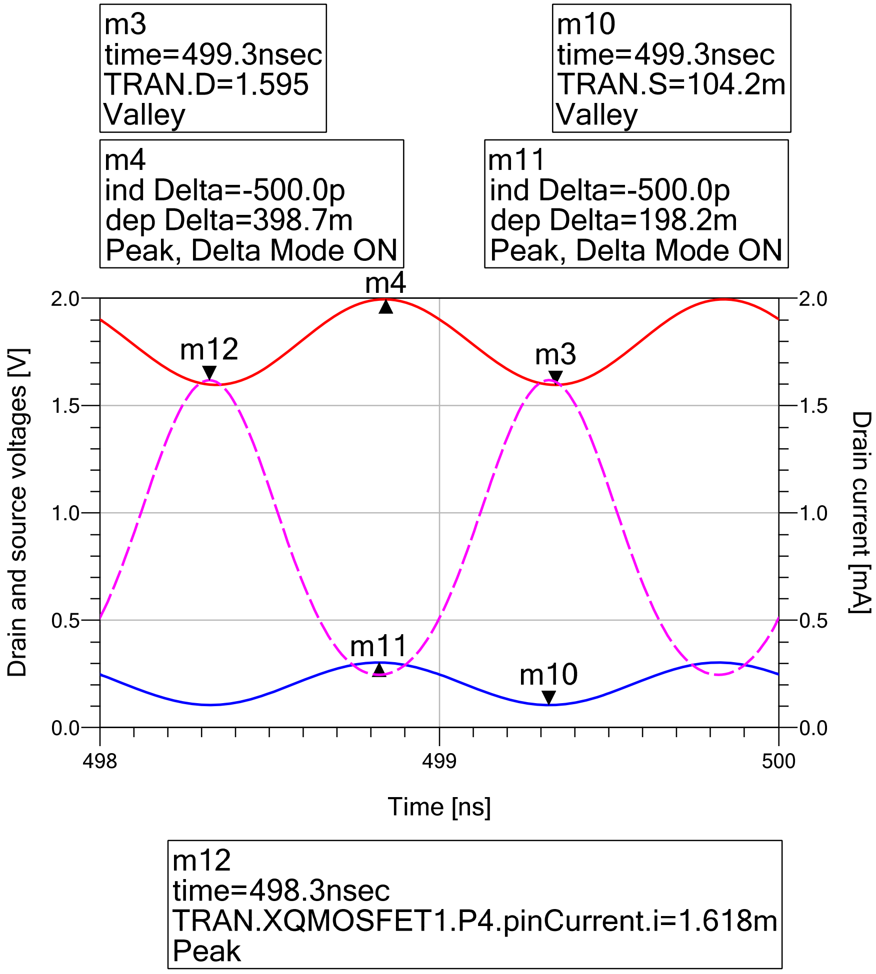
$$I_{spec} = \frac{2U_T \cdot G_{mscrit}}{v_{pscrit}} = \frac{U_T \cdot G_{mscrit}}{\sqrt{i_{bcrit}}} = 18.5\ \mu A$$

- The critical current and the required bias current are given by

$$I_{crit} = I_{spec} \cdot i_{bcrit} = 587.2\ \mu A \quad I_b = I_{spec} \cdot i_b = I_{spec} \cdot \left(\frac{v_{ps}}{2}\right)^2 = 621.7\ \mu A$$

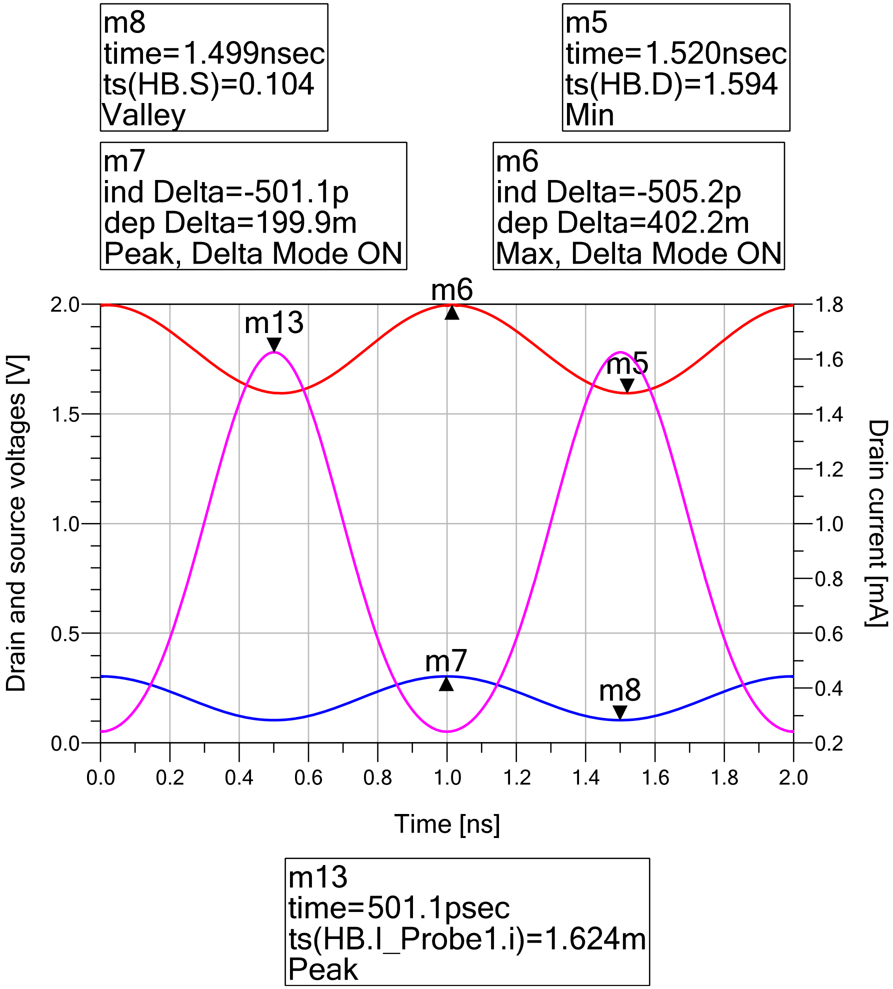
Colpitts Oscillator Example in SI – Transient Simulations

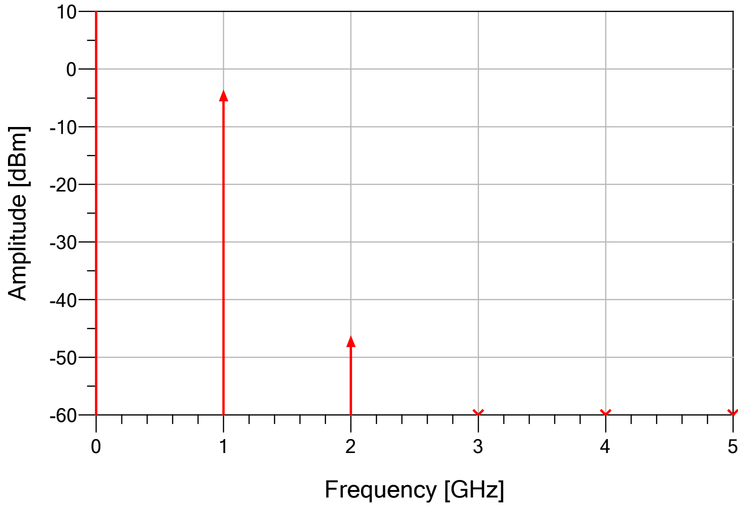




- Transient simulations performed with an ideal quadratic transconductor
- The amplitude at the source (99mV) and at the drain (199mV) are almost exactly 100mV and 200mV
- The above theory is based on the fundamental component only assuming a large Q and hence that the harmonics are negligible

Colpitts Oscillator Example in SI – HB Simulations

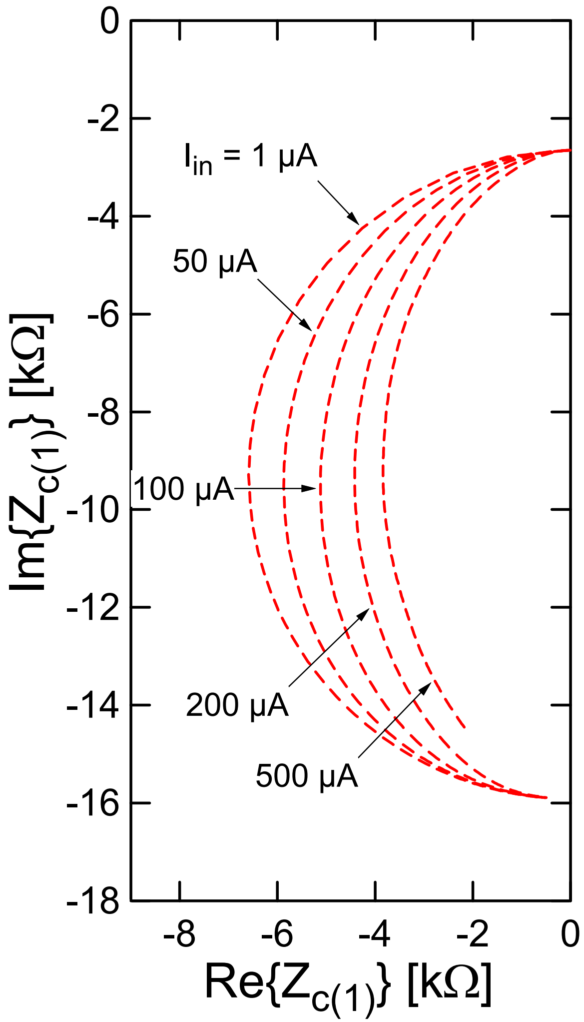


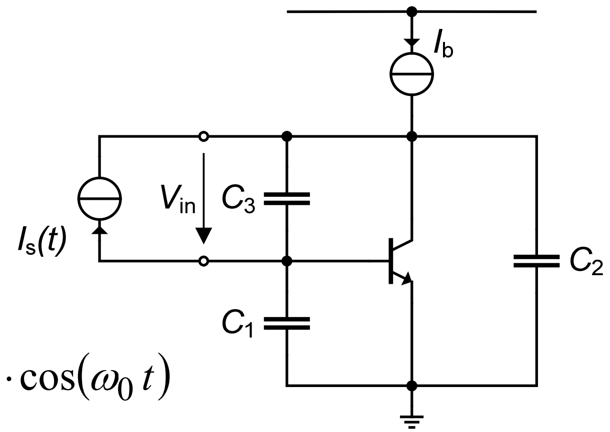


- Harmonic balance SS simulations performed with an ideal quadratic transconductor
- The amplitude at the gate is exactly 200mV, whereas the amplitude at the drain is slightly larger (235mV)

Nonlinear Effects

Sweep of the bias current I_b
for different amplitudes I_{in}





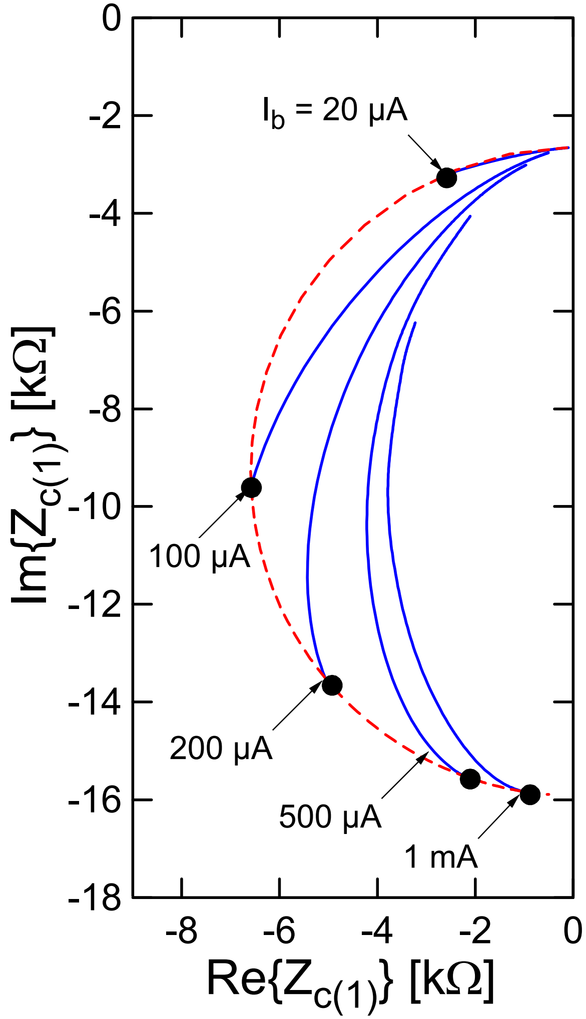
$$I_s(t) = I_{in} \cdot \cos(\omega_0 t)$$

$$Z_c = \frac{V_{in}}{I_{in}} \quad \text{and} \quad Z_{c(1)} = \frac{V_{in(1)}}{I_{in}}$$

- Plot of the input impedance $Z_{in(1)}$ evaluated for the fundamental component versus the bias current I_b swept from 1 µA to 1.7 mA for different current amplitudes I_{in} (1µA, 50µA, 100µA, 200µA and 500µA)
- For small amplitudes ($I_{in} = 1\ \mu A$), we get the circle obtained from the linear analysis, but for large amplitudes, the locus starts to deviate from the circle obtained for small amplitude due to nonlinear effects

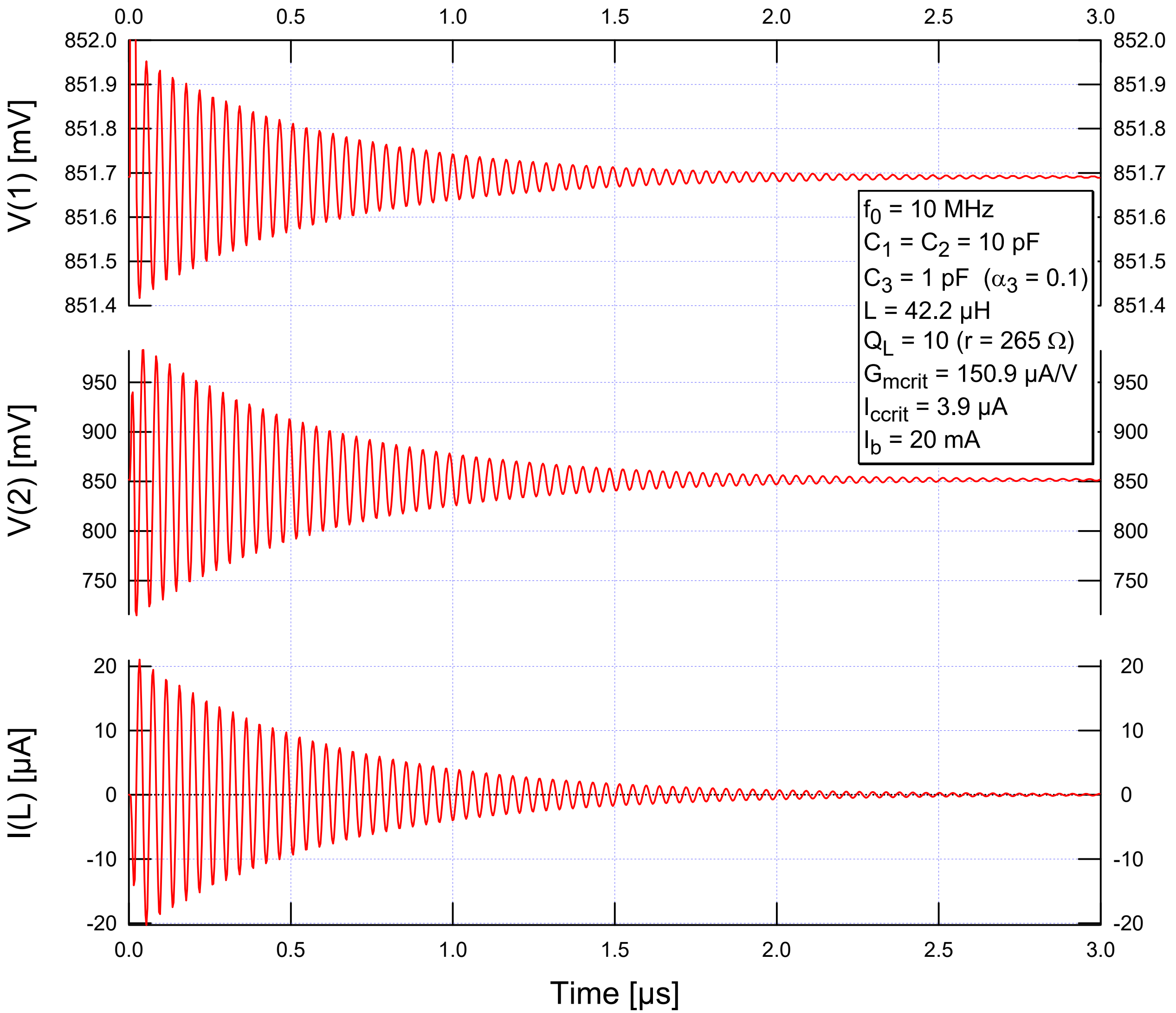
Nonlinear Effects

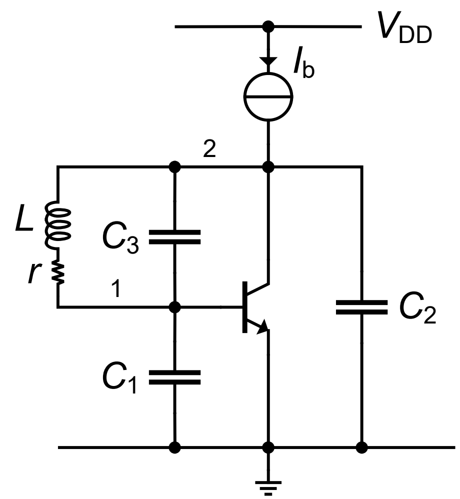
Sweep of the input current I_{in}
for different bias current I_b



- Plot of the input impedance $Z_{in(1)}$ evaluated for the fundamental component versus the current amplitude I_{in} swept from 1µA to 1mA for different bias currents I_b (20µA, 100µA, 200µA, 500µA and 1mA)
- The locus always starts on the circle obtained for small amplitude with a direction tangent to the circle and then deviates from it due to nonlinear effects
- The actual operating point can be quite far from the one obtained with the linear analysis
- There may eventually be **no operating point** when the bias current becomes too large, even though the small-signal analysis would show an intersection

Unstable Point at Very Large Bias Current



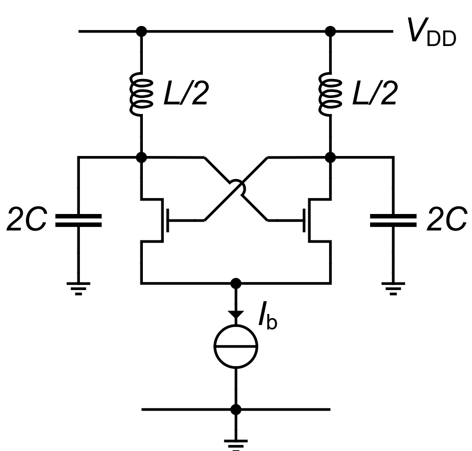


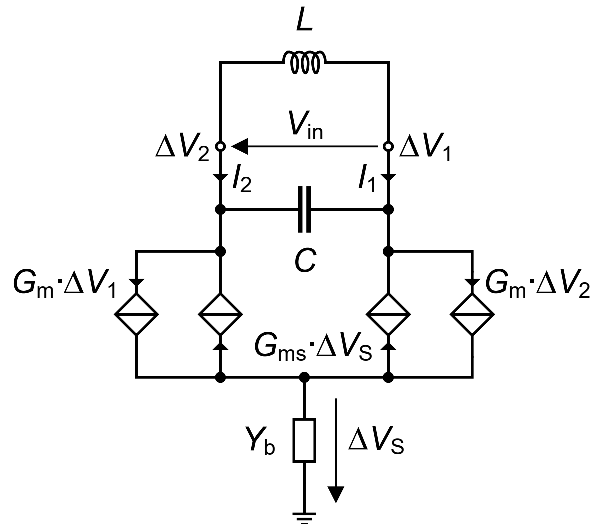
Outline

- General considerations
- The 3-points oscillator
- The cross-coupled pair oscillator

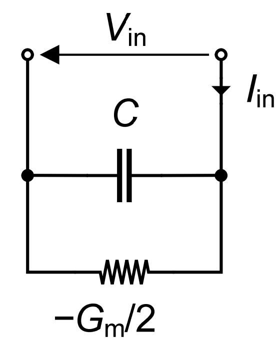
The Cross-coupled Pair Oscillator – Principle

- A balanced LO signal is most often required
- Can be generated by the cross-coupled pair oscillator shown below





$$\Delta V_S = \frac{G_m}{G_{ms} + Y_b/2} \cdot \frac{\Delta V_1 + \Delta V_2}{2}$$



- If fully balanced operation is assumed $\Delta V_1 = -\Delta V_2 = \dfrac{V_{in}}{2} \quad \rightarrow \quad \Delta V_S = 0$
- And hence

$$Y_c = \frac{I_{in}}{V_{in}} = -\frac{G_m}{2} + j\omega C$$

Critical G_m

- The circuit impedance Z_c is then given by

$$Z_c = \frac{1}{Y_c} = \frac{1}{-G_m/2 + j\omega C} = -\frac{G_m/2}{(G_m/2)^2 + (\omega C)^2} - \frac{j\omega C}{(G_m/2)^2 + (\omega C)^2}$$

- And hence

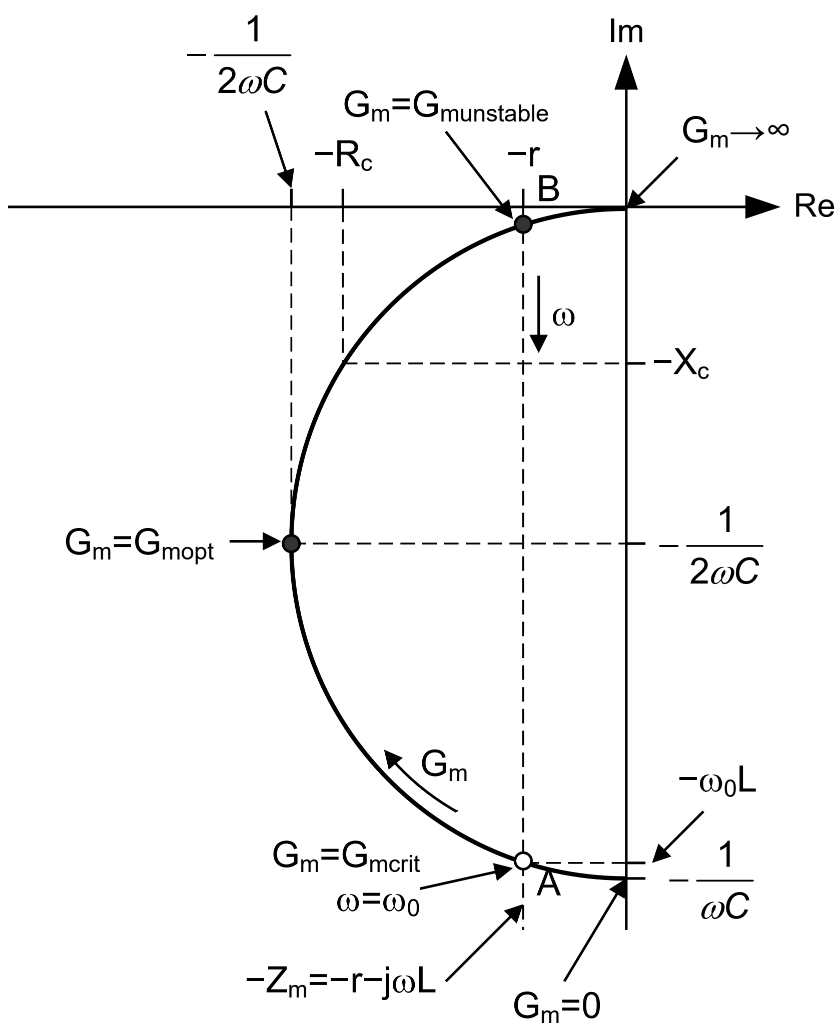
$$R_c = -\mathrm{Re}\{Z_c\} = \frac{G_m/2}{(G_m/2)^2 + (\omega C)^2} \quad \text{and} \quad X_c = -\mathrm{Im}\{Z_c\} = \frac{\omega C}{(G_m/2)^2 + (\omega C)^2}$$

- Solving for G_{mcrit} and ω_0 results in

$$\frac{X_c(\omega_0, G_{mcrit})}{R_c(\omega_0, G_{mcrit})} = Q_L \quad \Rightarrow \quad \frac{\omega_0 C}{G_{mcrit}/2} = Q_L \quad \Rightarrow \quad G_{mcrit} = \frac{2\omega_0 C}{Q_L}$$

$$X_c(\omega_0, G_{mcrit}) = \omega_0 L \quad \Rightarrow \quad \omega_0 = \frac{\omega_{LC}}{\sqrt{1 + \frac{1}{Q_L^2}}} \quad \text{with} \quad \omega_{LC} = \frac{1}{\sqrt{LC}}$$

Circuit Impedance

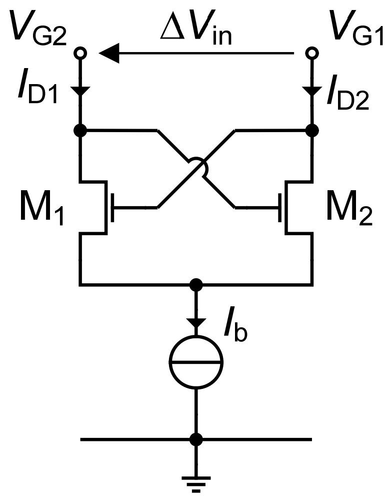


$$Z_c = \frac{-G_m/2 - j\omega C}{(G_m/2)^2 + (\omega C)^2}$$

$$G_{mcrit} = \frac{2C\omega_0}{Q_L}$$

$$\omega_0 = \frac{\omega_{LC}}{\sqrt{1+\frac{1}{Q_L^2}}} \quad \text{with} \quad \omega_{LC} = \frac{1}{\sqrt{LC}}$$

Large-signal Analysis (weak inversion)



- The differential current in WI is given by

$$\Delta I_{out} = I_{D2} - I_{D1} = -I_b \cdot \tanh\left(\frac{\Delta V_{in}}{2nU_T}\right)$$

- The output waveform for a sinusoidal differential voltage is then given by

$$\Delta V_{in}(t) = A \cdot \cos(\omega_0 t)$$

$$\Delta i_{out}(t) = \frac{\Delta I_{out}(t)}{I_b} = -\tanh\left(x \cdot \cos(\omega_0 t)\right) \quad \text{with} \quad x = \frac{A}{2nU_T}$$

- The output waveform is periodic and can be developed in a Fourier series

$$\Delta i_{out}(t) = \frac{\Delta I_{out}(t)}{I_b} = \sum_{\substack{n=1 \\ n\ odd}}^{+\infty} a_n(x) \cdot \cos(n\omega_0 t) = \sum_{\substack{n=1 \\ n\ odd}}^{+\infty} a_n(x) \cdot \cos(n\varphi)$$

with

$$a_n(x) = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} \tanh\left(x \cdot \cos(\varphi)\right) \cdot \cos(n\varphi) \cdot d\varphi$$

Large-signal Analysis (weak inversion)

- We are mostly interested in the fundamental component given by

$$a_1(x) = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} \tanh\left(x \cdot \cos(\varphi)\right) \cdot \cos(\varphi) \cdot d\varphi$$

- Unfortunately there is no analytical solution for this integral
- For $x \ll 1$, it can nevertheless be approximated by

$$a_1(x) \cong \frac{2\mathrm{I}_{B1}(x)}{\mathrm{I}_{B0}(x) + \mathrm{I}_{B2}(x)}$$

- For large signal amplitudes, the output waveform becomes a square wave and hence for $x \gg 1$ we have

$$a_1(x) \cong \frac{4}{\pi} \quad \text{for } x \gg 1$$

Transconductance for the Fundamental

- The transconductance for the fundamental component is then given by

$$G_{m(1)} = \frac{I_{D(1)}}{A} = \frac{a_1(x) \cdot I_b}{A} = \frac{I_b}{2nU_T} \cdot \frac{a_1(x)}{x} = G_m \cdot \frac{a_1(x)}{x}$$

with $G_m = \dfrac{I_b}{2nU_T}$

- Or in normalized form

$$\frac{G_{m(1)}}{G_m} = \frac{a_1(x)}{x} \cong \frac{2 \mathrm{I}_{B1}(x)}{x \cdot \left(\mathrm{I}_{B0}(x) + \mathrm{I}_{B2}(x)\right)}$$

Transconductance for the Fundamental