# Fundamentals of Electrical Circuits and Systems

## Chapter 5: AC Circuits 2

Farhad Rachidi
École Polytechnique Fédérale de Lausanne
Lausanne, Switzerland

EPFL

# AC Circuits 2

- Phasors and complex numbers
- Complex numbers
  - Notions of complex algebra
  - Euler's formula
  - Derivation and integration
- Phasors
  - Definition
  - Basic operations
- Impedance and admittance

# Phasors and complex numbers

- Phasor : simple way to represent sinusoidal voltages and currents. This method has been proposed by C.P. Steinmetz and is based on Euler's relation.

Charles Proteus **Steinmetz** (1865-1923), American electrical engineer of German origin. He developed the symbolic method for AC calculations.

# Complex numbers: definition

- We call complex number z any expression of the form

$$z = a + \mathrm{j}b$$

where $\mathrm{j} = \sqrt{-1}$ $\mathrm{j}^2 = -1$

and $\mathrm{j}^3 = -\mathrm{j}$ $\mathrm{j}^4 = 1$ Etc.

Carl Friedrich **Gauss** (1777-1855), astronomer, mathematician and German physicist. He introduced the complex calculus in 1801.

# Notions of complex algebra

- Equality of two complex numbers

$$z_1 = a_1 + \mathrm{j}b_1 \qquad z_2 = a_2 + \mathrm{j}b_2$$

$$z_1 = z_2 \quad \Leftrightarrow \quad a_1 = a_2 \text{ et } b_1 = b_2$$

- Complex conjugate of $z = a + \mathrm{j}b$

$$z^* = a - \mathrm{j}b$$

$$z + z^* = 2a$$

# Notions of complex algebra

- Addition and substraction

$$z_1 + z_2 = (a_1 + a_2) + \mathrm{j}(b_1 + b_2)$$

$$z_1 - z_2 = (a_1 - a_2) + \mathrm{j}(b_1 - b_2)$$

- Multiplication

$$z_1 \cdot z_2 = (a_1 \cdot a_2 - b_1 \cdot b_2) + \mathrm{j}(b_1 \cdot a_2 + a_1 \cdot b_2)$$

$$z \cdot z^* = a^2 + b^2$$

# Notions of complex algebra

- Division

$$\frac{z_1}{z_2} = \frac{a_1 \cdot a_2 + b_1 \cdot b_2}{a_2^2 + b_2^2} + j\frac{b_1 \cdot a_2 - a_1 \cdot b_2}{a_2^2 + b_2^2}$$

- Complex Conjugate of Elementary Operations

$$(z_1 + z_2)^* = z_1^* + z_2^*$$

$$(z_1 - z_2)^* = z_1^* - z_2^*$$

$$(z_1 \cdot z_2)^* = z_1^* \cdot z_2^*$$

$$(z_1 / z_2)^* = z_1^* / z_2^*$$

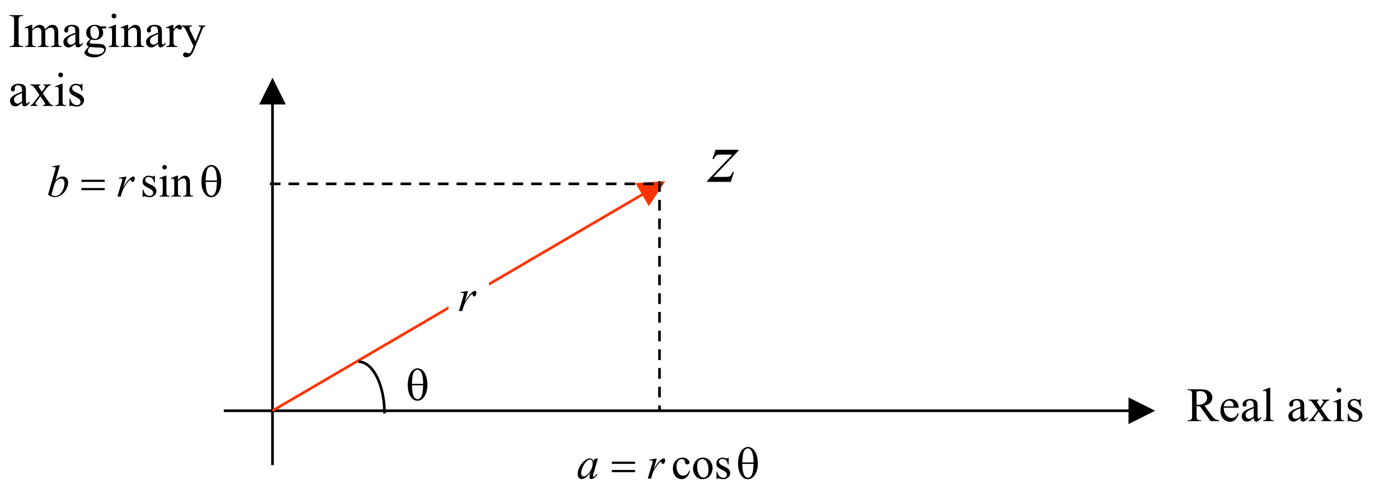
# Complex Numbers: Geometric Representation

$$z = a + \mathrm{j}b$$

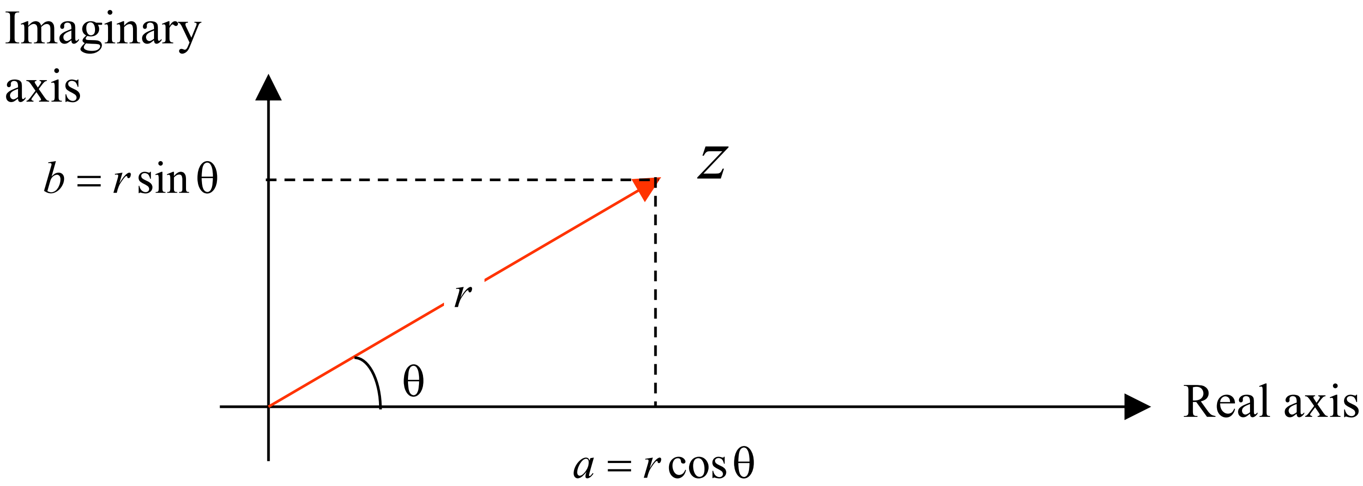
- We call **a** the real part, and **b** the imaginary part of the complex number z.

$$a = \mathrm{Re}(z) \qquad b = \mathrm{Im}(z)$$

- Representation in the complex plane:



# Complex Numbers: Geometric Representation




$$r = |z| = \sqrt{a^2 + b^2}$$ : complex number modulus

$$\theta = \arg(z) = \arctan\frac{b}{a}$$ : argument of the complex number

# Complex numbers: Other properties

$$|z_1 z_2| = |z_1| \cdot |z_2|$$

$$|z_1 + z_2| \le |z_1| + |z_2|$$

$$|z_1 - z_2| \ge |z_1| - |z_2|$$

Distance between two complex numbers:

$$|z_2 - z_1| = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2}$$

# Complex numbers: Euler's Formula

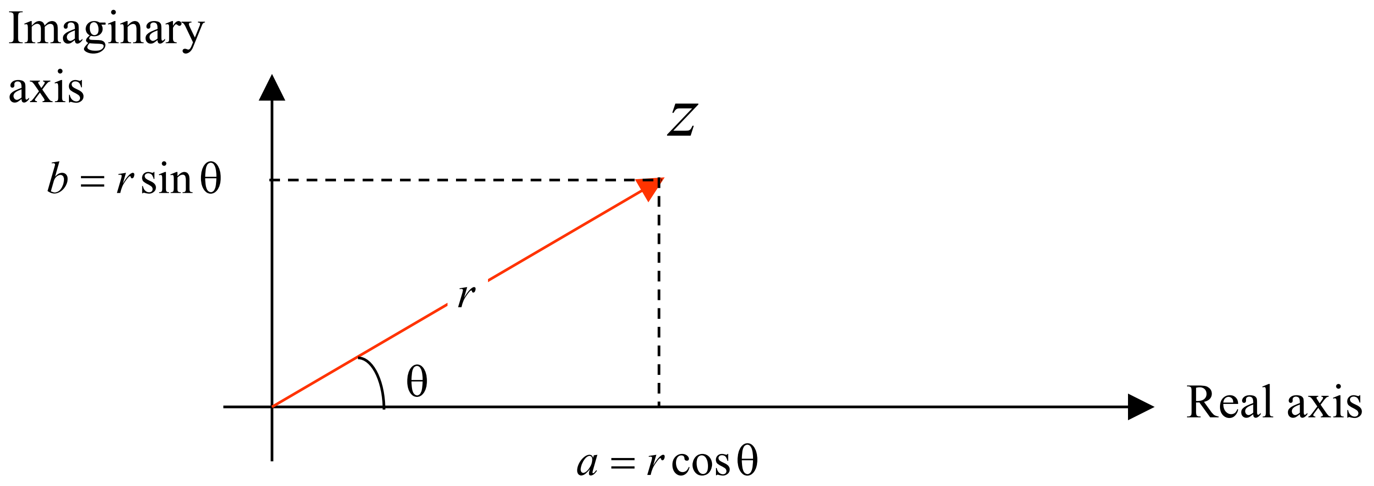
$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\Longleftrightarrow \begin{cases} \cos\theta = \dfrac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right) \\ \sin\theta = \dfrac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right) \end{cases}$$

Leonhard **Euler** (1707-1783), Swiss mathematician died in St. Petersburg.

# Complex numbers: Euler's Formula

$$z = a + jb = re^{j\theta} = r\cos\theta + jr\sin\theta$$



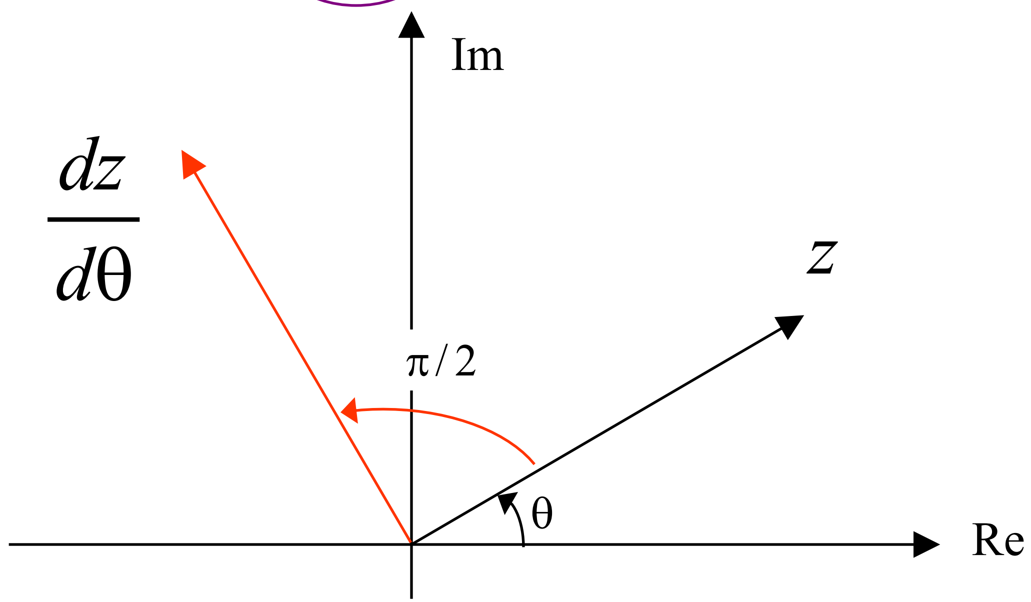

Particular case: $j = e^{j\pi/2}$

# Complex numbers: Derivation from the argument

$$\boxed{e^{j\pi/2}e^{j\theta} = e^{j(\theta+\pi/2)}}$$

$$z = re^{j\theta}$$

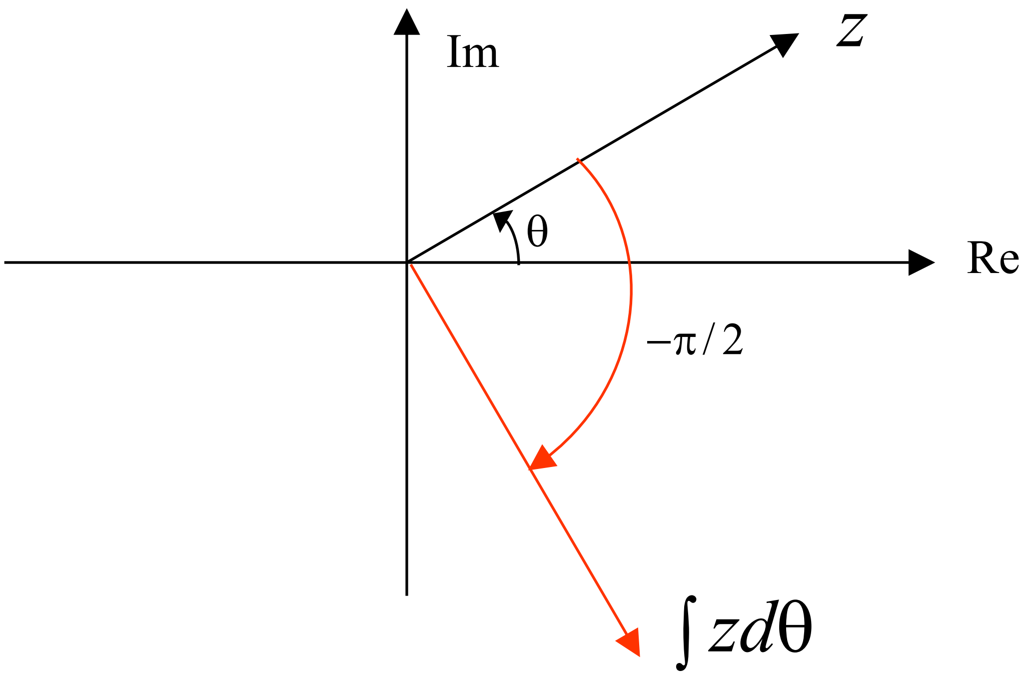
$$\frac{dz}{d\theta} = rje^{j\theta} = re^{j\pi/2}e^{j\theta} = re^{j(\theta+\pi/2)}$$



# Complex numbers: Integration with the argument

$$z = re^{j\theta}$$

$$\int z d\theta = \int re^{j\theta} d\theta = rj^{-1}e^{j\theta} = re^{j(\theta - \pi/2)}$$



# Complex Numbers: Powers and Roots

Power :

$$z = a + jb = re^{j\theta}$$

$$z^n = r^n e^{jn\theta}$$

Roots:

$$w = \sqrt[n]{z} = \rho e^{j\psi}$$

$$\rho = \sqrt[n]{r}$$

$$\psi = \frac{\theta}{n} + \frac{k}{n} 2\pi$$

# Complex representation of a sinusoidal quantity

Reminder: Euler's formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

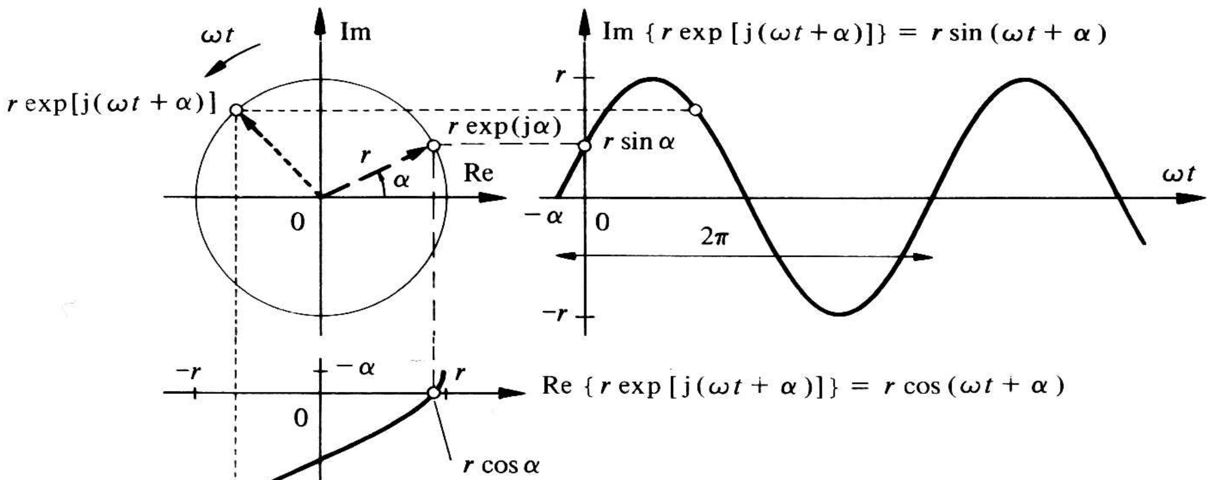
Let a sinusoidal function

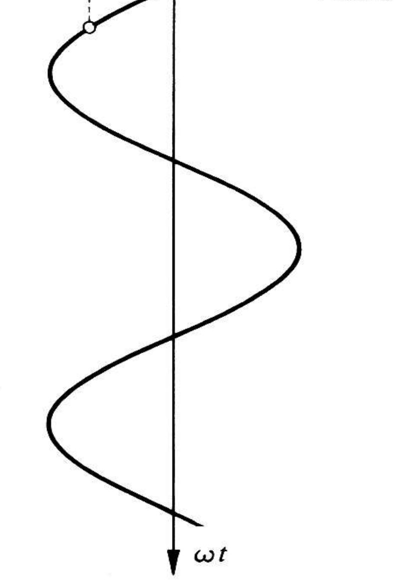
$$x(t) = \sqrt{2}X\cos(\omega t + \theta)$$

$$\Rightarrow \quad x(t) = \mathrm{Re}\left\{\sqrt{2}Xe^{j(\omega t+\theta)}\right\}$$

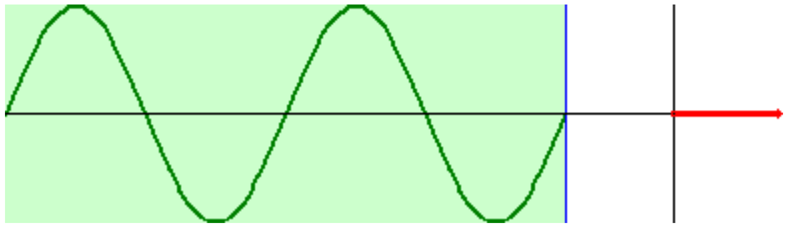
$\underline{x} = \sqrt{2}Xe^{j(\omega t+\theta)}$ : complex instantaneous value

# Complex representation of a sinusoidal quantity




Animation



# The Phasor

$$x(t) = \sqrt{2}X\cos(\omega t + \theta) \Longleftrightarrow \underline{x} = \sqrt{2}Xe^{j(\omega t+\theta)}$$

$$\underline{x} = \sqrt{2}Xe^{j(\omega t+\theta)} = \sqrt{2}Xe^{j\theta}e^{j\omega t}$$

In a linear electrical circuit with sinusoidal sources, all currents and voltages have the same pulsation ω. The term exp(jωt) is therefore common to all quantities (currents and tensions) of the circuit. Any size can be characterized only by its amplitude (rms value) X and its phase θ.

$\Rightarrow$ The phasor associated with x(t): $\boxed{\underline{X} = Xe^{j\theta}}$

# The Phasor

$$x(t) = \sqrt{2}X\cos(\omega t + \theta) \quad \Longleftrightarrow \quad \underline{x} = \sqrt{2}Xe^{j(\omega t + \theta)}$$

$$\underline{x} = \sqrt{2}X \ldots$$

**Phasor does not depend on time**

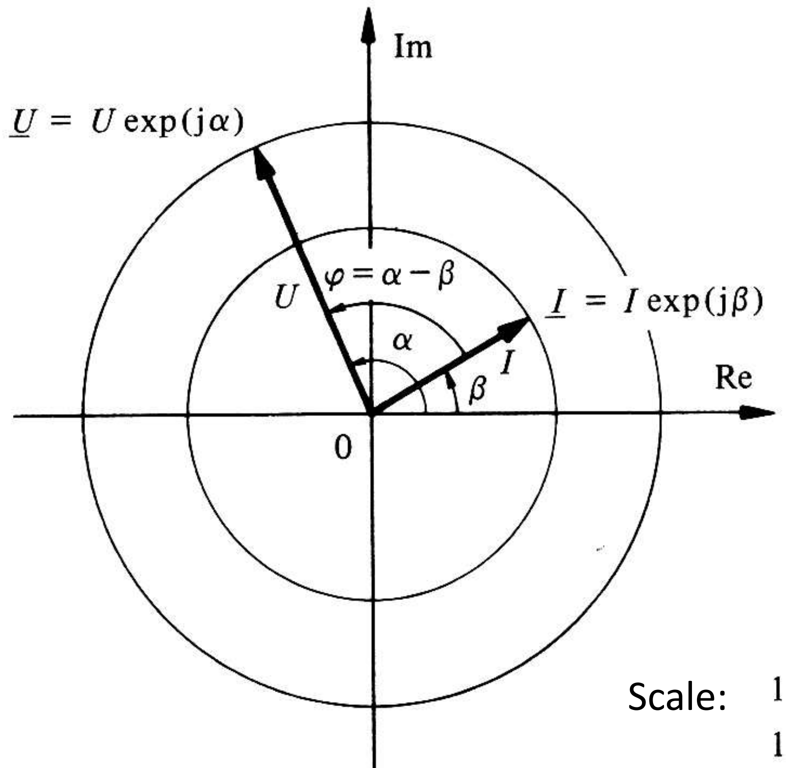
In a linear el… …ents and voltages hav… …therefore common to … …rcuit. Any size can be characterized only by its amplitude (rms value) X and its phase θ.

The phasor associated with x(t): $\boxed{\underline{X} = Xe^{j\theta}}$

# Phasor diagram

$$u(t) = \sqrt{2}U\cos(\omega t + \alpha) \Longleftrightarrow \underline{U} = Ue^{j\alpha}$$

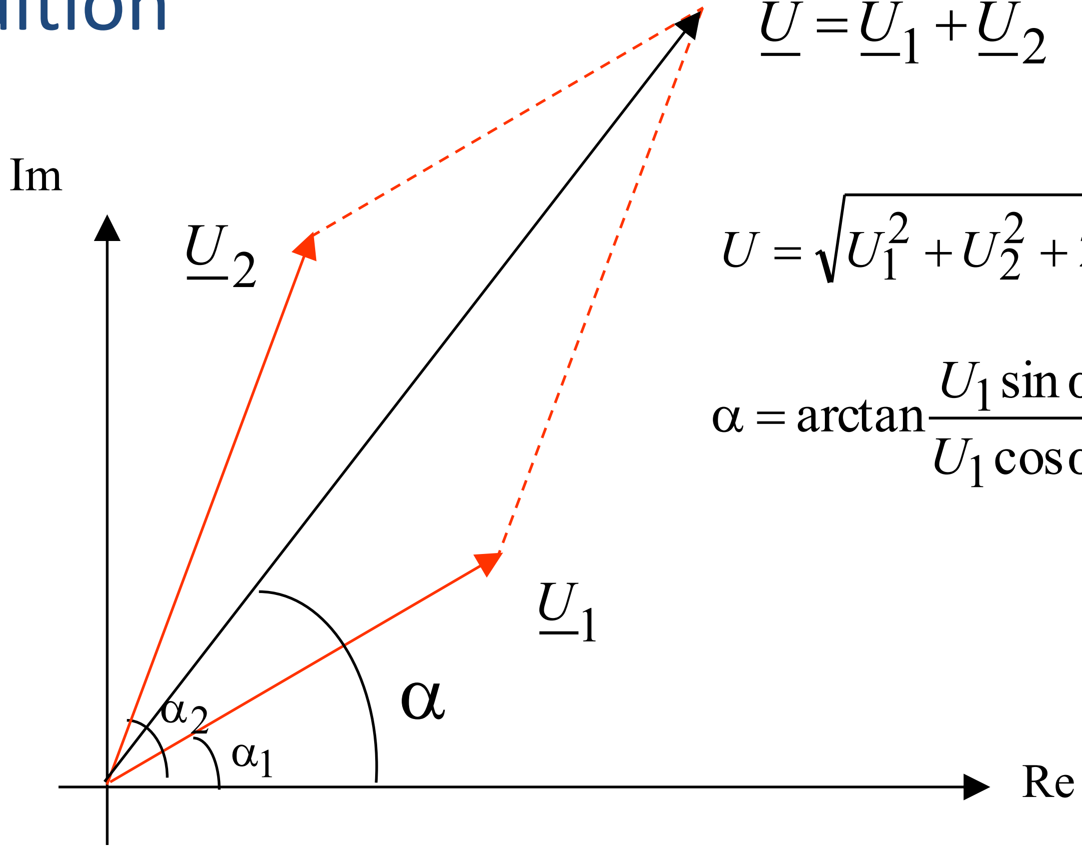
$$i(t) = \sqrt{2}I\cos(\omega t + \beta) \Longleftrightarrow \underline{I} = Ie^{j\beta}$$




Scale: 1 cm ≙ 10 V
1 cm ≙ 1 A

# Basic operations on phasors

- Addition




$$\underline{U} = \underline{U}_1 + \underline{U}_2$$

$$U = \sqrt{U_1^2 + U_2^2 + 2U_1U_2\cos(\alpha_1 - \alpha_2)}$$

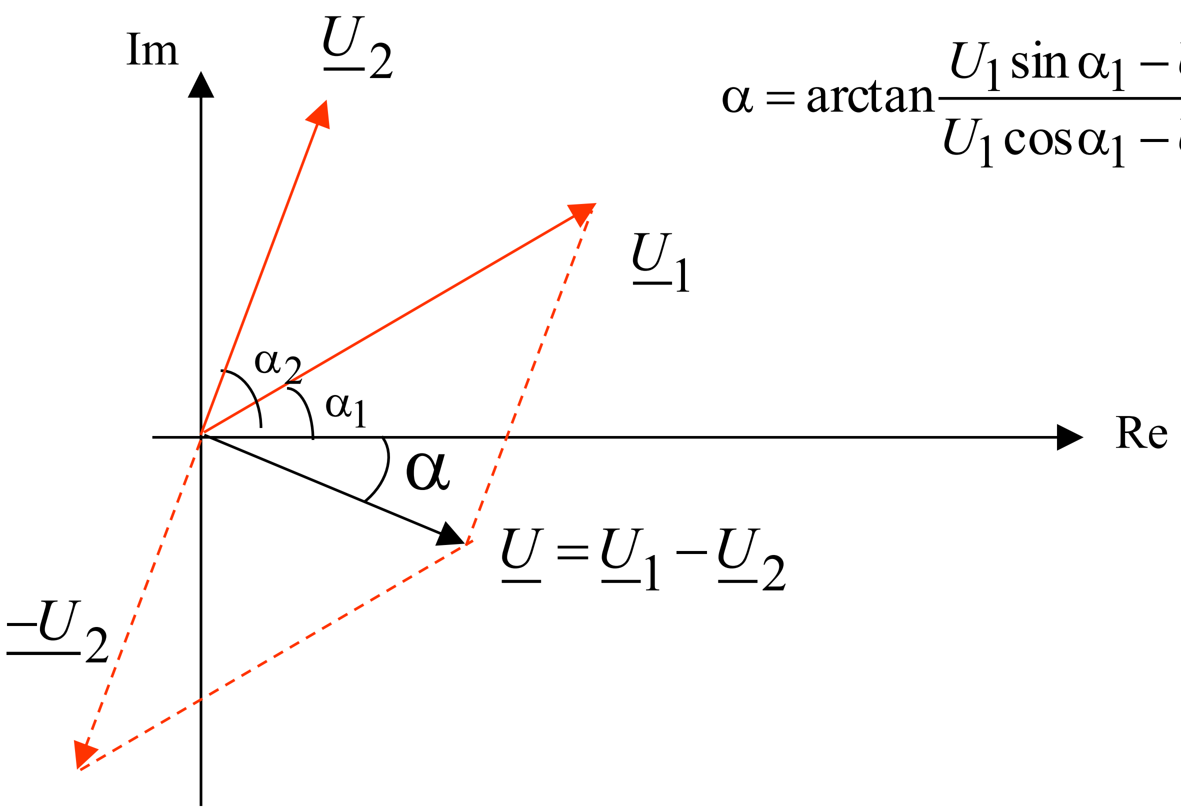
$$\alpha = \arctan\frac{U_1\sin\alpha_1 + U_2\sin\alpha_2}{U_1\cos\alpha_1 + U_2\cos\alpha_2}$$

# Basic operations on phasors

- Substraction

$$U = \sqrt{U_1^2 + U_2^2 - 2U_1U_2\cos(\alpha_1 - \alpha_2)}$$

$$\alpha = \arctan\frac{U_1\sin\alpha_1 - U_2\sin\alpha_2}{U_1\cos\alpha_1 - U_2\cos\alpha_2}$$



# Basic operations on phasors

- Multiplication

$$\underline{X} = Xe^{j\alpha} \qquad \underline{Y} = Ye^{j\beta}$$

$$\underline{Z} = \underline{X} \cdot \underline{Y} = Xe^{j\alpha} Ye^{j\beta} = Ze^{j\varphi}$$

with $Z = XY$ and $\varphi = \alpha + \beta$

# Basic operations on phasors

- Quotient

$$\underline{X} = Xe^{j\alpha} \qquad \underline{Y} = Ye^{j\beta}$$

$$\underline{Z} = \frac{\underline{X}}{\underline{Y}} = \frac{Xe^{j\alpha}}{Ye^{j\beta}} = Ze^{j\varphi}$$

with $Z = \dfrac{X}{Y}$ and $\varphi = \alpha - \beta$

# Derivation of a sinusoidal quantity

$$x(t) = \sqrt{2} X \cos(\omega t + \theta)$$ Sinusoidal quantity

$$\underline{x} = X e^{j\theta} e^{j\omega t}$$ Complex instantaneous value

$$\underline{X} = X e^{j\theta}$$ Phasor

Derivation with respect to time: $y = \dfrac{dx}{dt}$

$$\underline{y} = \frac{d\underline{x}}{dt} = X e^{j\theta} j\omega e^{j\omega t} = j\omega \underline{x} \quad \Longleftrightarrow \quad \underline{Y} = j\omega \underline{X}$$

Equivalent to a multiplication by jω !

# Derivation of a sinusoidal quantity

$$\underline{y} = \frac{d\underline{x}}{dt} = Xe^{j\theta} j\omega e^{j\omega t} = j\omega \underline{x} \quad \Longleftrightarrow \quad \underline{Y} = j\omega \underline{X}$$

Generalization :

$$\underline{z} = \frac{d^n \underline{x}}{dt^n} = (j\omega)^n \underline{x} \quad \Longleftrightarrow \quad \underline{Z} = (j\omega)^n \underline{X}$$

# Integration of a sinusoidal quantity

Integration with respect to time: $y = \int x(t)\,dt$

$$\underline{y} = \int \underline{x}\,dt = \int X e^{j\theta} e^{j\omega t}\,dt = \frac{X}{j\omega} e^{j\theta} e^{j\omega t} = \frac{1}{j\omega}\underline{x}$$

$$\Longleftrightarrow \quad \underline{Y} = \frac{1}{j\omega}\underline{X}$$

Equivalent to a division by jω !

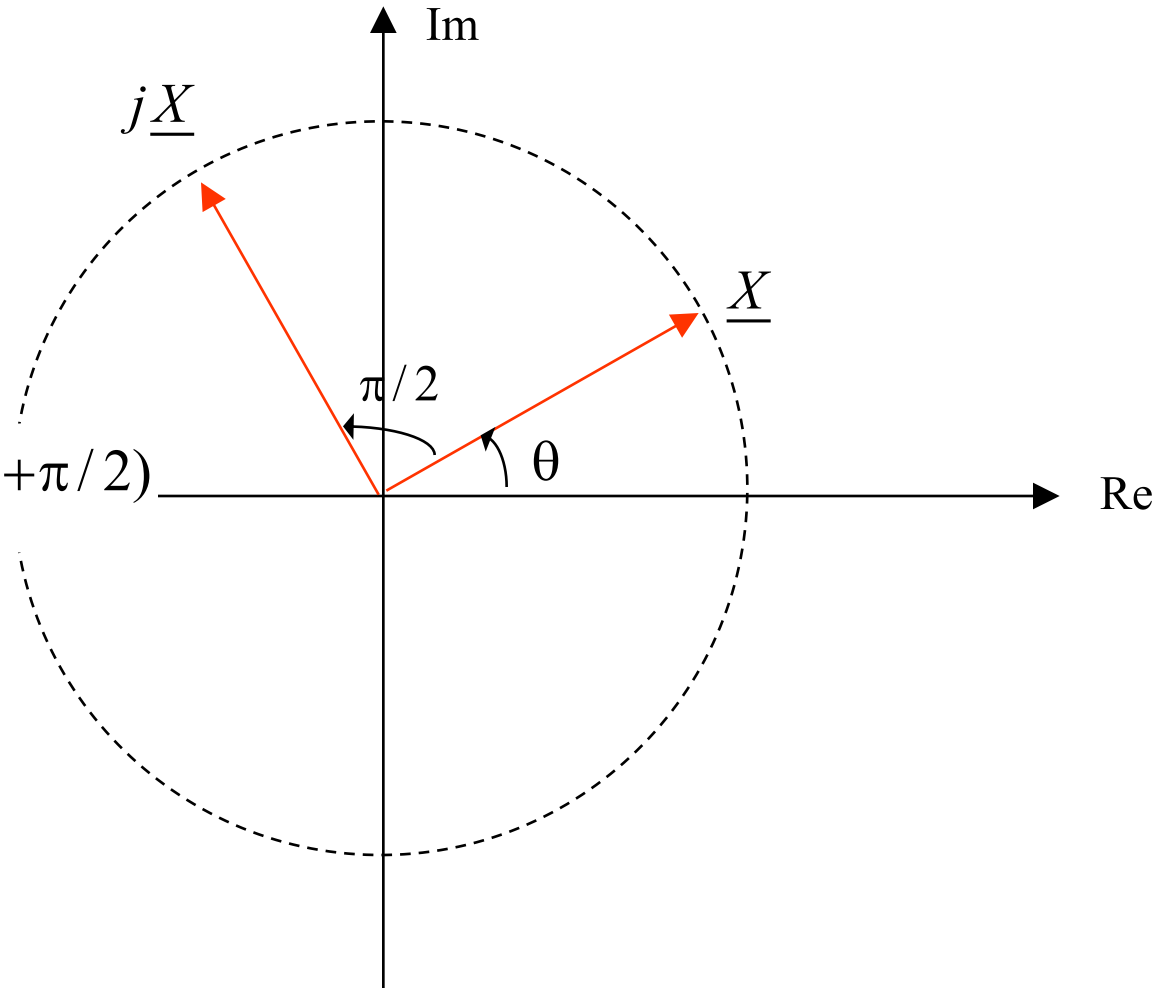
# Derivation and integration of a sinusoidal quantity

- The use of a complex representation of sinusoidal quantities makes it possible to replace the derivation and integration operations by multiplication or division by $j\omega$.

- Thus an integro-differential equation is transformed into an algebraic equation !

# Geometric interpretation

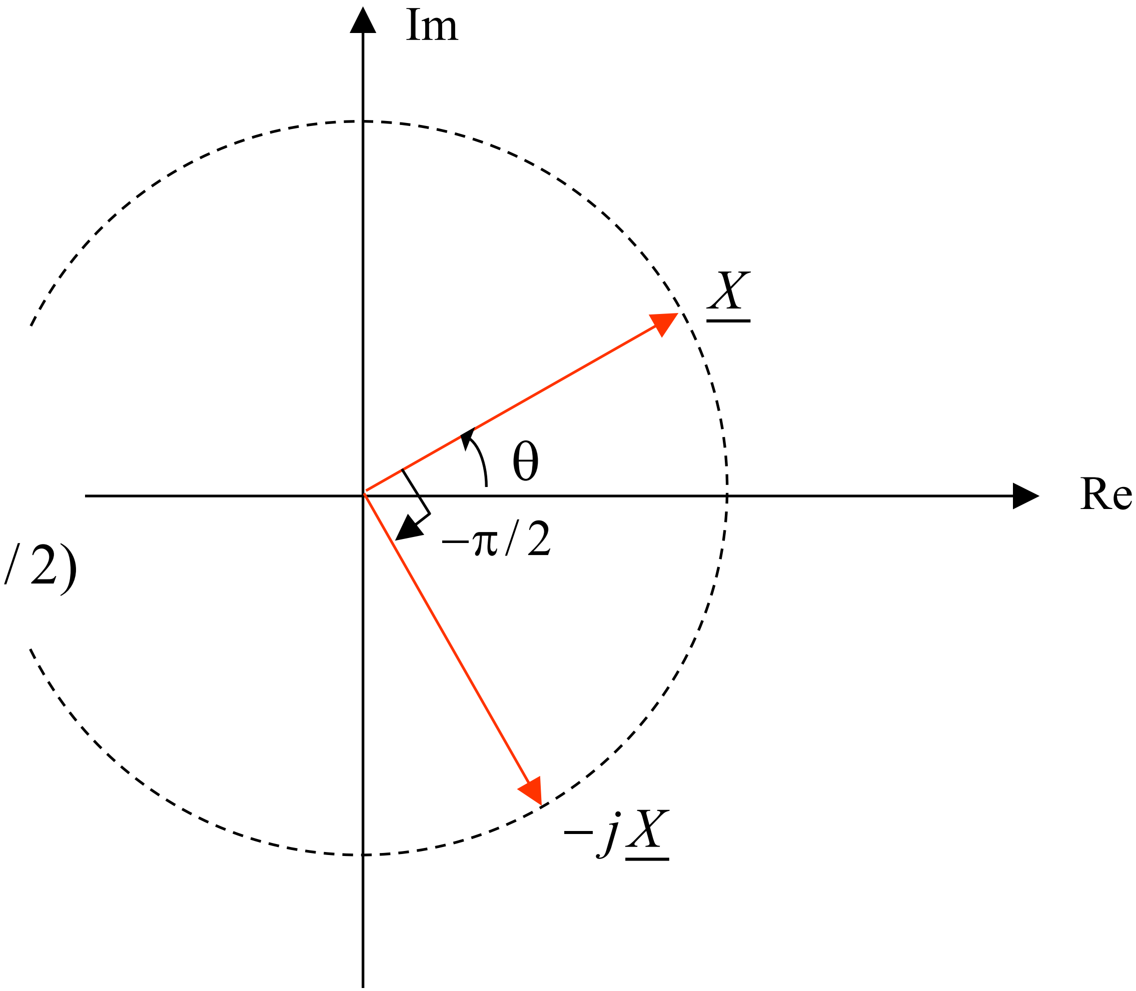
$$j\underline{X} = e^{j\pi/2}\underline{X} = Xe^{j(\theta+\pi/2)}$$



# Geometric interpretation

$$\frac{\underline{X}}{j} = -j\,\underline{X}$$

$$= e^{-j\pi/2}\,\underline{X} = Xe^{j(\theta-\pi/2)}$$



# Impedance and admittance

- The complex impedance of a bipole in a sinusoidal circuit :

$$\underline{Z} = \frac{\underline{u}}{\underline{i}} = \frac{\underline{U}}{\underline{I}}$$

- The complex admittance of a bipole in a sinusoidal circuit :

$$\underline{Y} = \frac{\underline{i}}{\underline{u}} = \frac{\underline{I}}{\underline{U}} = \frac{1}{\underline{Z}}$$

# Impedance and admittance

$$\underline{U} = Ue^{j\alpha} \qquad \underline{I} = Ie^{j\beta}$$

$$\underline{Z} = \frac{\underline{U}}{\underline{I}} = \frac{Ue^{j\alpha}}{Ie^{j\beta}} = Ze^{j\varphi}$$

$$Z = \frac{U}{I} \qquad \varphi = \alpha - \beta$$

$$\underline{Y} = \frac{\underline{I}}{\underline{U}} = \frac{Ie^{j\beta}}{Ue^{j\alpha}} = Ye^{-j\varphi}$$

$$Y = \frac{I}{U} \qquad \varphi = \alpha - \beta$$

Z expressed in ohm

Y expressed in siemens

# Impedance and admittance

$$\underline{Z} = Ze^{j\varphi} = R + jX$$

R: resistance $R = Z\cos\varphi$

X: reactance $X = Z\sin\varphi$

$$Z = \sqrt{R^2 + X^2}$$

$$\varphi = \arctan(X / R)$$

$$\underline{Y} = Ye^{-j\varphi} = G + jB$$

G: conductance $G = Y\cos\varphi$

B: susceptance $B = -Y\sin\varphi$

$$Y = \sqrt{G^2 + B^2}$$

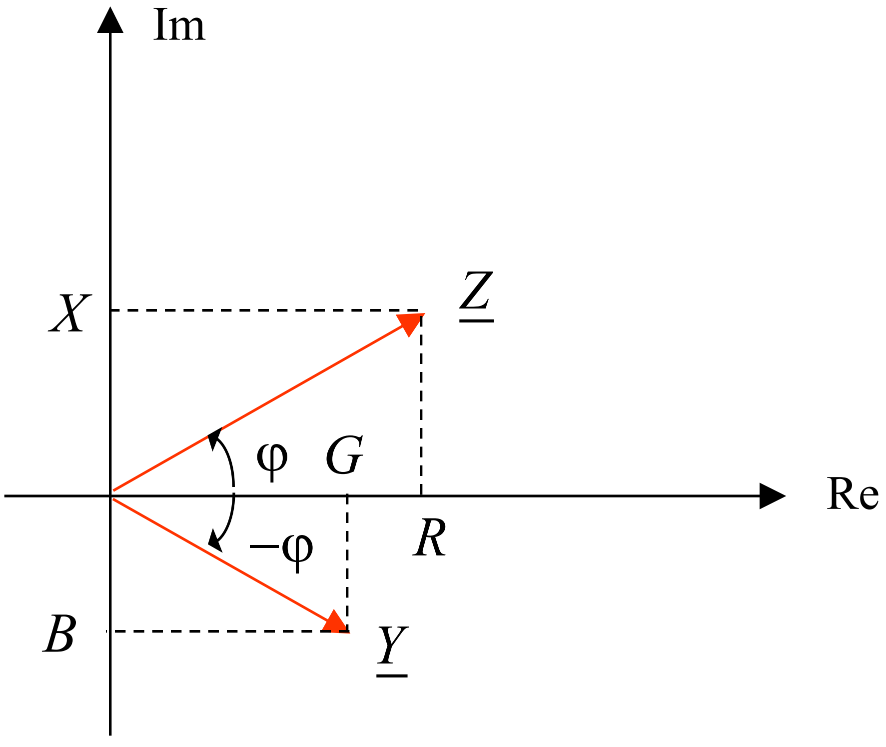
$$\varphi = \arctan(-B / G)$$

# Impedance and admittance

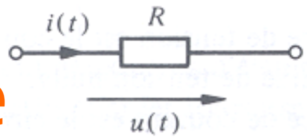
$$\underline{Z} = \frac{\underline{U}}{\underline{I}} = \frac{Ue^{j\alpha}}{Ie^{j\beta}} = Ze^{j\varphi} \qquad \underline{Y} = \frac{\underline{I}}{\underline{U}} = \frac{Ie^{j\beta}}{Ue^{j\alpha}} = Ye^{-j\varphi}$$

$$\underline{Z} = R + jX$$

$$\underline{Y} = G + jB$$

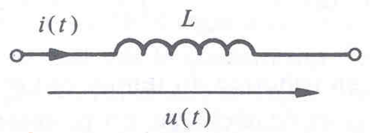


# Application to the resistance




| Instantaneous values | Phasors |
|---|---|
| $u(t) = \sqrt{2}U\cos(\omega t + \alpha)$ | $\underline{U} = Ue^{j\alpha}$ |
| $i(t) = \sqrt{2}I\cos(\omega t + \beta)$ | $\underline{I} = Ie^{j\beta}$ |
| $u(t) = Ri(t)$ | $\underline{U} = R\underline{I}$ |

$$\boxed{\underline{Z}_R = R, \quad Z_R = R, \quad \varphi_R = 0}$$




# Application to inductance

Instantaneous values

$$u(t)=\sqrt{2}U\cos(\omega t+\alpha)$$

$$i(t)=\sqrt{2}I\cos(\omega t+\beta)$$

$$u(t)=L\frac{di}{dt}$$

Phasors

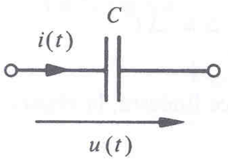
$$\underline{U}=Ue^{j\alpha}$$

$$\underline{I}=Ie^{j\beta}$$

$$\underline{U}=j\omega L\underline{I}$$

$$\boxed{\underline{Z}_L=j\omega L,\quad Z_L=\omega L,\quad \varphi_L=\pi/2}$$

$$R_L=0,\quad X_L=\omega L$$




# Application to capacitance

| Instantaneous values | Phasors |
| --- | --- |
| $u(t)=\sqrt{2}U\cos(\omega t+\alpha)$ | $\underline{U}=Ue^{j\alpha}$ |
| $i(t)=\sqrt{2}I\cos(\omega t+\beta)$ | $\underline{I}=Ie^{j\beta}$ |
| $i(t)=C\dfrac{du}{dt}$ | $\underline{I}=j\omega C\underline{U}$ |

$$\boxed{\underline{Z}_C=1/j\omega C,\quad Z_C=1/\omega C,\quad \varphi_C=-\pi/2}$$

$$R_C=0,\quad X_C=-1/\omega C$$

# Example

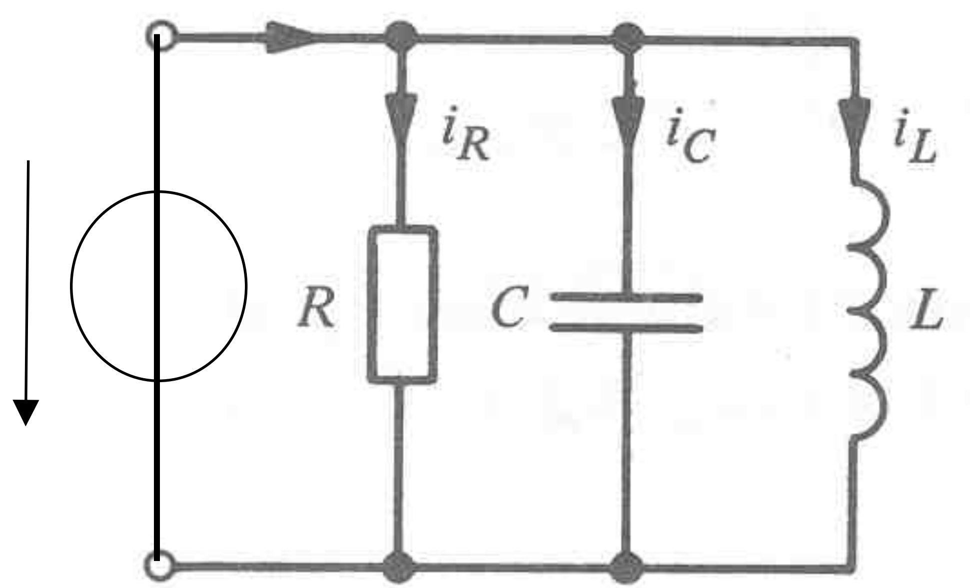
- Evaluate rms value, frequency and period of total current i(t)
- Analytically determine the current in each element and that provided by the source
- Draw each of these currents

$$u(t) = 170\cos(157.1t + \pi/6)$$

$$R = 170\Omega$$

$$C = 46.8\mu F$$

$$L = 0.722H$$



# Example

- Evaluate rms value, frequency and period of total current i(t)
- Analytically determine the current in each element and that provid
- Draw e

**Calculation on the board!**
**But this time using phasors!**

$$u(t) = 170\cos(157.1t + \pi/6)$$

$R = 170\Omega$

$C = 46.8\mu F$

$L = 0.722 H$