Fundamentals of Electrical Circuits and Systems

Chapter 4: AC Circuits 1

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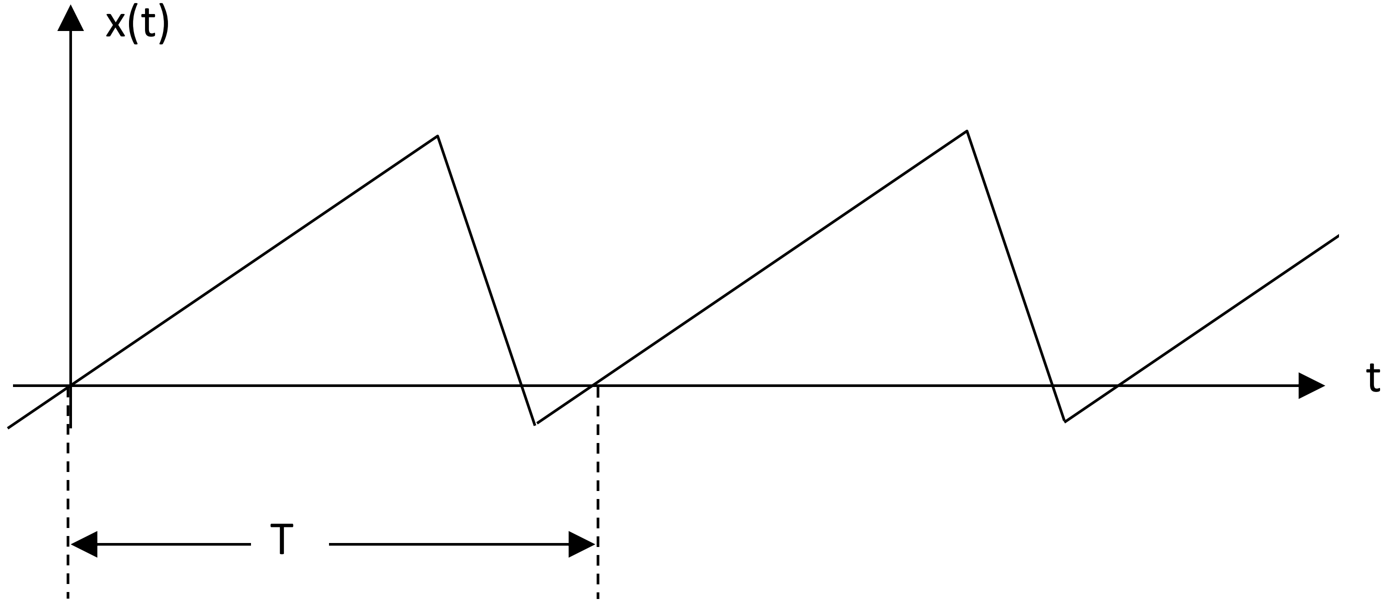
AC Circuits 1

- Periodic functions
- Sinusoidal quantities
- Importance of AC circuits
- Response of a linear circuit to a sinusoidal excitation
- Example

Periodic functions

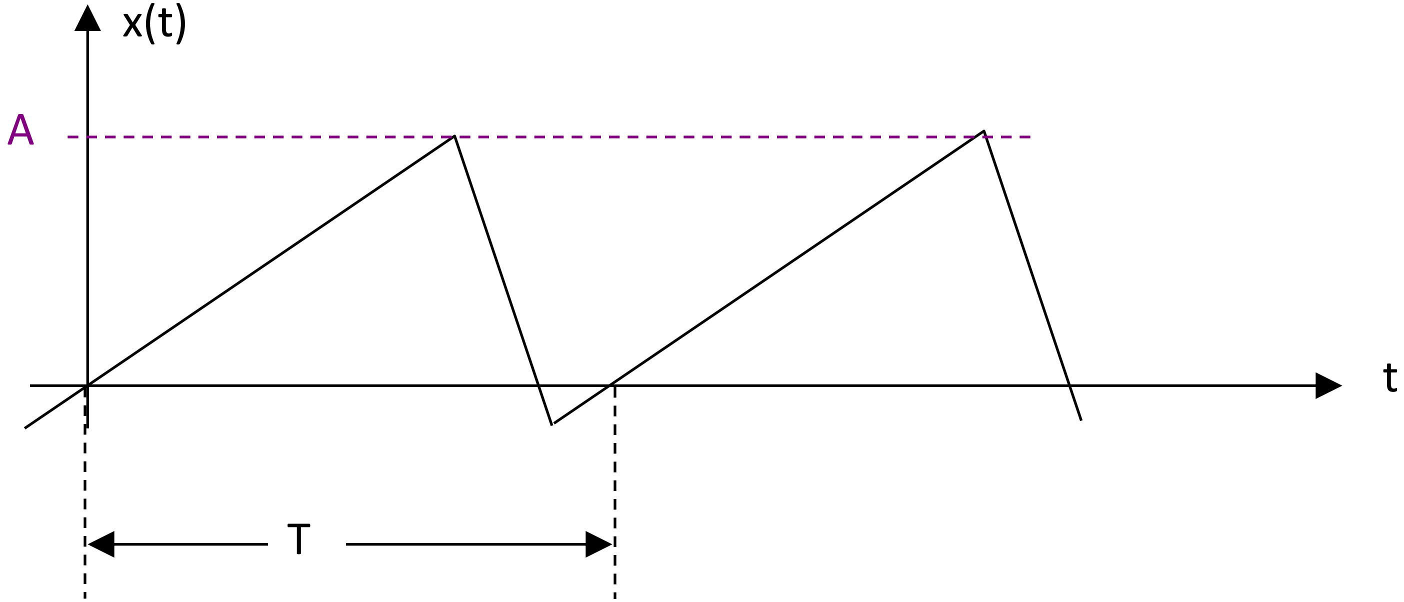
- A periodic function is a function that satisfies the relation $f(t) = f(t + nT)$, where n is an integer and T is the period measured in unit of time.



Some definitions

- Peak value or amplitude A :

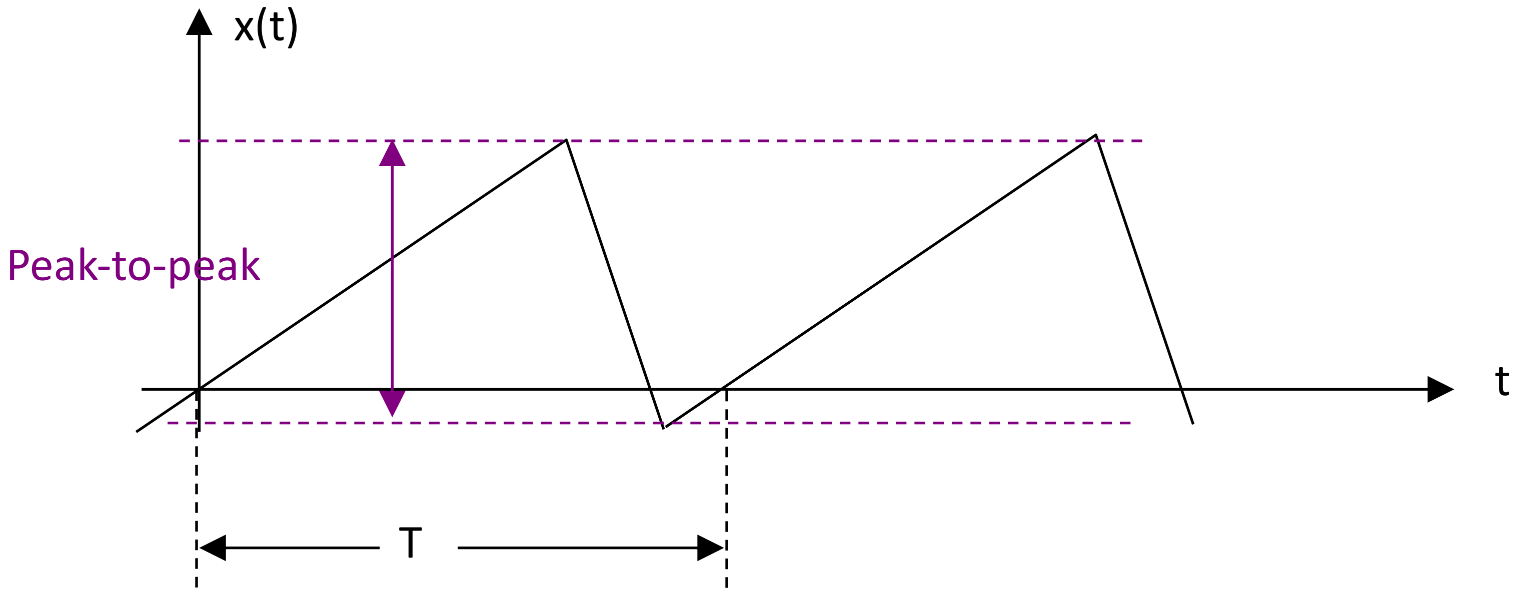
  maximum value of a periodic function



Some definitions

- Peak-to-peak value :

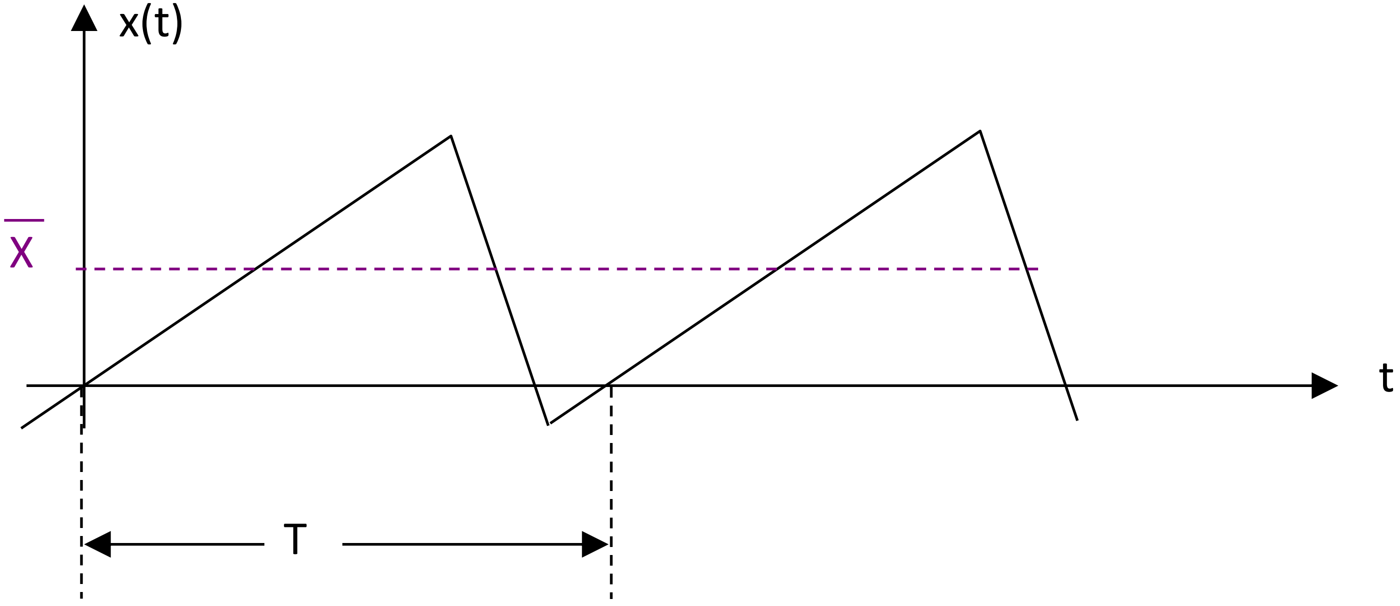
  maximum amplitude difference reached during a period



Some definitions

- Mean value:

$$\overline{X} = \frac{1}{T}\int_{t}^{t+T} x(\tau)\,d\tau$$



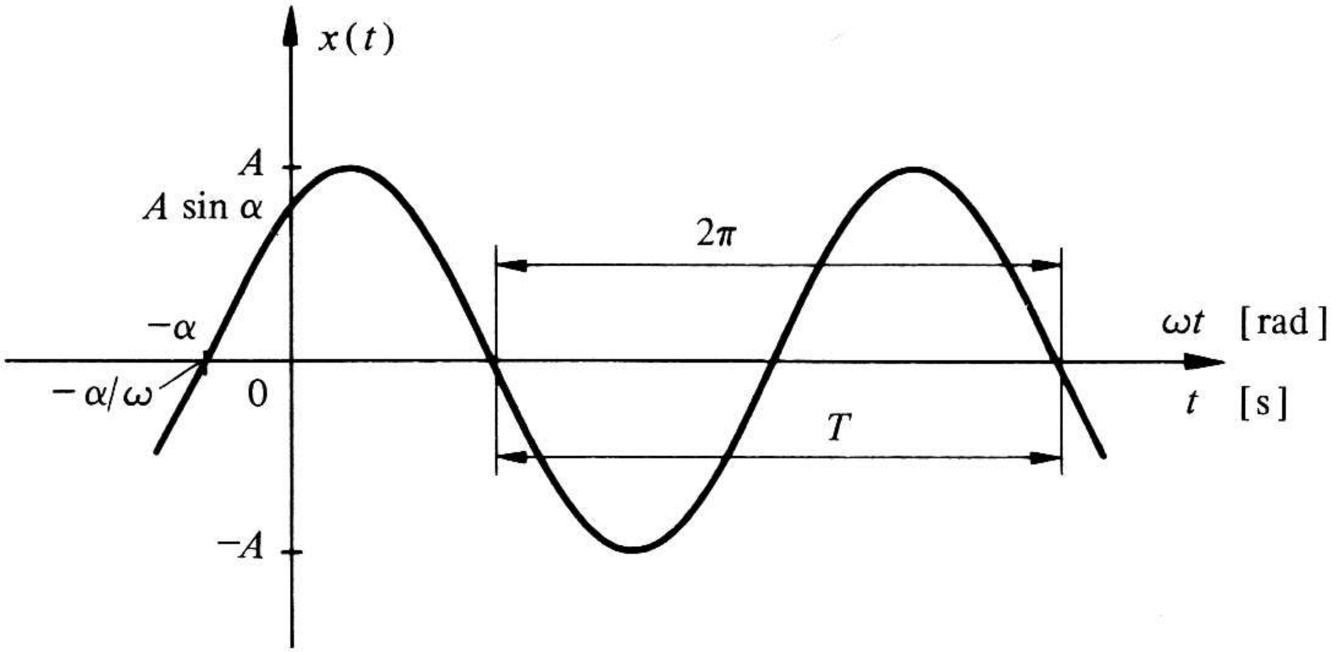
Quelques définitions

- Root Mean Square (RMS): $$X = \sqrt{\frac{1}{T}\int_{t}^{t+T} x^2(\tau)\,d\tau}$$

The rms value is always positive!

Sinusoidal function

$$x(t) = A\sin(\frac{2\pi}{T}t + \alpha)$$



Sinusoidal function: frequency

- Frequency: Number of cycles per unit of time

$$f = 1/T$$

$$x(t) = A\sin(\frac{2\pi}{T}t + \alpha) = A\sin(2\pi f t + \alpha)$$

Sinusoidal function: frequency

- The unit of measure of the frequency: hertz (Hz)
- A hertz corresponds to the frequency of a periodic phenomenon whose period T is a second

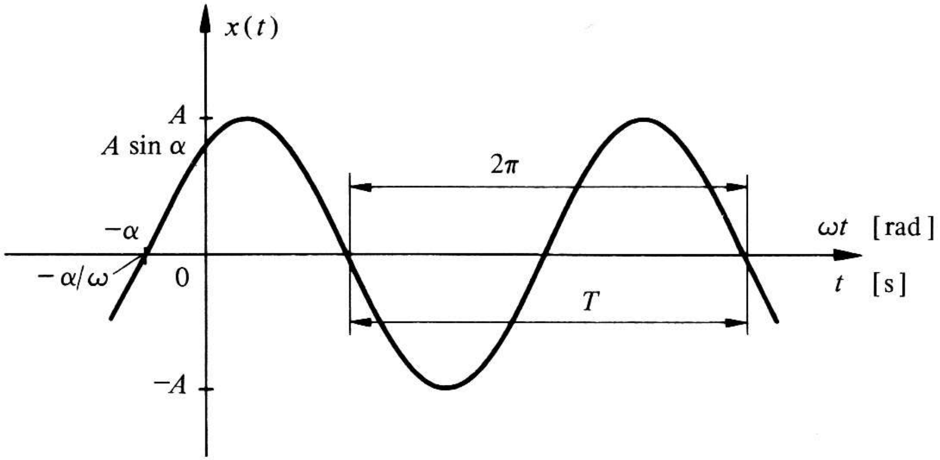
Heinrich **Hertz** (1857-1894), German physicist. His work confirmed Maxwell's electromagnetic theory of light.

Sinusoidal function: angular frequency or pulsation

- Angular frequency or pulsation : ω
- Unit rad/s

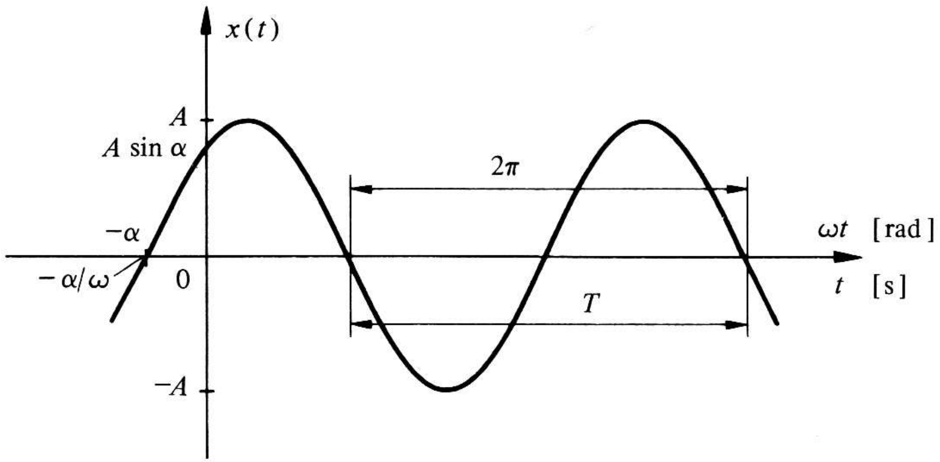
$$\omega = 2\pi f = \frac{2\pi}{T}$$

$$\begin{aligned} x(t) &= A\sin\left(\frac{2\pi}{T}t + \alpha\right) \\ &= A\sin(2\pi f t + \alpha) \\ &= A\sin(\omega t + \alpha) \end{aligned}$$



Sinusoidal function: mean and rms values

$$x(t) = A\sin(\omega t + \alpha)$$



- Mean value

$$\overline{X} = 0$$

- Rms value

$$\sin^2 a = \frac{1}{2} - \frac{1}{2}\cos 2a$$

$$X = \sqrt{\frac{A^2}{T}\int_0^T \sin^2(\omega t + \alpha)dt} = \sqrt{\frac{A^2}{T}\int_0^T \frac{1}{2}dt - \frac{A^2}{2T}\int_0^T \cos(2\omega t + 2\alpha)dt} = \frac{A}{\sqrt{2}}$$

Rms value of a sinusoidal quantity:

$$u(t) = \hat{U}\cos(\omega t + \alpha)$$

$$i(t) = \hat{I}\cos(\omega t + \beta)$$

- Rms values

$$U = \hat{U}/\sqrt{2} \qquad I = \hat{I}/\sqrt{2}$$

$$\Rightarrow \begin{cases} u(t) = \sqrt{2}U\cos(\omega t + \alpha) \\ i(t) = \sqrt{2}I\cos(\omega t + \beta) \end{cases}$$

Question

- When we talk about a sinusoidal voltage of 230 V, it is about

A. Its peak value

B. Its mean value

C. Its rms value

D. None of the above

Phase difference between two sinusoidal quantities

$$u(t) = \hat{U}\cos(\omega t + \alpha)$$

$$i(t) = \hat{I}\cos(\omega t + \beta)$$

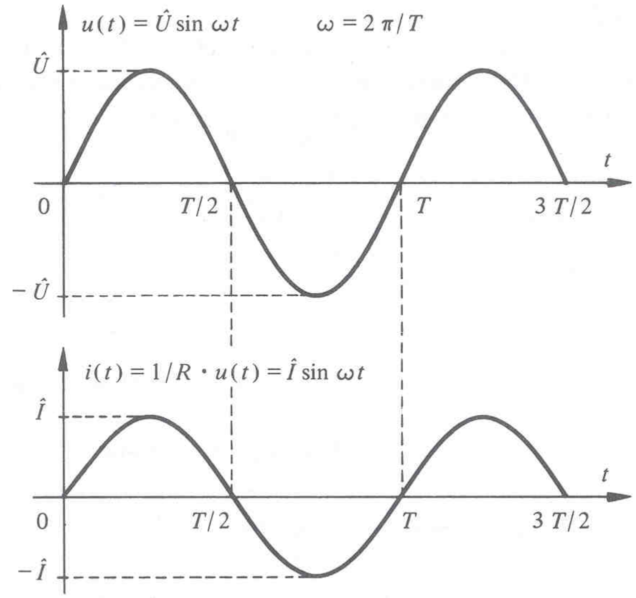
- Phase difference between *u*(*t*) and *i*(*t*): $\varphi = \alpha - \beta$
- Note: we always consider the main value of the phase difference between -π and π.
- $\varphi > 0$: voltage ahead of the current
- $\varphi < 0$: voltage behind the current

Heating of a resistor

- When the voltage is sinusoidal, the average power dissipated in a resistor is equal to the integral, over a period of time, of the product of instantaneous current and voltage.

$$u(t) = \hat{U}\sin(\omega t)$$

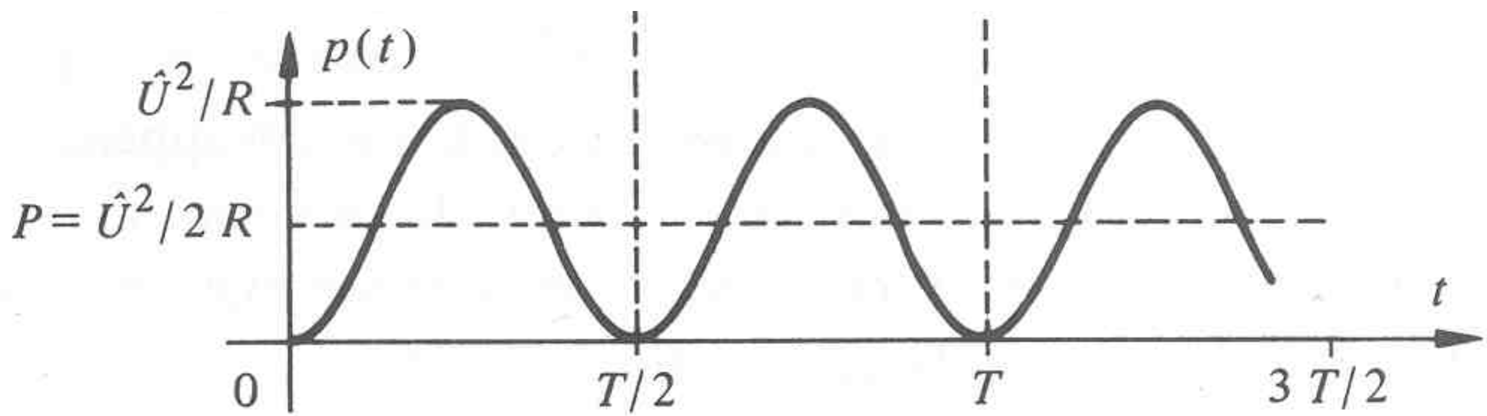
$$i(t) = \frac{\hat{U}}{R}\sin(\omega t)$$



Heating of a resistor

$$u(t) = \hat{U}\sin(\omega t) \qquad\qquad i(t) = \frac{\hat{U}}{R}\sin(\omega t)$$

$$p(t) = u(t)i(t) = Ri^2(t) = \frac{u^2(t)}{R} = \frac{\hat{U}^2}{R}\sin^2(\omega t)$$



Heating of a resistor

$$P = \frac{1}{T}\int_0^T u(t)i(t)dt = \frac{1}{T}\frac{\hat{U}^2}{R}\int_0^T \sin^2(\omega t)dt = \frac{\hat{U}^2}{2R}$$

$$U = \frac{\hat{U}}{\sqrt{2}} \Rightarrow P = \frac{U^2}{R}$$

$$\left.\begin{array}{l} \hat{U} = R\hat{I} \\ I = \dfrac{\hat{I}}{\sqrt{2}} \end{array}\right\} \Rightarrow P = RI^2$$

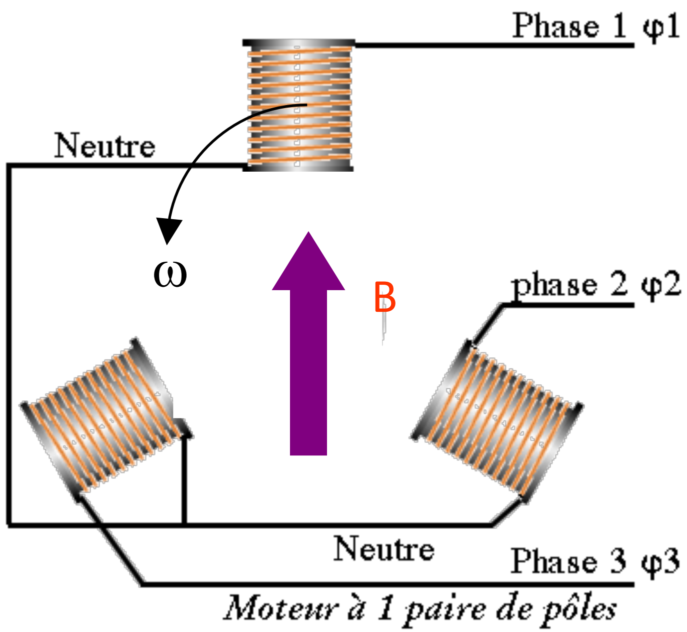
Heating of a resistor

$$P = \frac{U^2}{R} \qquad\qquad P = RI^2$$

- The rms value has been defined so that a continuous volt or 1 volt alternating produces the same heating in a resistor!

Importance of AC circuits

- The production of electrical energy provides a sinusoidal voltage: conversion mechanical energy - electrical energy: rotation of a coil placed in a magnetic field



Importance of AC circuits

- The only periodic function that has a derivative and a similar integral

$$x(t) = A\cos(\omega t + \alpha)$$

$$\frac{dx}{dt} = -A\omega\sin(\omega t + \alpha) = A\omega\cos(\omega t + \alpha + 90^{o})$$

$$\int x(t)dt = \frac{A}{\omega}\sin(\omega t + \alpha) = \frac{A}{\omega}\cos(\omega t + \alpha - 90^{o})$$

Importance of AC circuits

- The sum of two sinusoidal functions is a sinusoidal function

$$x(t) = \cos(\omega t + \alpha)$$

$$y(t) = \cos(\omega t + \beta)$$

$$x(t) + y(t) = 2\cos(\frac{\alpha - \beta}{2})\cos(\omega t + \frac{\alpha + \beta}{2})$$

Importance of AC circuits

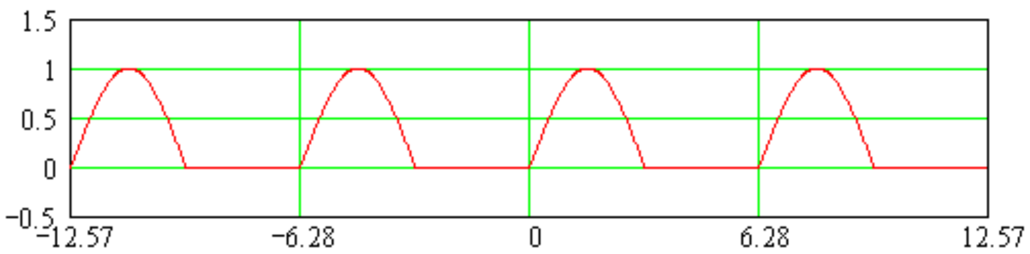
- Fourier series development: representation of a periodic signal f(t) by sinusoidal functions

$$f(t) = A_o + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + b_n \sin(n\omega t)$$

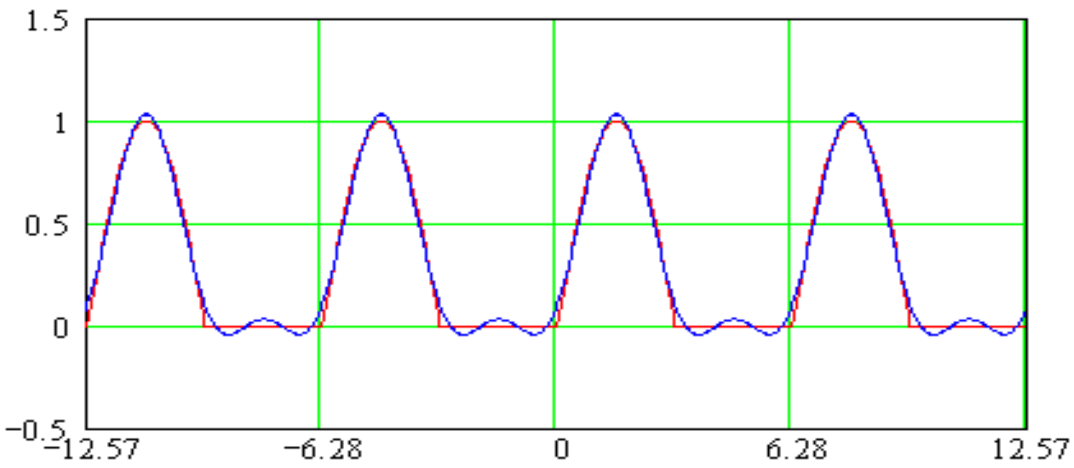
with
$$\begin{cases} A_o = \dfrac{1}{T} \displaystyle\int_{t_o}^{t_o+T} f(t)\,dt \\ a_n = \dfrac{1}{T} \displaystyle\int_{t_o}^{t_o+T} f(t)\cos(n\omega t)\,dt \\ b_n = \dfrac{1}{T} \displaystyle\int_{t_o}^{t_o+T} f(t)\sin(n\omega t)\,dt \end{cases}$$

Representation of a periodic signal by sinusoidal functions: Example 1

Rectified sinusoidal function:

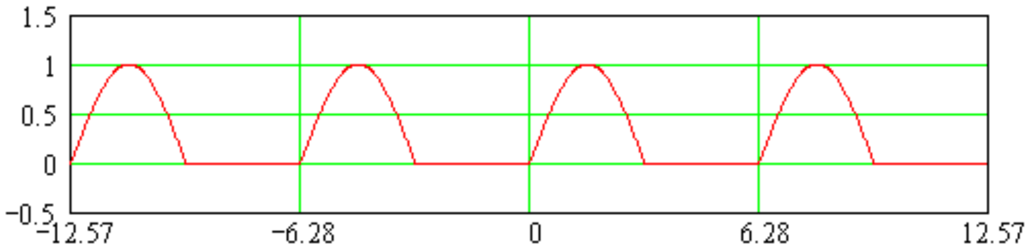


2 first terms of the Fourier series:

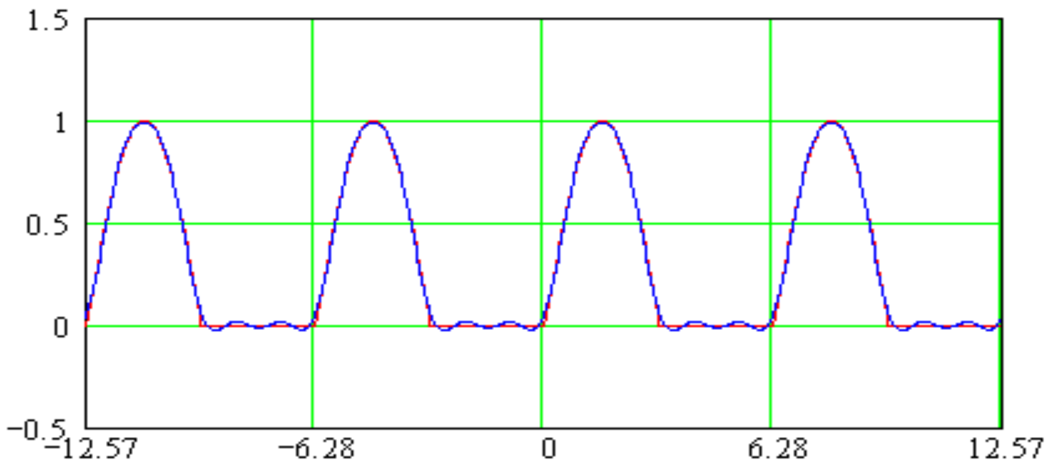


Representation of a periodic signal by sinusoidal functions: Example 1

Rectified sinusoidal function:

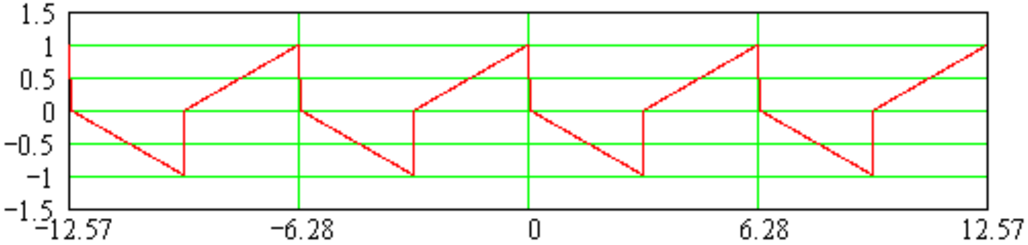


4 first terms of the Fourier series:

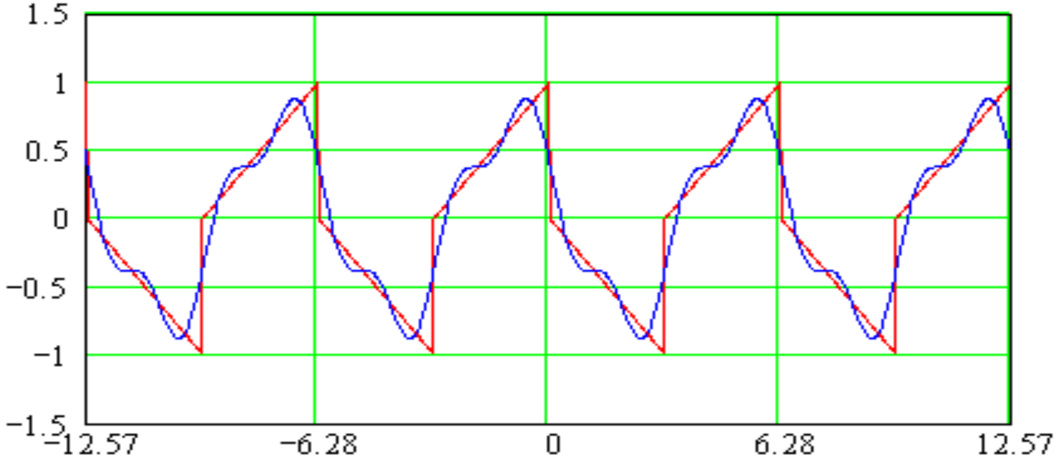


Representation of a periodic signal by sinusoidal functions: Example 2

Triangular function:



4 first terms of the Fourier series:



Representation of a periodic signal by sinusoidal functions

- Java Applet
  http://www.jhu.edu/~signals/fourier2/index.html

Importance of AC circuits

- Fourier transform:

  Generalization of the Fourier series

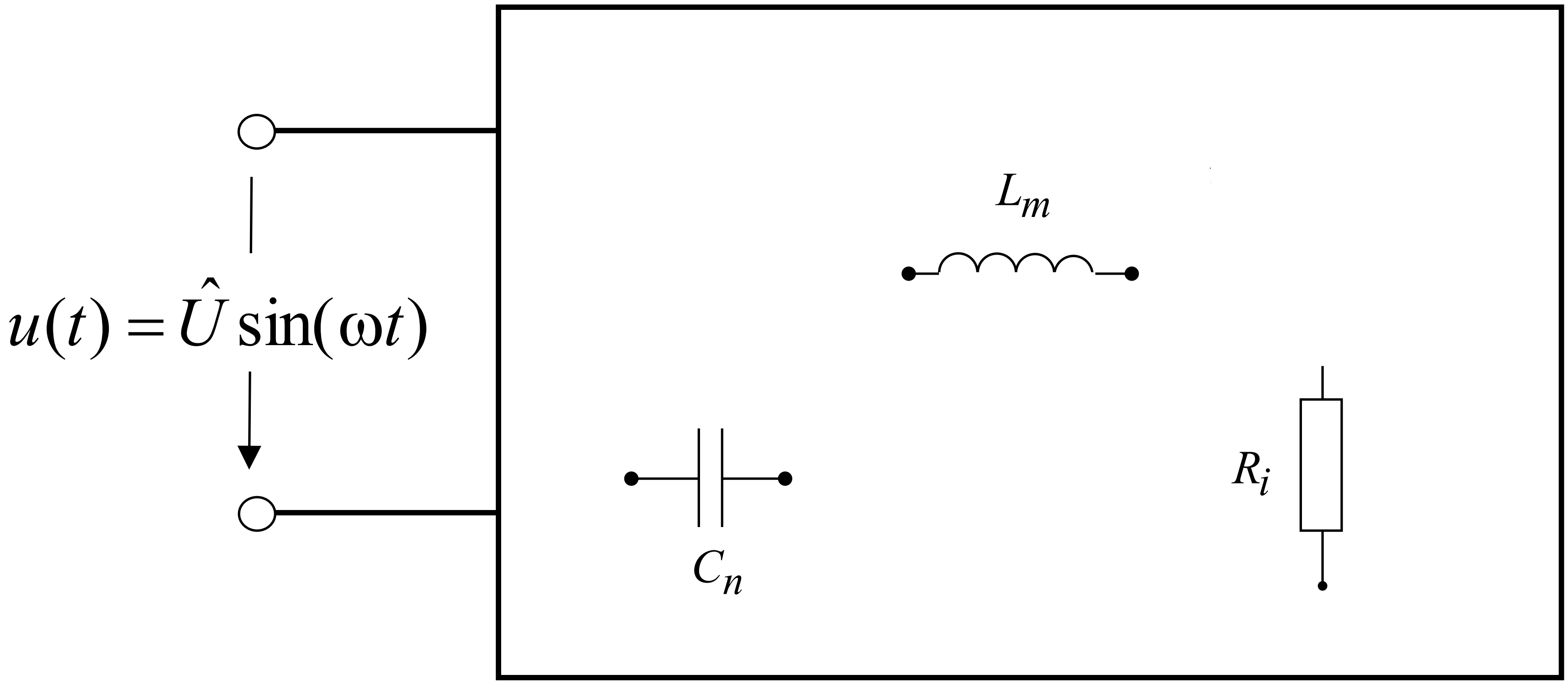
  Frequency analysis of non-periodic signals

Question

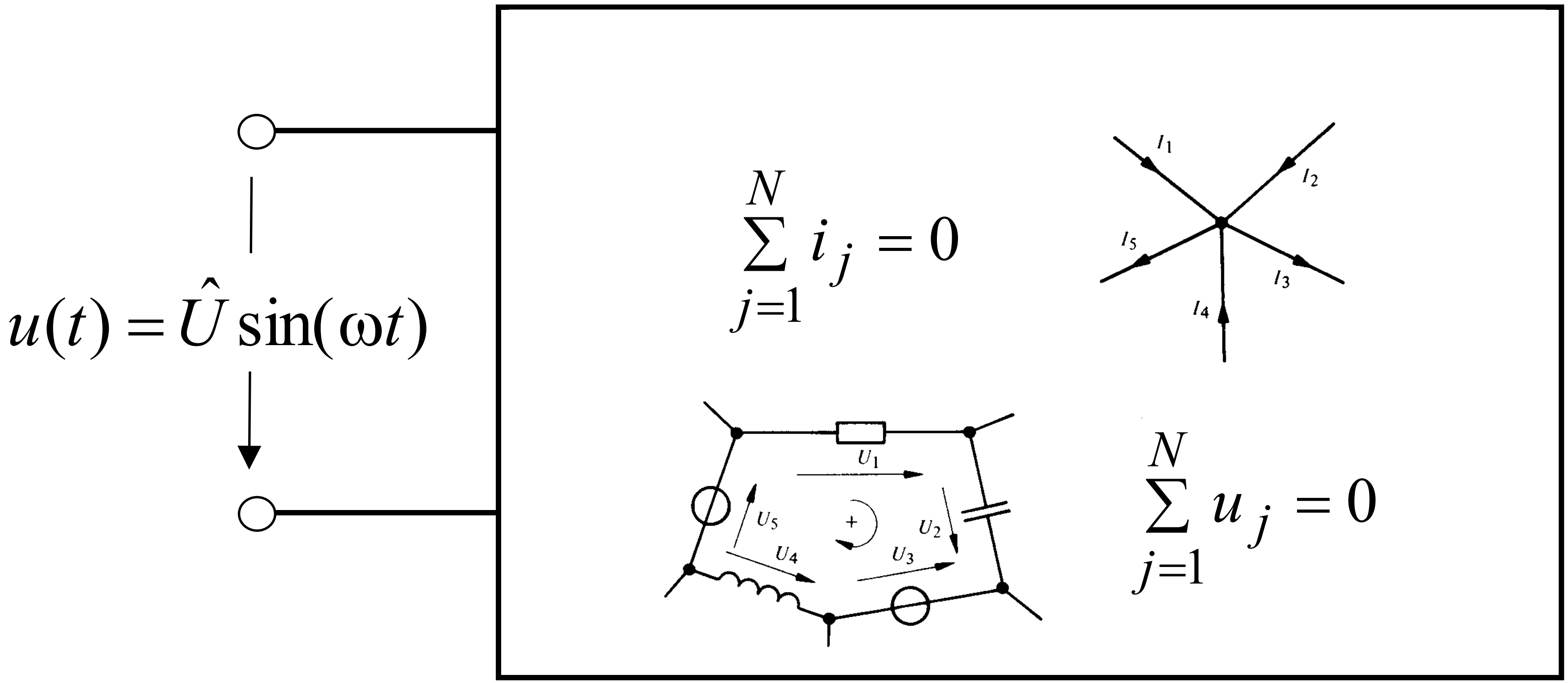
- A sinusoidal voltage is applied to a linear circuit (R, L, C). What can be said of the currents and voltages in the circuit?

A. They are all sinusoidal
B. They are all sinusoidal with the same frequency (as that of the source)
C. They are all periodic
D. None of the above

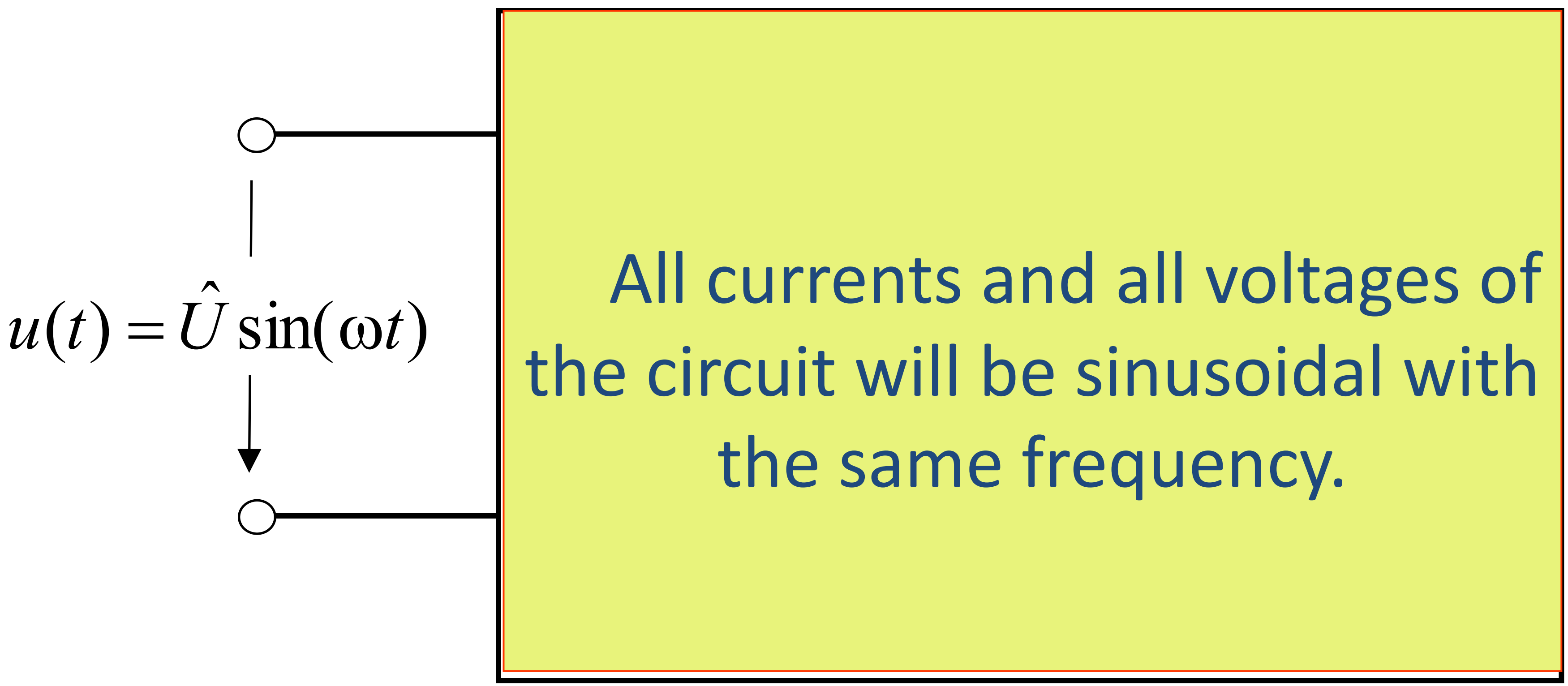
Response of a linear circuit to a sinusoidal excitation



Response of a linear circuit to a sinusoidal excitation



Response of a linear circuit to a sinusoidal excitation



Example

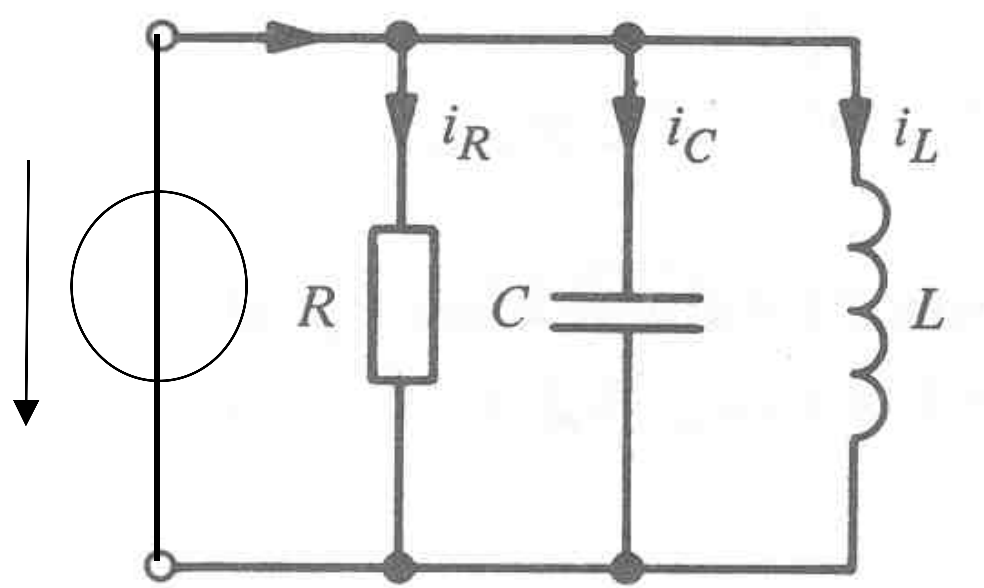
- Evaluate rms value, frequency and period of total current i(t)
- Analytically determine the current in each element and that provided by the source
- Draw each of these currents

$$u(t) = 170\cos(157.1t + \pi/6)$$

$$R = 170\Omega$$

$$C = 46.8\mu F$$

$$L = 0.722H$$



Example

- Evaluate rms value, frequency and period of total current i(t)
- Analytically determine the current in each element and that provided by the source
- Draw e

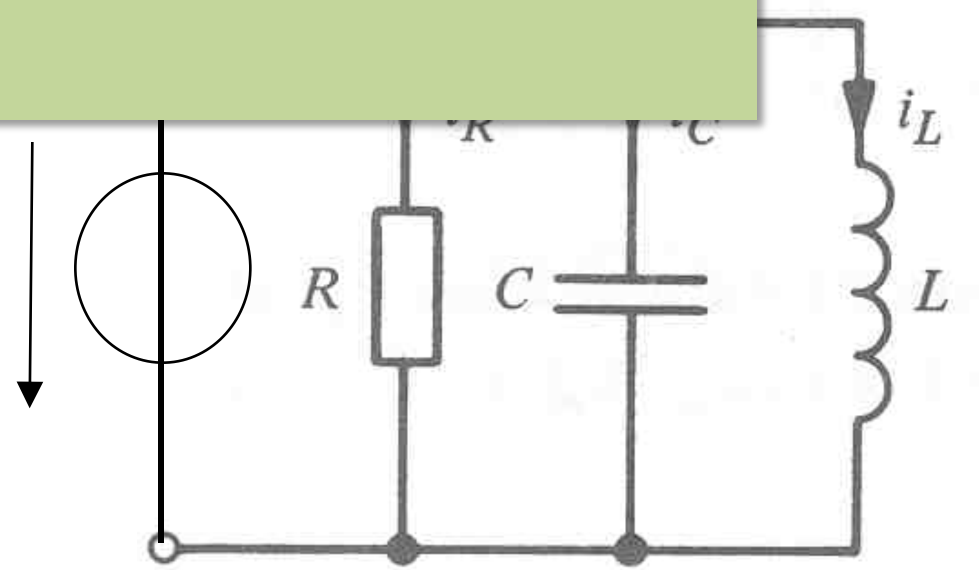
Calculation on the board!

$$u(t) = 170\cos(157.1t + \pi/6)$$

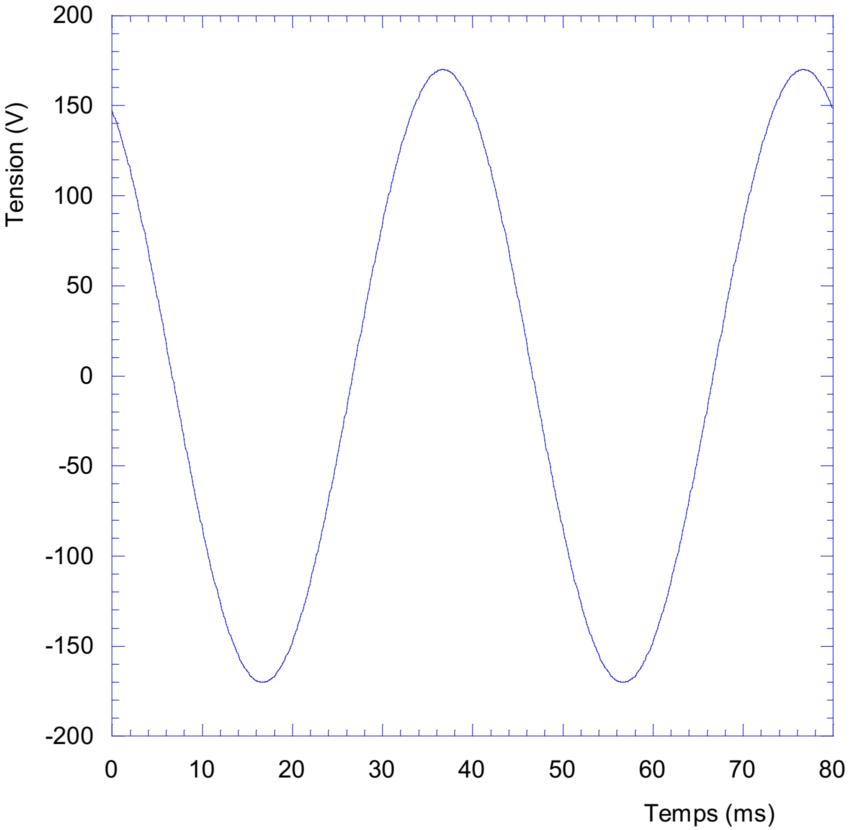
$R = 170\Omega$

$C = 46.8\mu F$

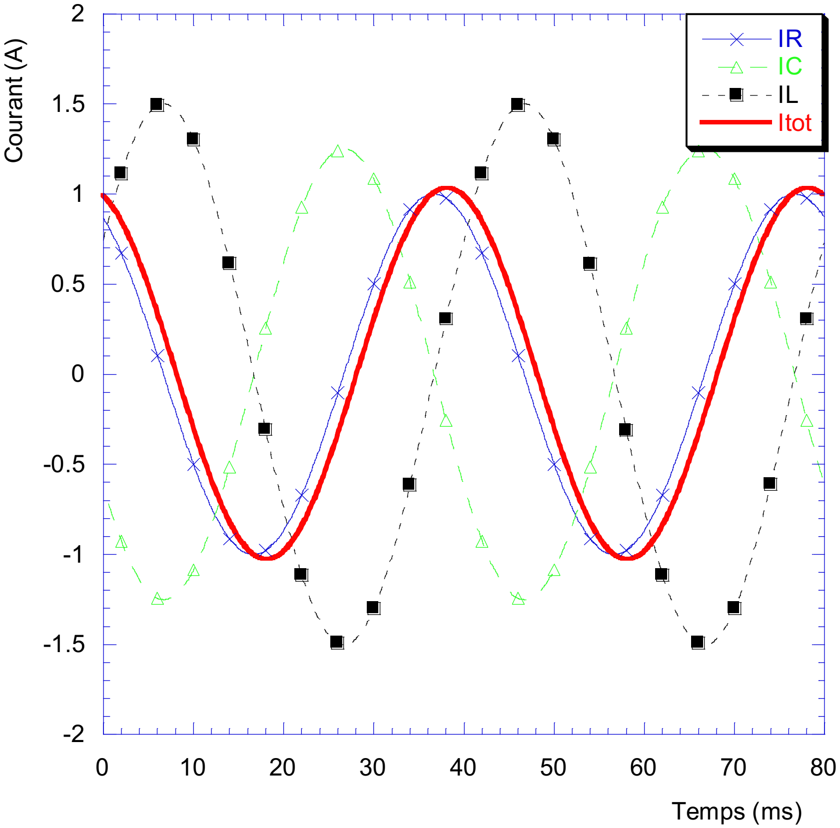
$L = 0.722H$



Example



Tension (V)
200
150
100
50
0
-50
-100
-150
-200
0
10
20
30
40
50
60
70
80
Temps (ms)



Courant (A)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
10
20
30
40
50
60
70
80
Temps (ms)
IR
IC
IL
Itot

Example

- This example shows us that the solution for a simple circuit is already laborious and would quickly become unusable for complex problems.

→ Phasors!