Fundamentals of Electrical Circuits and Systems

Chapter 3: Methods of Analysis

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Analysis Methods

- Nodal analysis
  - Based on the systematic application of Kirchhoff's law on currents
- Mesh analysis
  - Based on the sytematic application of Kirchhoff's law on voltages

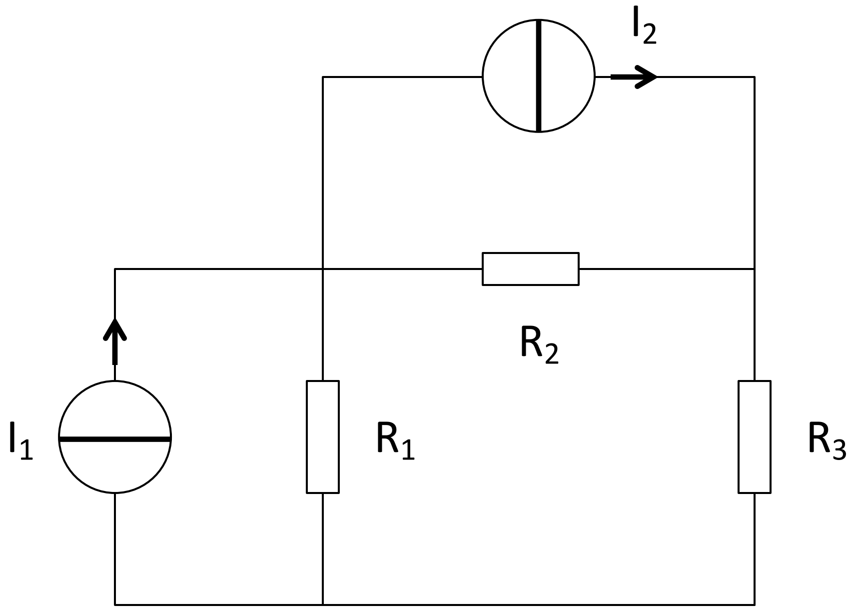
Nodal analysis

- General circuit analysis procedure using node voltages as circuit variables.
- In the nodal analysis, we are interested to find the node voltages.
- We will first consider a resistive circuit with n nodes and without voltage source.

Nodal analysis: Steps to determine node voltages

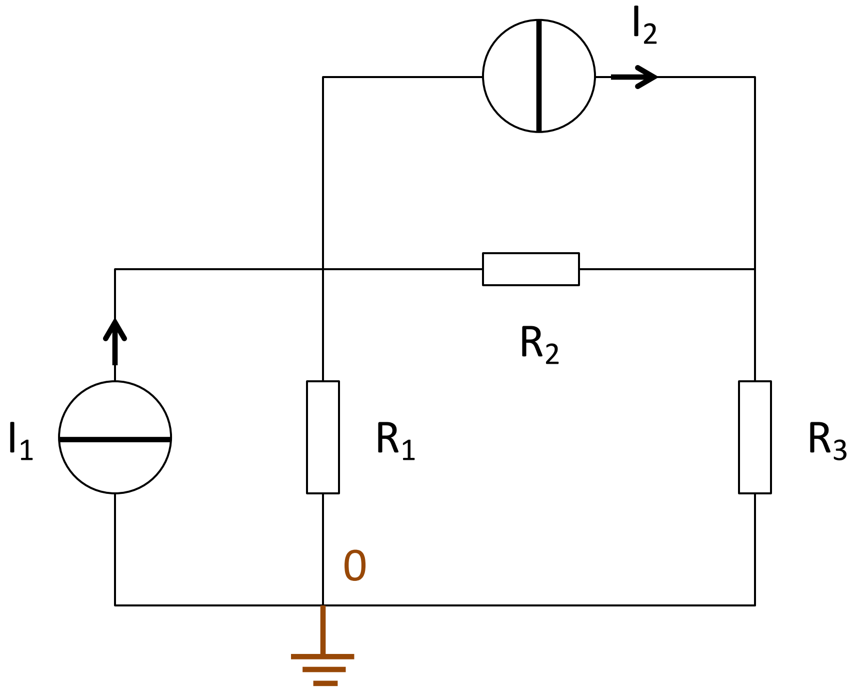
1. Select node n as the reference node. Assign the voltages v_1, v_2, …v_{n-1} to the other (n-1) nodes. Potential differences are established relative to the reference node.
2. Apply Kirchoff's law (currents) to each of the n-1 nodes. Express the currents of the branches.
3. Solve the system of equations to obtain the values of unknown voltages.

Nodal analysis: Application example



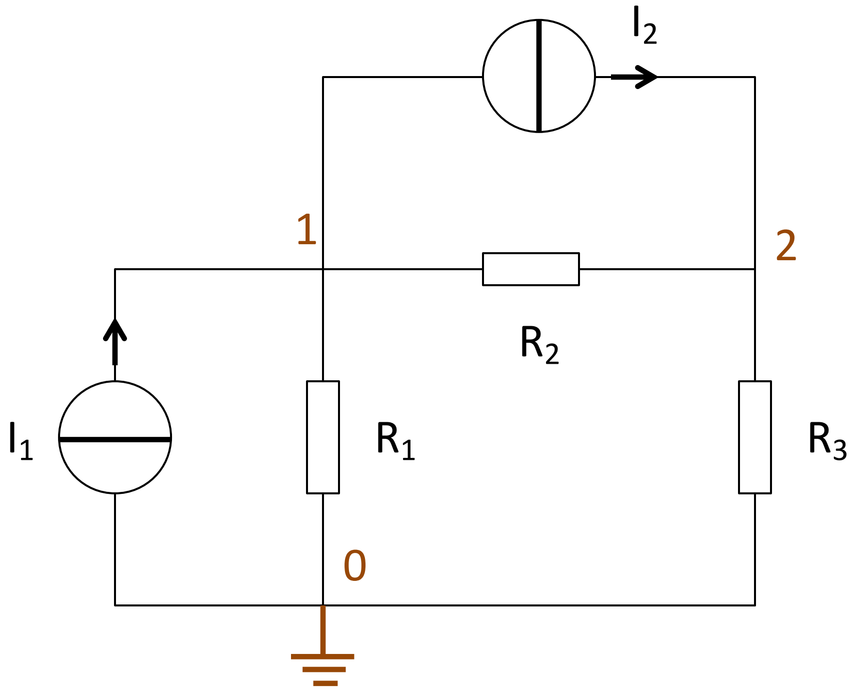
Nodal analysis: Application example

Step 1: selection of the reference node



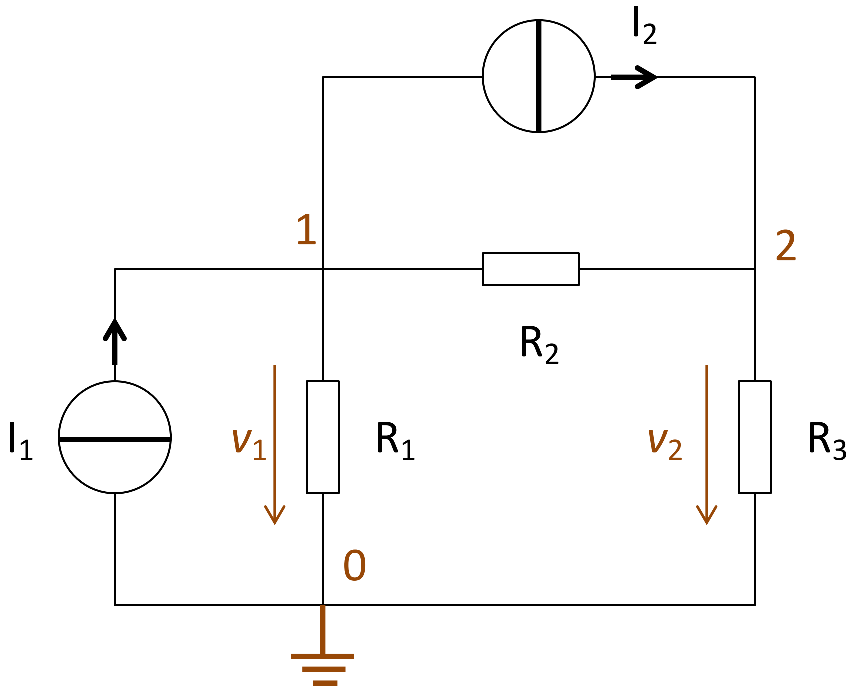
Nodal analysis: Application example

Step 1: the other nodes



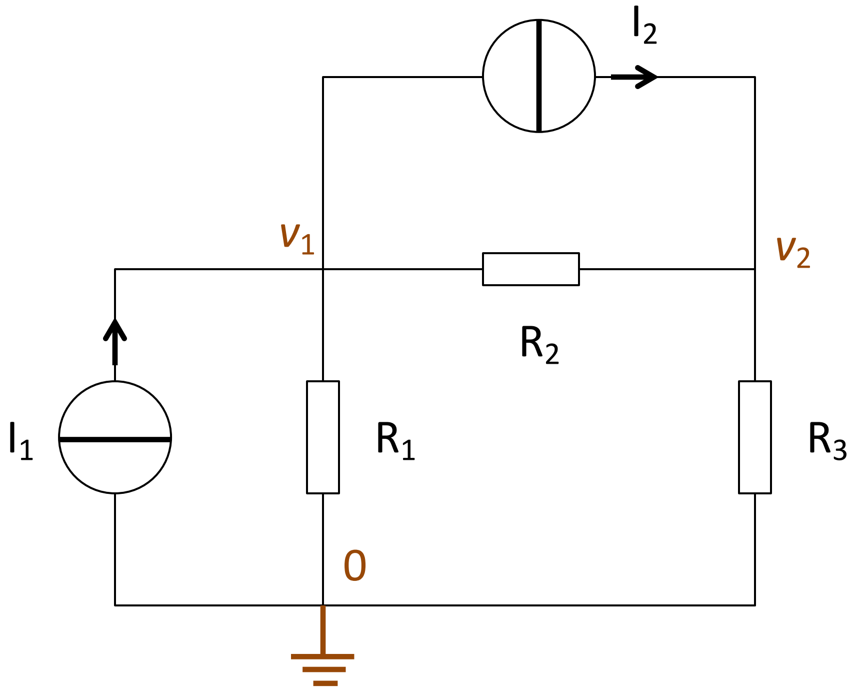
Nodal analysis: Application example

Step 1: allocation of voltages to the nodes



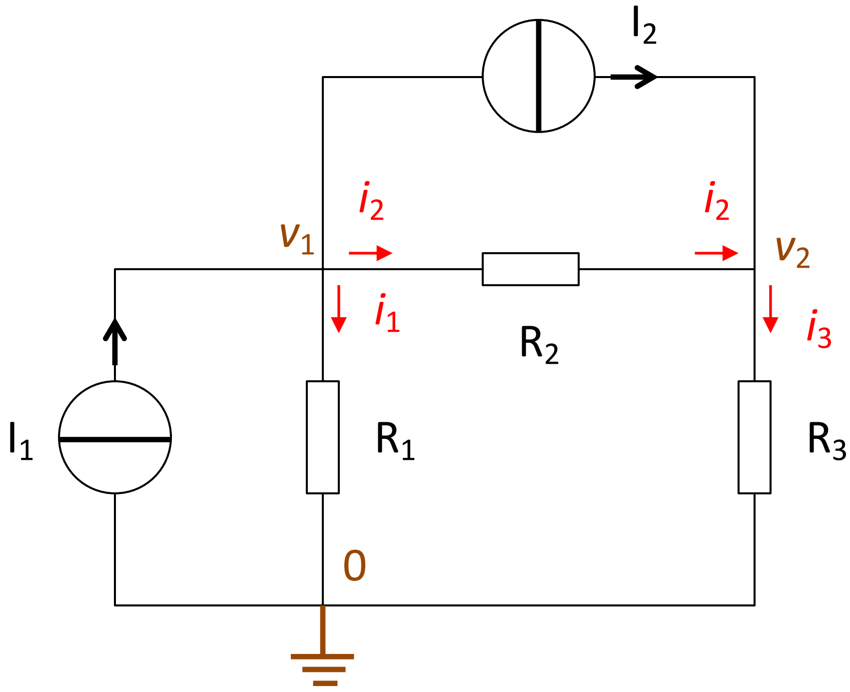
Nodal analysis: Application example

Step 1: allocation of voltages at the nodes



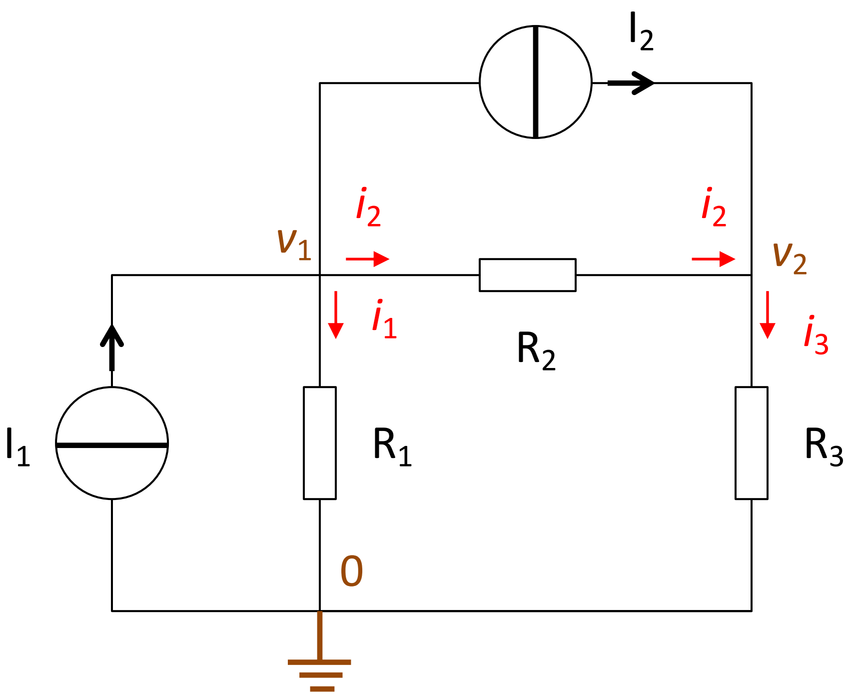
Nodal analysis: Application example

Step 2: application of Kirchoff's law (currents)



Nodal analysis: Application example

Step 2: application of Kirchoff's law (currents)



Node 1:

$$I_1 = I_2 + i_1 + i_2$$

Node 2:

$$I_2 + i_2 = i_3$$

Current-voltage relations:

$$i_1 = \frac{v_1}{R_1} \qquad i_2 = \frac{v_1 - v_2}{R_2} \qquad i_3 = \frac{v_2}{R_3}$$

By substituting these relations in the Kirchoff equations at nodes 1 and 2, we obtain (see following slide)

Nodal analysis: Application example

$$I_1 = I_2 + \frac{v_1}{R_1} + \frac{v_1 - v_2}{R_2} \qquad I_2 + \frac{v_1 - v_2}{R_2} = \frac{v_2}{R_3}$$

In terms of conductances, these equations become:

$$I_1 = I_2 + G_1 v_1 + G_2(v_1 - v_2) \qquad I_2 + G_2(v_1 - v_2) = G_3 v_2$$

In matrix form:

$$\begin{bmatrix} G_1 + G_2 & -G_2 \\ -G_2 & G_2 + G_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} I_1 - I_2 \\ I_2 \end{bmatrix}$$

Nodal analysis: Application example

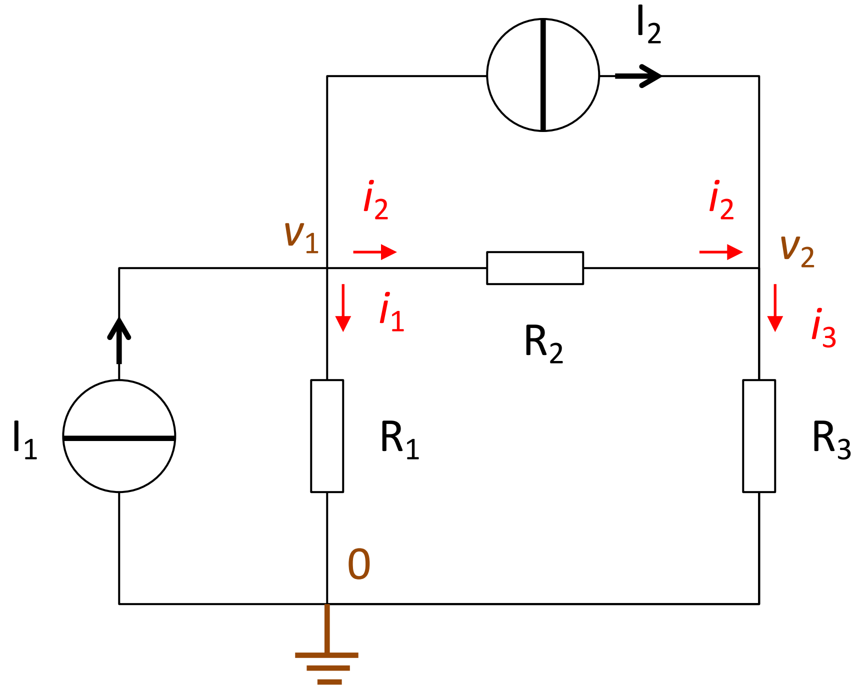
$$\begin{bmatrix} G_1 + G_2 & -G_2 \\ -G_2 & G_2 + G_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} I_1 - I_2 \\ I_2 \end{bmatrix}$$

Solution:

- Substitution method
- Elimination method
- Cramer's rule
- Matrix inversion

Nodal analysis by circuit inspection

$$\begin{bmatrix} G_1 + G_2 & -G_2 \\ -G_2 & G_2 + G_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} I_1 - I_2 \\ I_2 \end{bmatrix}$$



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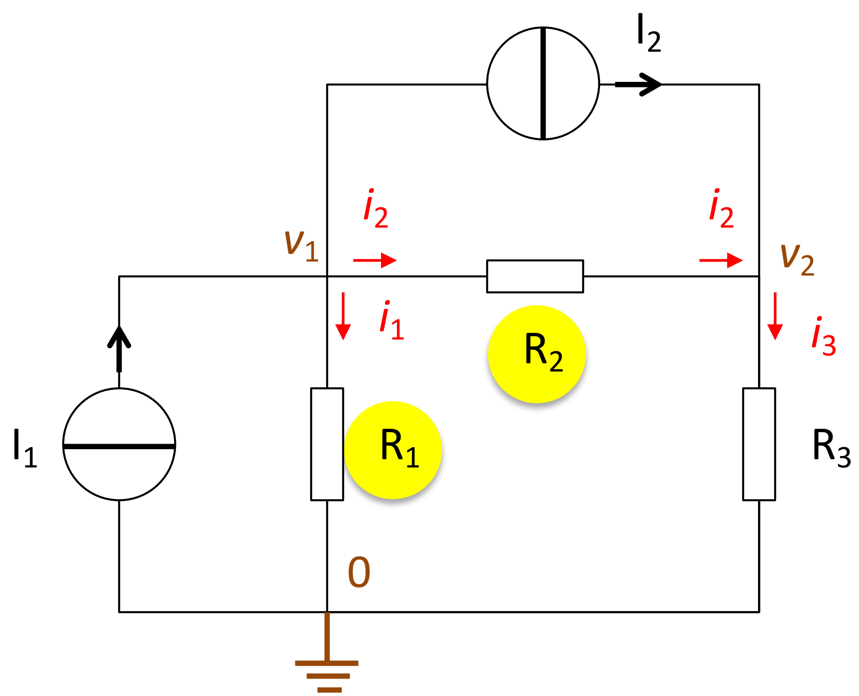
Sum of the conductances connected to node 1



Nodal analysis by circuit inspection

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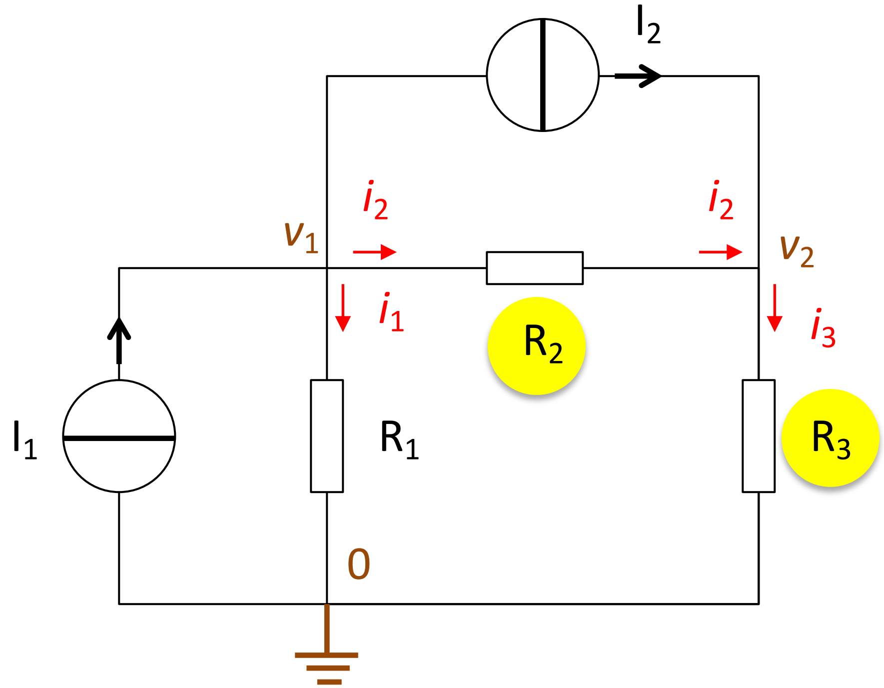
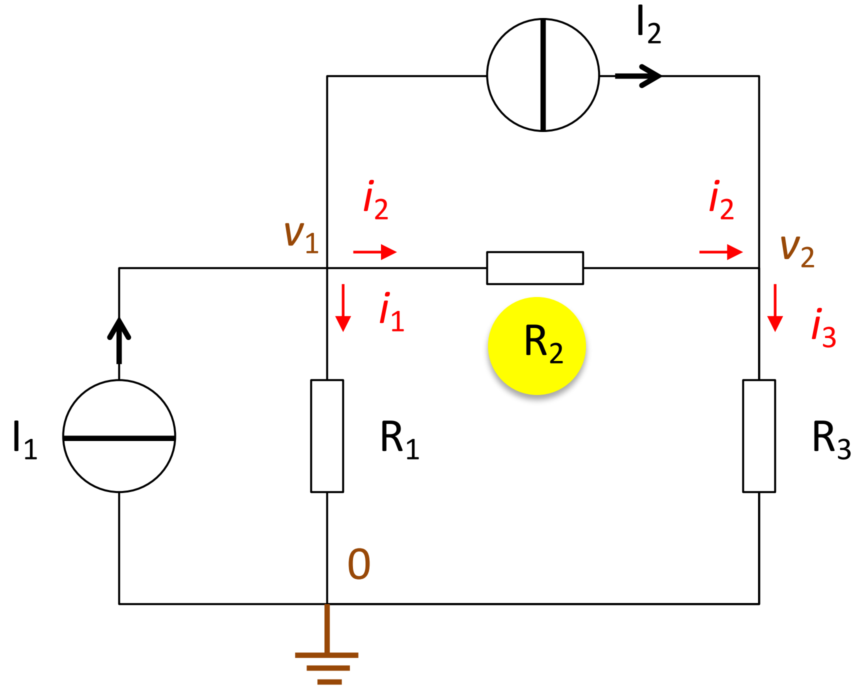
Sum of the conductances connected to node 2



Nodal analysis by circuit inspection

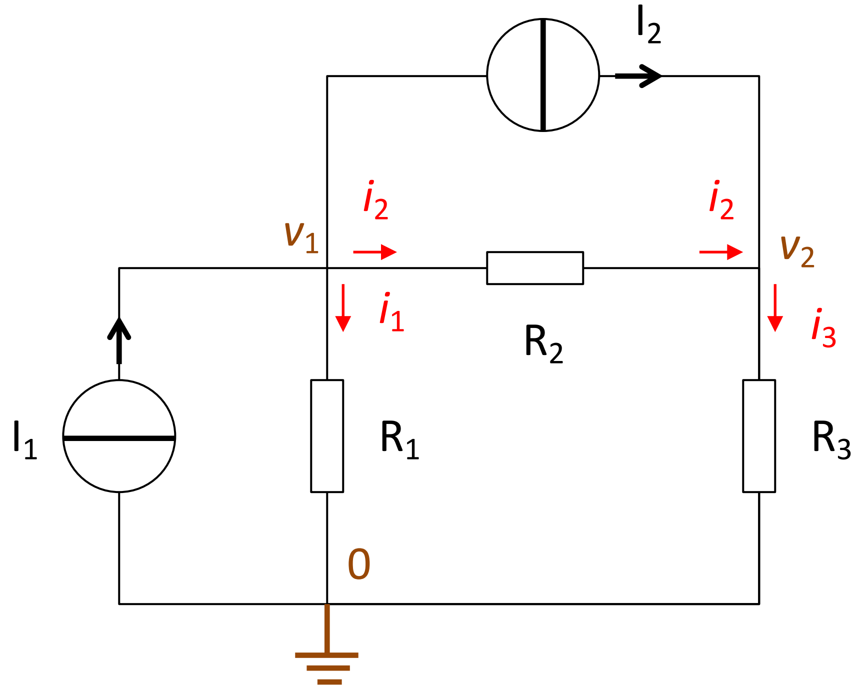
$$\begin{bmatrix} G_1 + G_2 & -G_2 \\ -G_2 & G_2 + G_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} I_1 - I_2 \\ I_2 \end{bmatrix}$$



Conductance connected between nodes 1 and 2, with a - sign.

Nodal analysis by circuit inspection

$$\begin{bmatrix} G_1 + G_2 & -G_2 \\ -G_2 & G_2 + G_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} I_1 - I_2 \\ I_2 \end{bmatrix}$$



Sum of currents injected at node 1

Nodal analysis by circuit inspection

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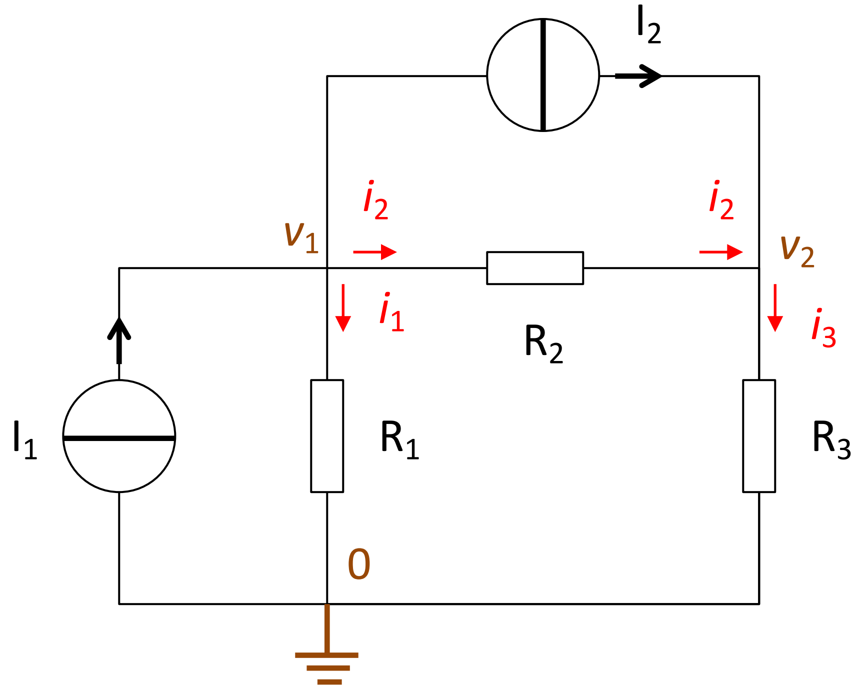

Sum of currents injected at node 2

Nodal analysis by circuit inspection: General case

- In general, for an N-node circuit with independent current sources, the equations for node voltages can be written in terms of conductances, as follows :

$$\begin{bmatrix} G_{11} & G_{12} & \dots & G_{1N} \\ G_{21} & G_{22} & \dots & G_{2N} \\ \dots & \dots & \dots & \dots \\ G_{N1} & G_{N2} & \dots & G_{NN} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix} = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_N \end{bmatrix}$$

with:

G_{kk}: sum of the conductances connected to the node k

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with:

$G_{kj}=G_{jk}$: sum with negative sign of the conductances directly connecting the nodes k and j (with k ≠ j).

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with:

v_k: unknown voltage at node k

Nodal analysis by circuit inspection: General case

- In general, for an N-node circuit with independent current sources, the equations for node voltages can be written in terms of conductances, as follows :

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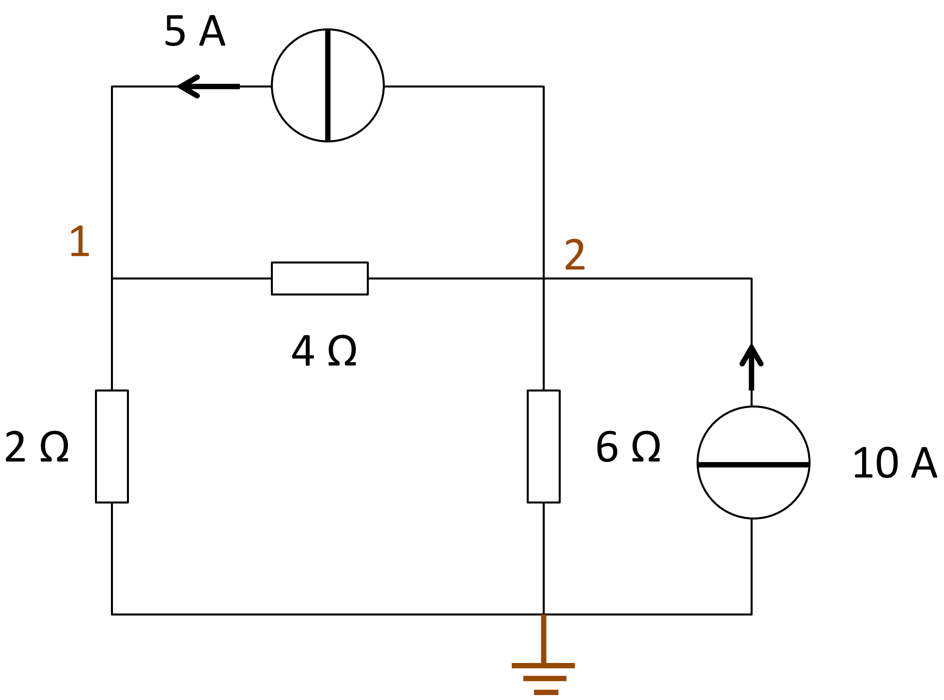
with:

i_k: sum of all independent current sources directly connected to the node k, with currents entering the node considered positive.

Nodal Analysis: Numerical Example



Solution on the board!

Mesh analysis

- General procedure for analyzing circuits using mesh currents as circuit variables.

- Nodal analysis uses Kirchhoff's law for *currents* to find unknown voltages, while mesh analysis uses Kirchhoff's law for *voltages* to find unknown currents.

Mesh analysis

- General procedure for analyzing circuits using mesh currents as circuit variables.

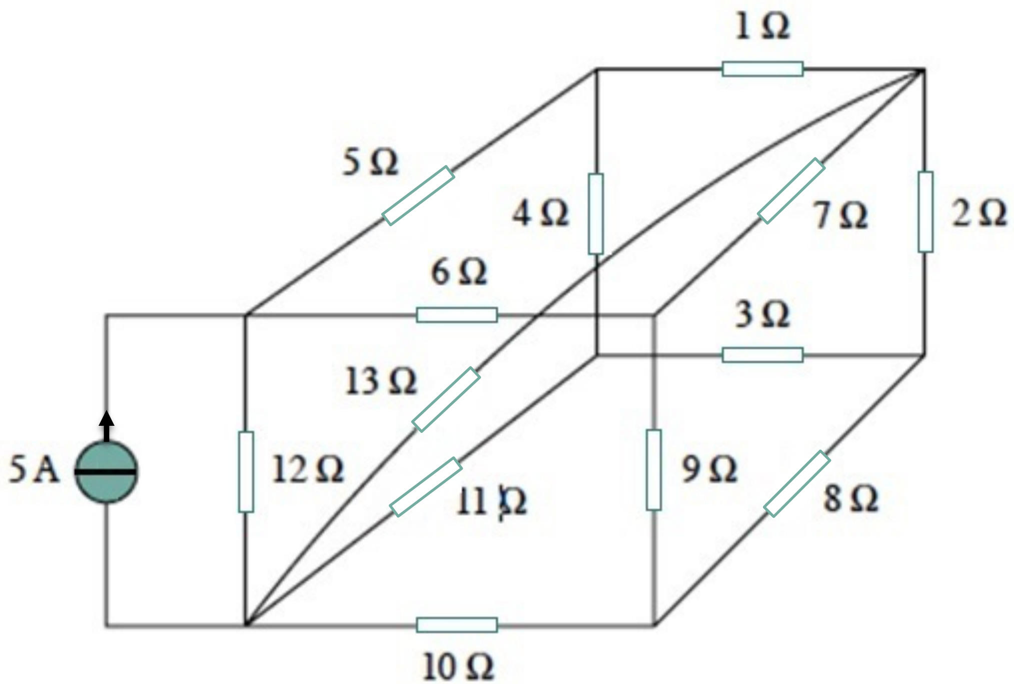
- Nodal analysis uses Kirchhoff's law for *currents* to find unknown voltages, while mesh analysis uses Kirchhoff's law for *voltages* to find unknown currents.

Mesh analysis is not as general as nodal analysis because it is applicable only to a planar circuit! (see following slide)

Planar circuit

- Definition: a planar circuit is a circuit for which the branches do not intersect.
- Otherwise, the circuit is of the non-planar or spatial type.

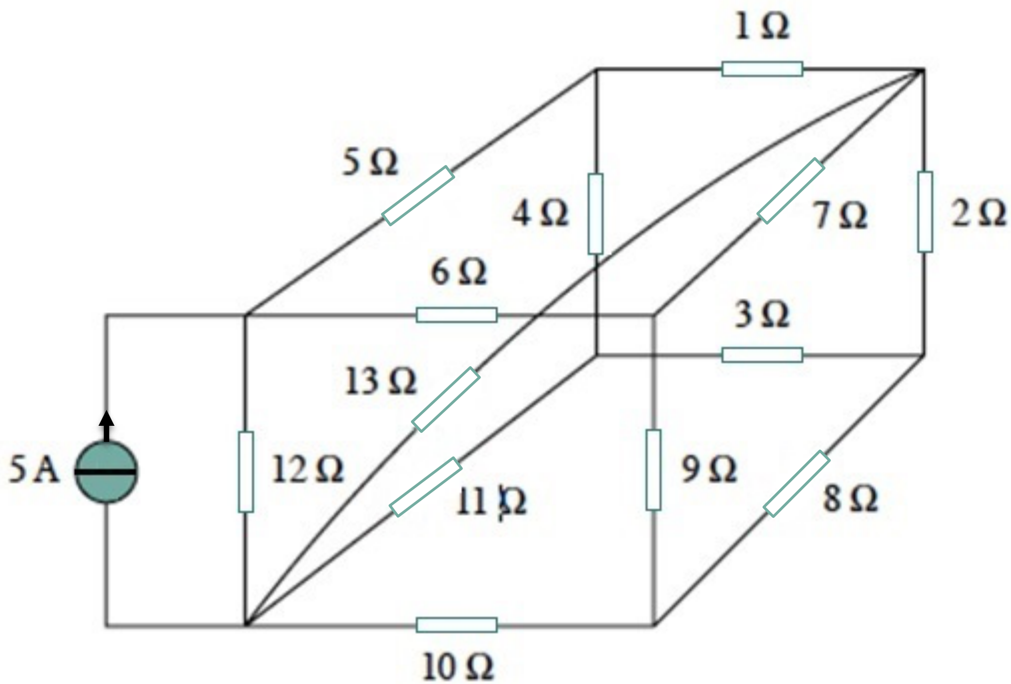
Planar and non-planar circuits



☐ A. planar

☐ B. non-planar

Planar and non-planar circuits



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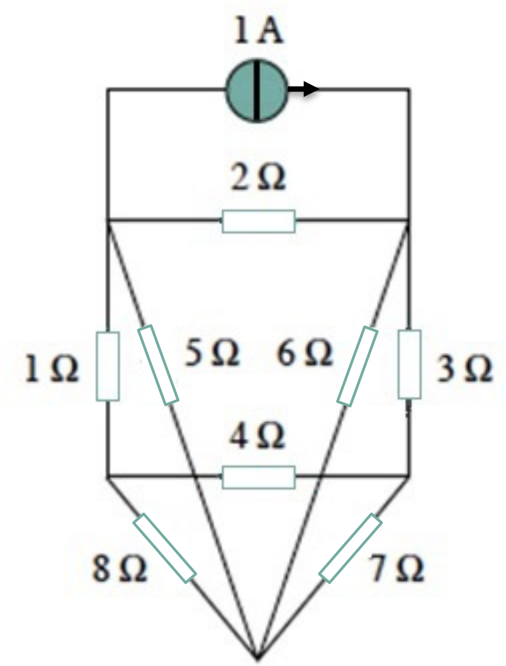
Planar and non-planar circuits



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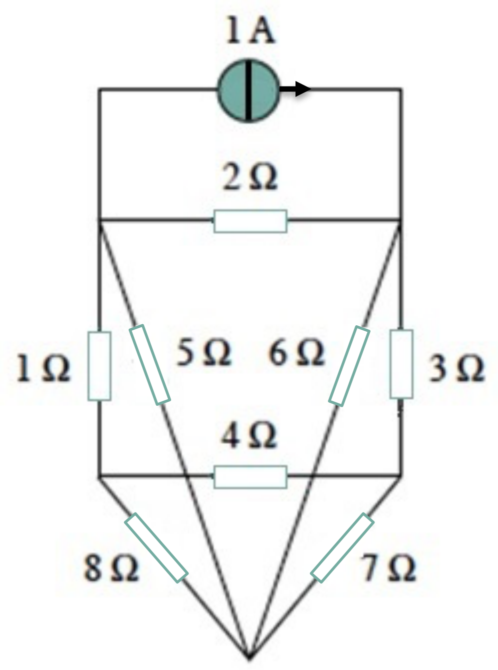
Planar and non-planar circuits



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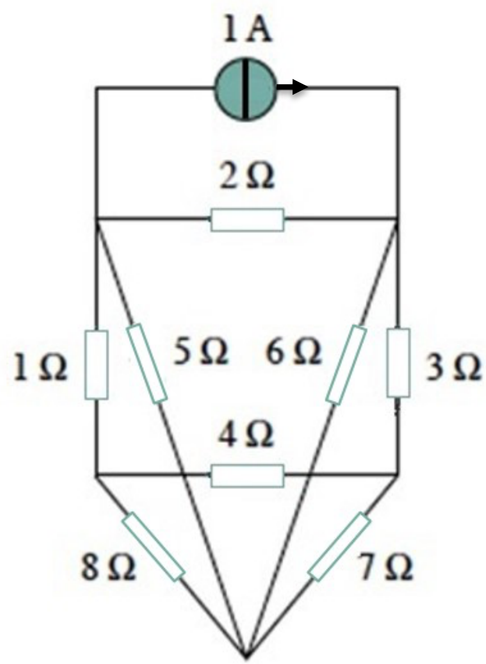
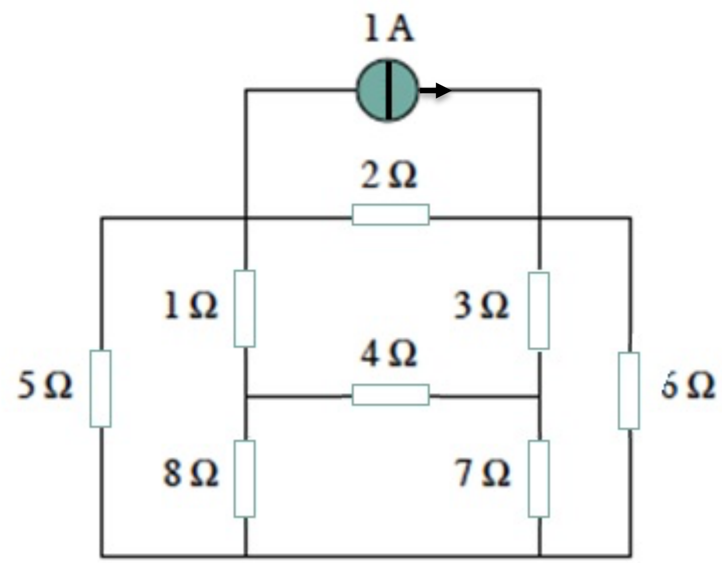
Planar and non-planar circuits



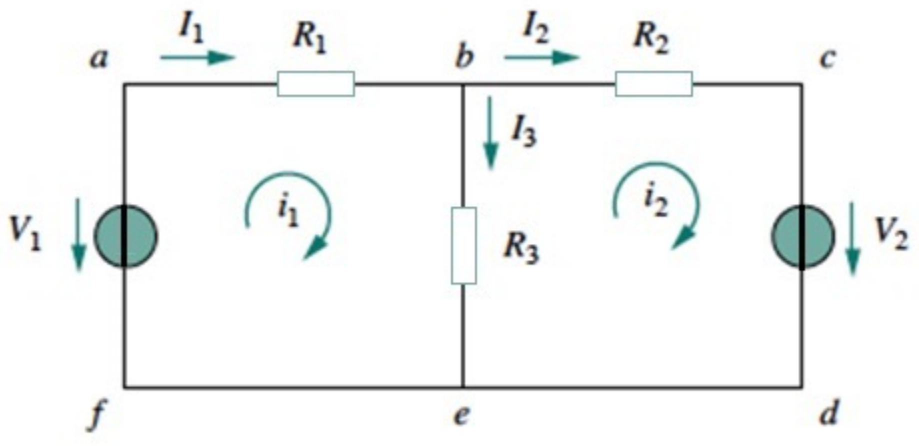


☑ A. planar

☐ B. non-planar

Definition of a loop

- A loop is a mesh that does not contain other meshes within it.



Example: circuit with two loops

Definition of a loop

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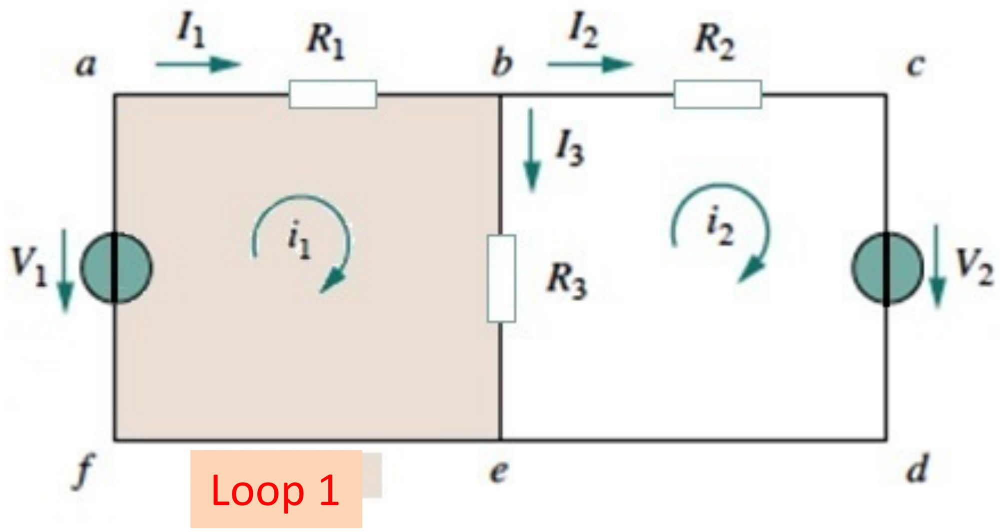

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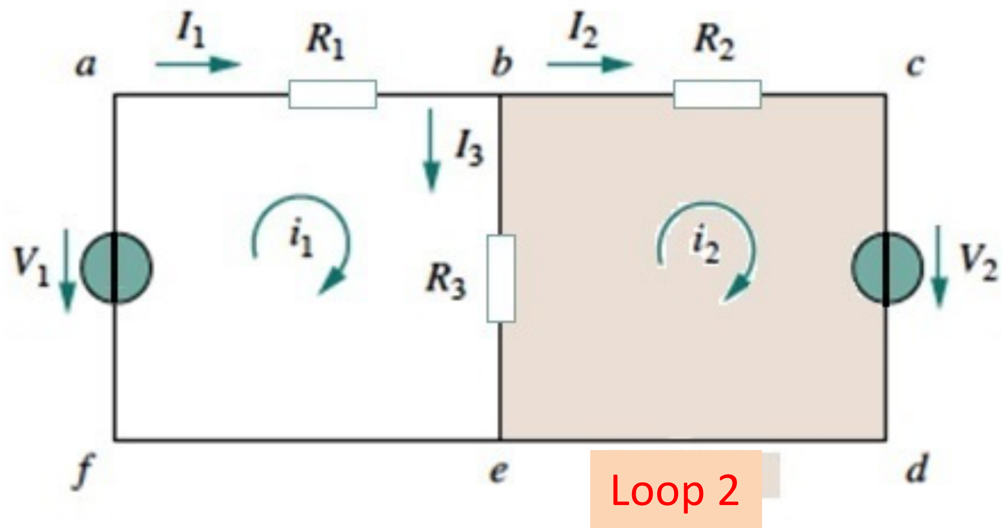

Definition of a loop

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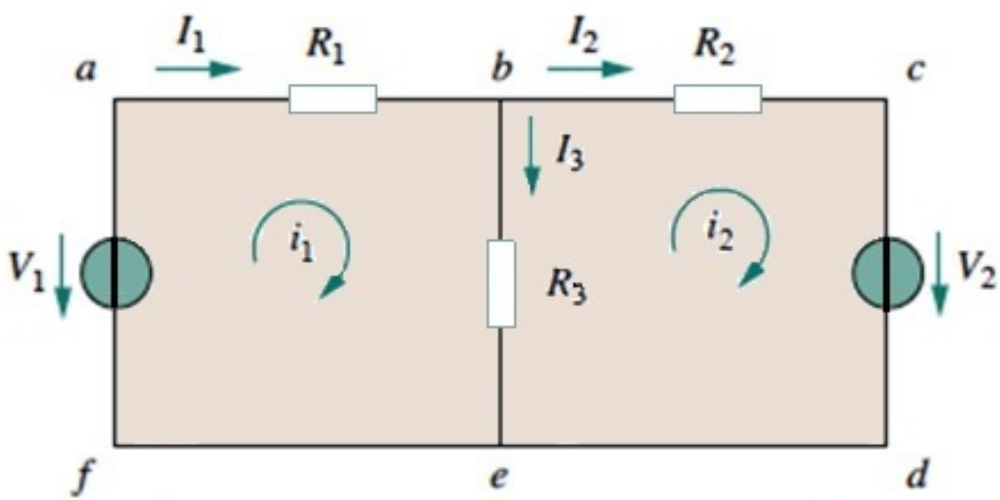
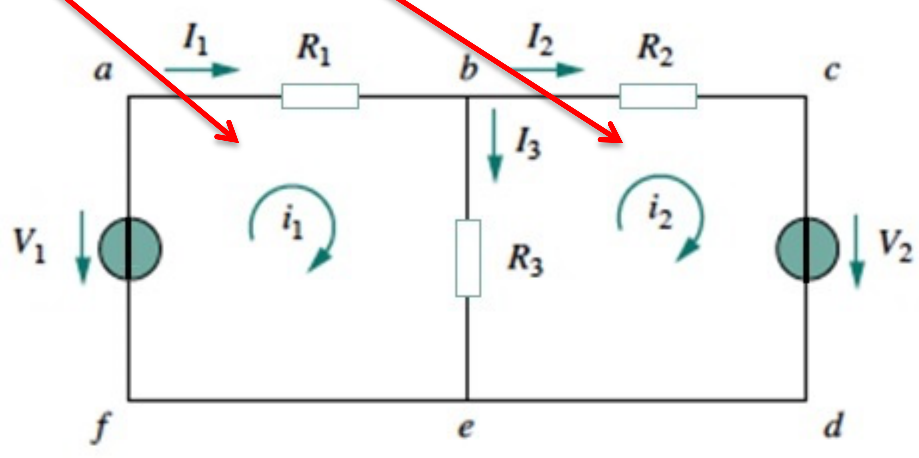


Warning: this is not a loop!

But it is a mesh!

Definition of a loop

i_1 and i_2 are the loop currents (mesh)

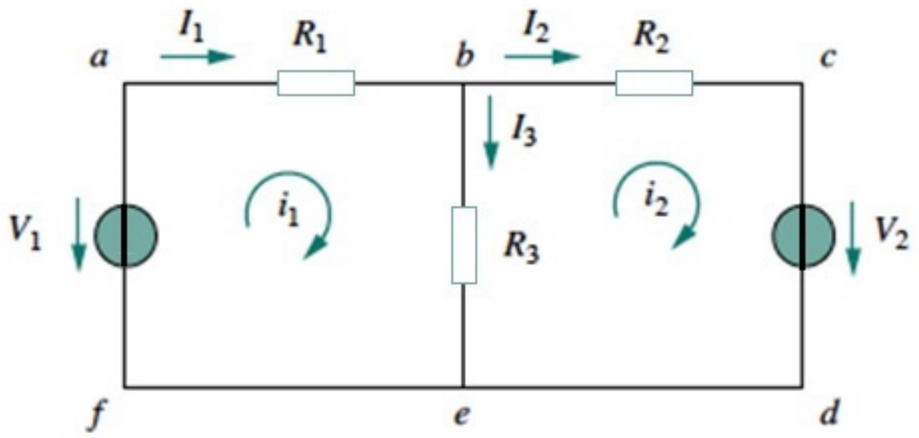


Mesh analysis: Steps to determine mesh currents

1. Identify mesh currents i_1, i_2, …i_n for the n loops of the circuit.
2. Apply Kirchoff's law (voltages) to each of the n loops. Use Ohm's law to express voltages in terms of mesh currents.
3. Solve the n resulting equations to obtain the currents of the meshes.

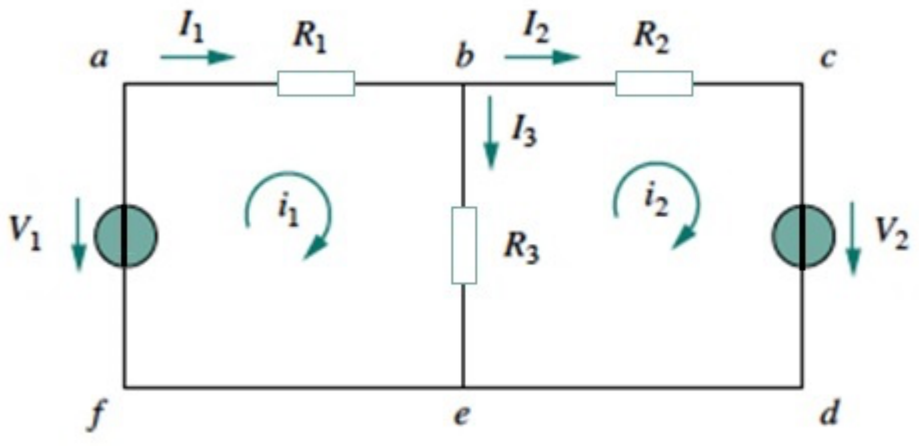
Mesh Analysis: Application Example

Step 1: identification of mesh currents



Mesh Analysis: Application Example

Step 2: application of Kirchhoff's law (voltages)



Mesh 1: $-V_1 + R_1 i_1 + R_3(i_1 - i_2) = 0$ or: $(R_1 + R_3)i_1 - R_3 i_2 = V_1$

Mesh 2: $R_2 i_2 + V_2 + R_3(i_2 - i_1) = 0$ or: $-R_3 i_1 + (R_2 + R_3)i_2 = -V_2$

Mesh Analysis: Application Example

Step 3: Solving equations

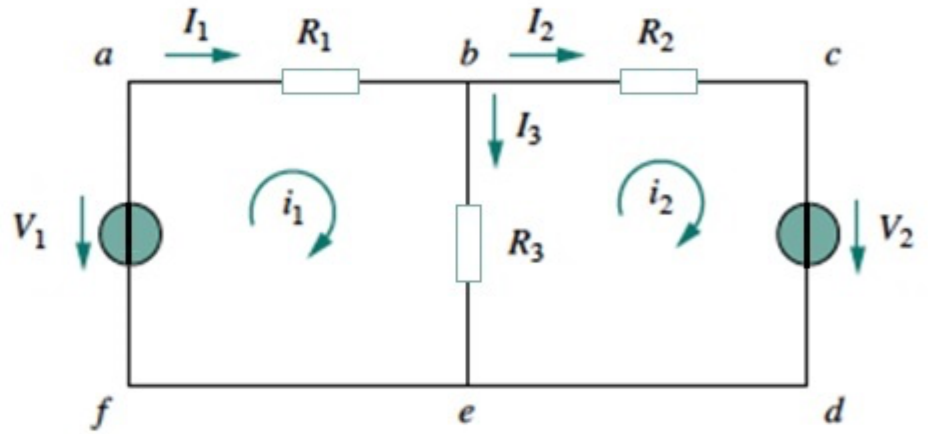
Equations written in matrix form

$$\begin{bmatrix} R_1 + R_3 & -R_3 \\ -R_3 & R_2 + R_3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ -V_2 \end{bmatrix}$$

which can be solved using conventional methods.

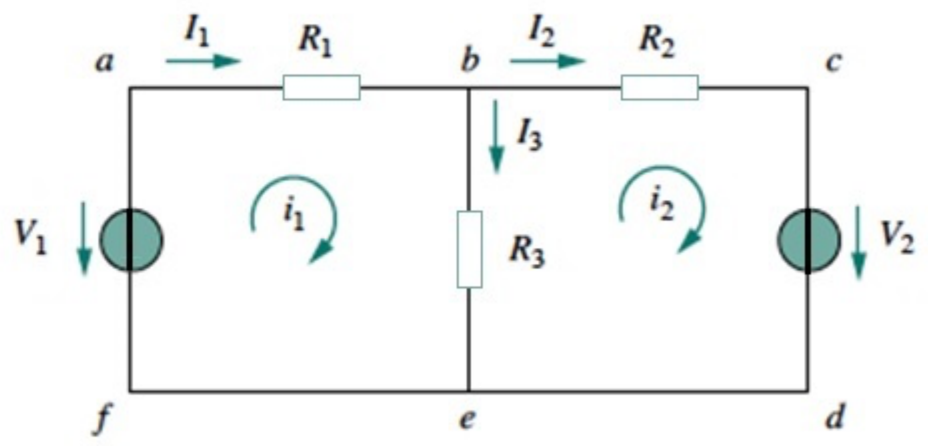
Branch currents are, in general, different from mesh currents.

Mesh Analysis: Application Example



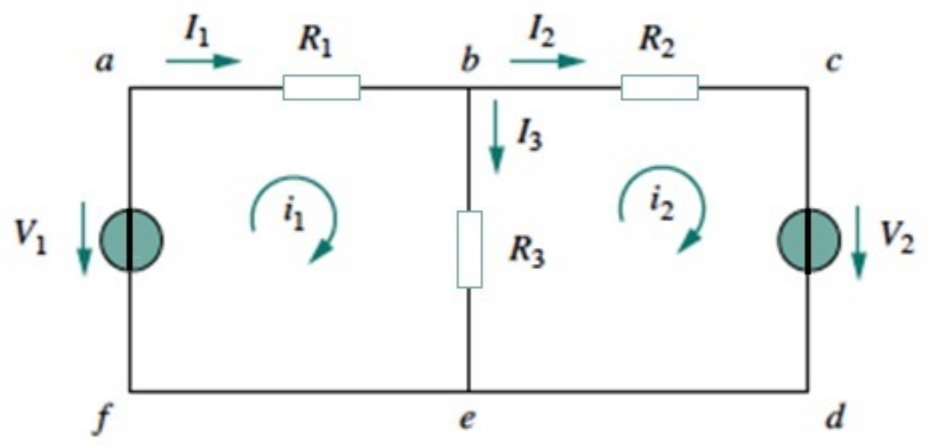
$$I_1 = i_1 \qquad I_2 = i_2 \qquad I_3 = i_1 - i_2$$

Mesh analysis by circuit inspection



$$\begin{bmatrix} R_1 + R_3 & -R_3 \\ -R_3 & R_2 + R_3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ -V_2 \end{bmatrix}$$

Mesh analysis by circuit inspection



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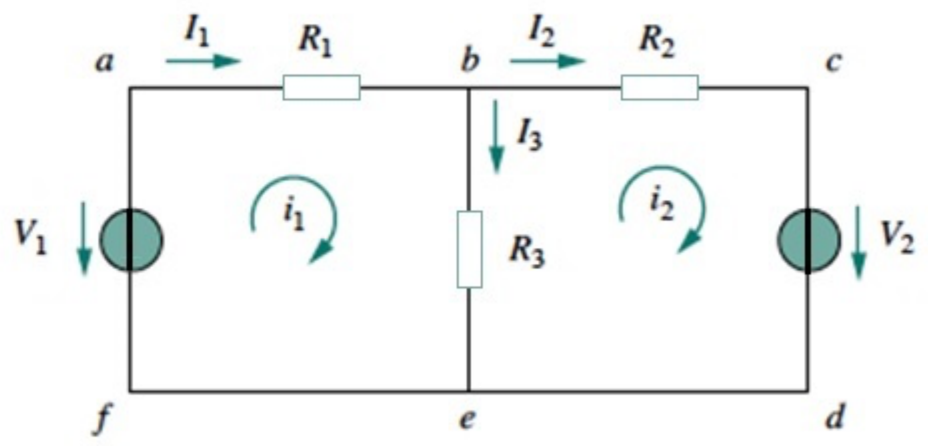
Sum of the resistances belonging to the mesh 1

Mesh analysis by circuit inspection



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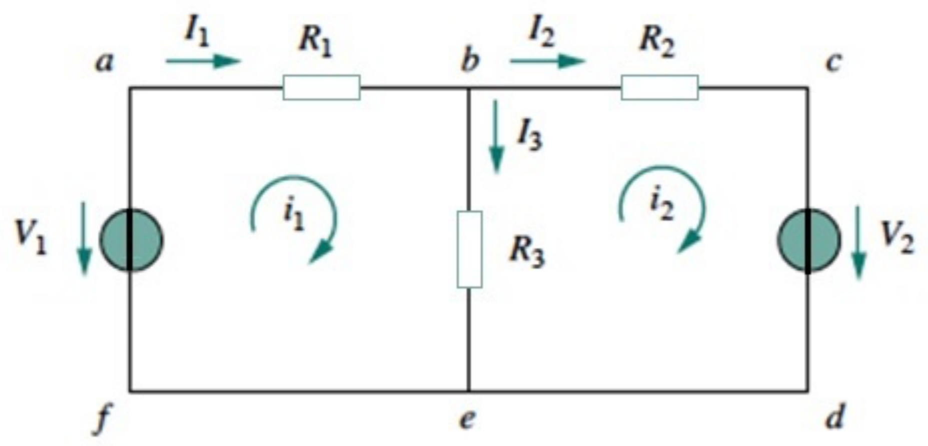
Sum of the resistances belonging to the mesh 2

Mesh analysis by circuit inspection



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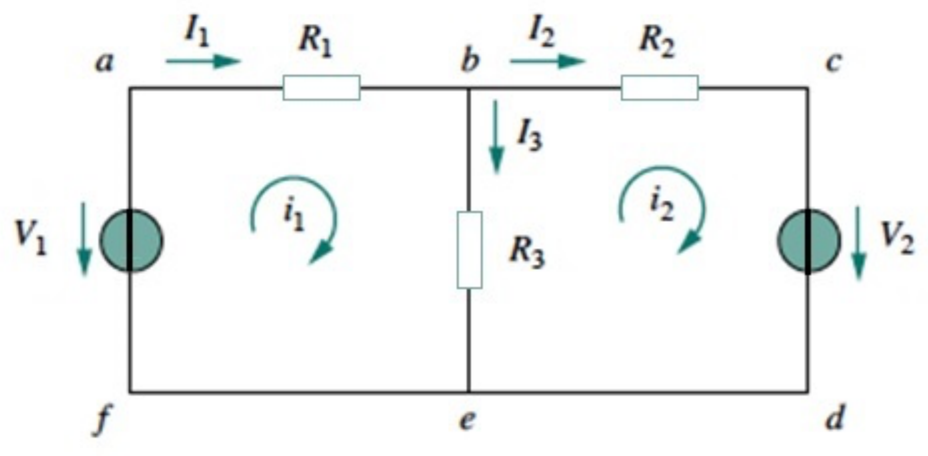
Sum with negative sign of resistances shared between meshes 1 and 2

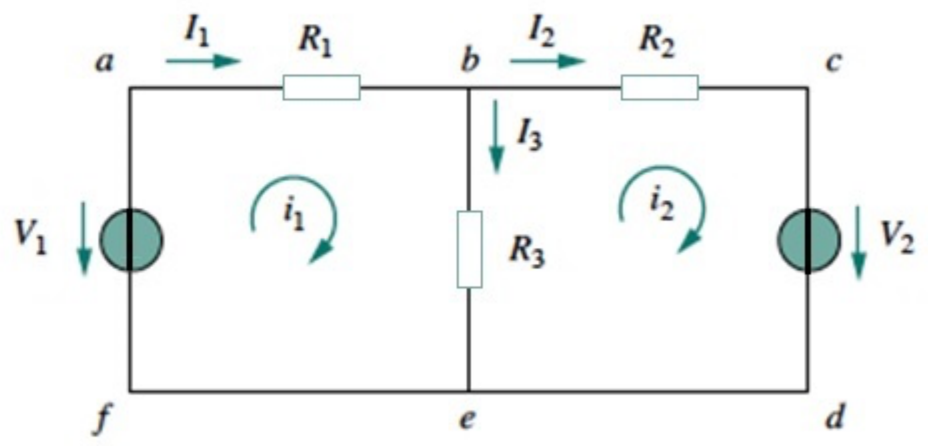
Mesh analysis by circuit inspection



$$\begin{bmatrix} R_1 + R_3 & -R_3 \\ -R_3 & R_2 + R_3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ -V_2 \end{bmatrix}$$

Sum of independent voltage sources of mesh 1

Mesh analysis by circuit inspection



$$\begin{bmatrix} R_1 + R_3 & -R_3 \\ -R_3 & R_2 + R_3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ -V_2 \end{bmatrix}$$

Sum of independent voltage sources of mesh 2

Mesh analysis by circuit inspection: General case

- In general, for a N-cell circuit with independent voltage sources, the equations for the mesh currents can be written in terms of resistances, as follows :

$$\begin{bmatrix} R_{11} & R_{12} & \dots & R_{1N} \\ R_{21} & R_{22} & \dots & R_{2N} \\ \dots & \dots & \dots & \dots \\ R_{N1} & R_{N2} & \dots & R_{NN} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_N \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix}$$

with:

R_{kk}: sum of the resistances belonging to mesh k.

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with:

i_k: unknown current of mesh k

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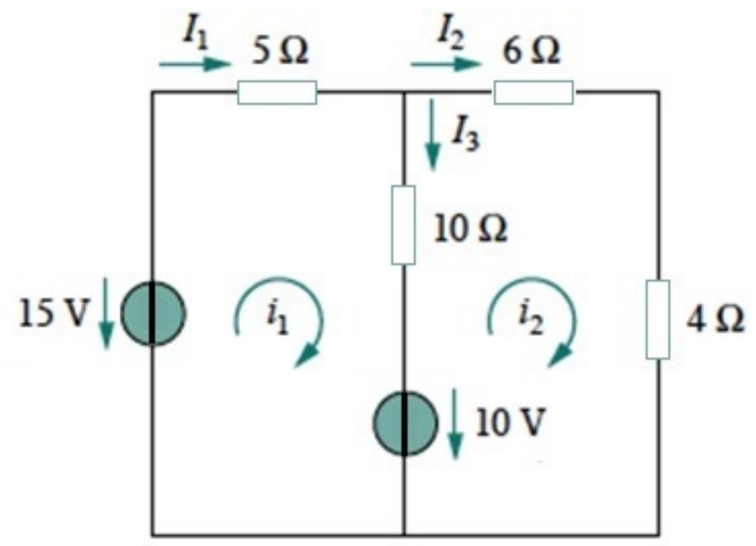
with: v_k: sum of all the independent voltage sources of mesh k

Mesh Analysis: Numerical Example



Solution on the board!