Fundamentals of Electrical Circuits and Systems

Chapter 2: Fundamental Theorems of Linear Circuits

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Electrical Circuits: Fundamental Theorems

- Linear circuits
- Superposition theorem
- Source transformation
- Thévenin's theorem
- Norton's theorem
- Maximum power transfer theorem
- Tellegen's theorem

Linear circuits

- A linear circuit is one whose output is linearly connected to its input.

- A linear circuit has only linear elements, linear dependent and independent sources.

Question

- A resistor is a linear element

A. Yes
B. No

Question

- A capacitor is a linear element

A. Yes
B. No

Question

- An inductor is a linear element

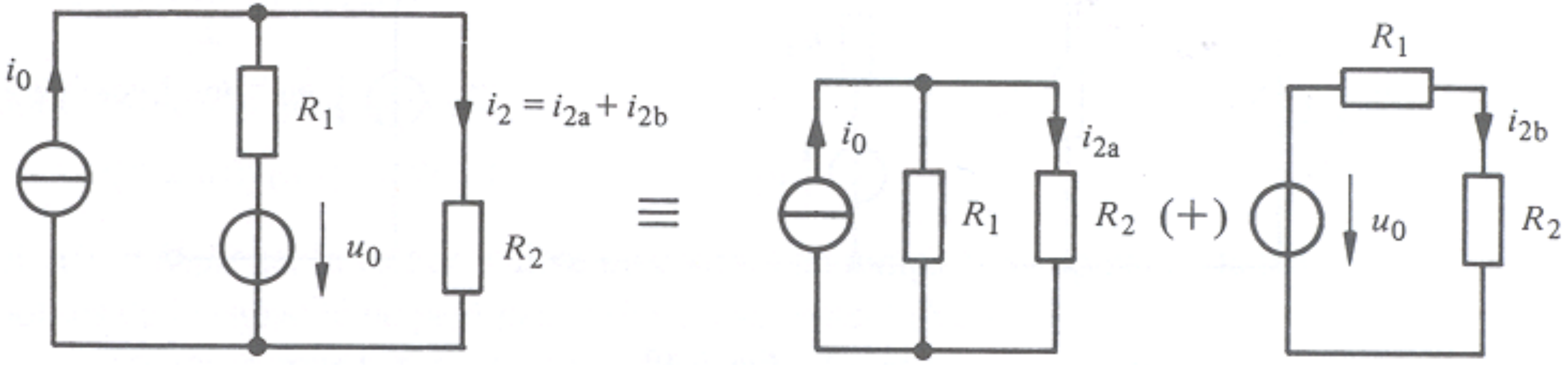
A. Yes
B. No

Superposition theorem

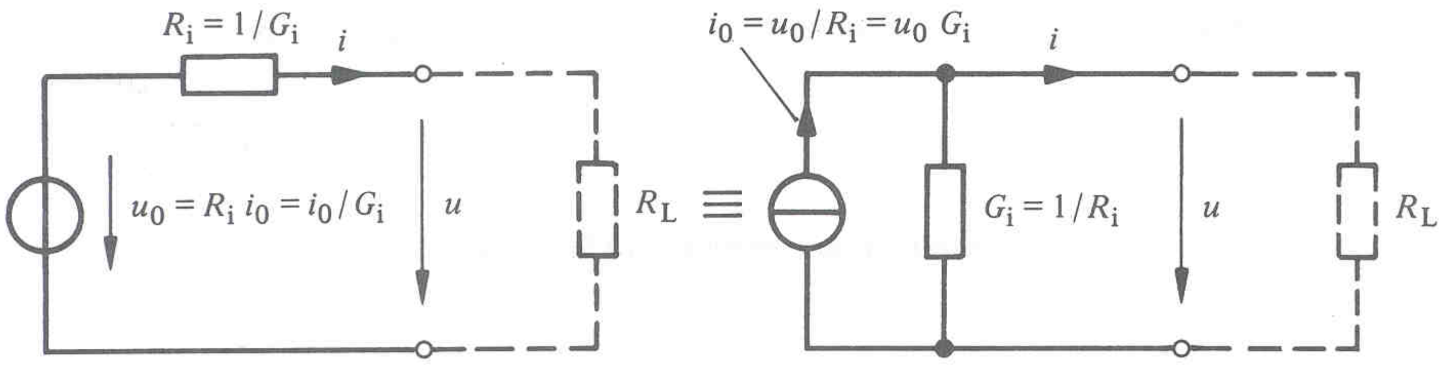
- The current and voltages in any element of a linear circuit are given by the sum of the voltages and currents produced in that element by each source taken separately.

- Note: this theorem does not apply to power.

Superposition theorem: example



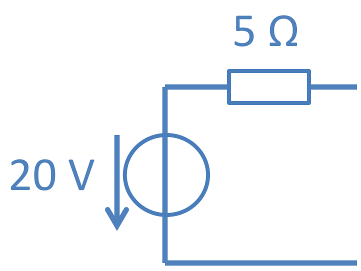
Equivalence voltage source / current source



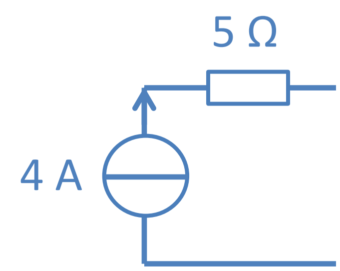
Question

- Which pair of circuits are equivalent circuits?

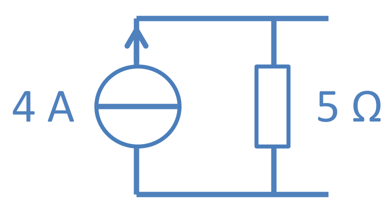
A. (a) and (b)
B. (b) and (d)
C. (a) and (c)
D. (c) and (d)



(a)



(b)



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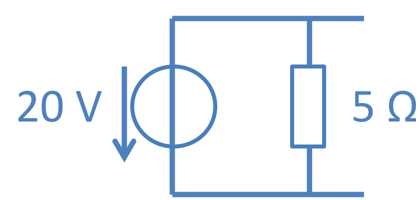
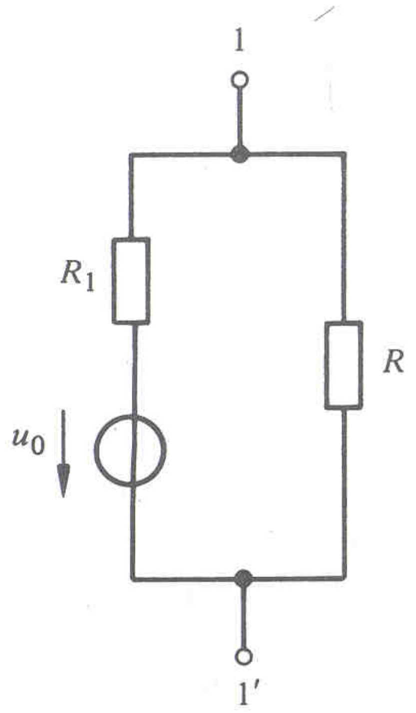


(d)

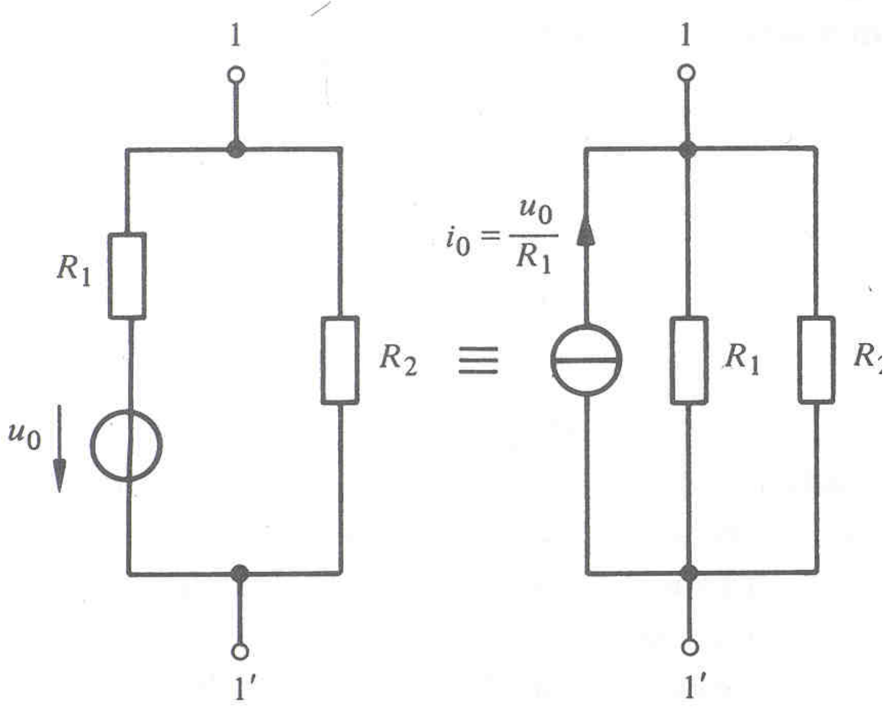
Equivalence voltage source / current source

Example:



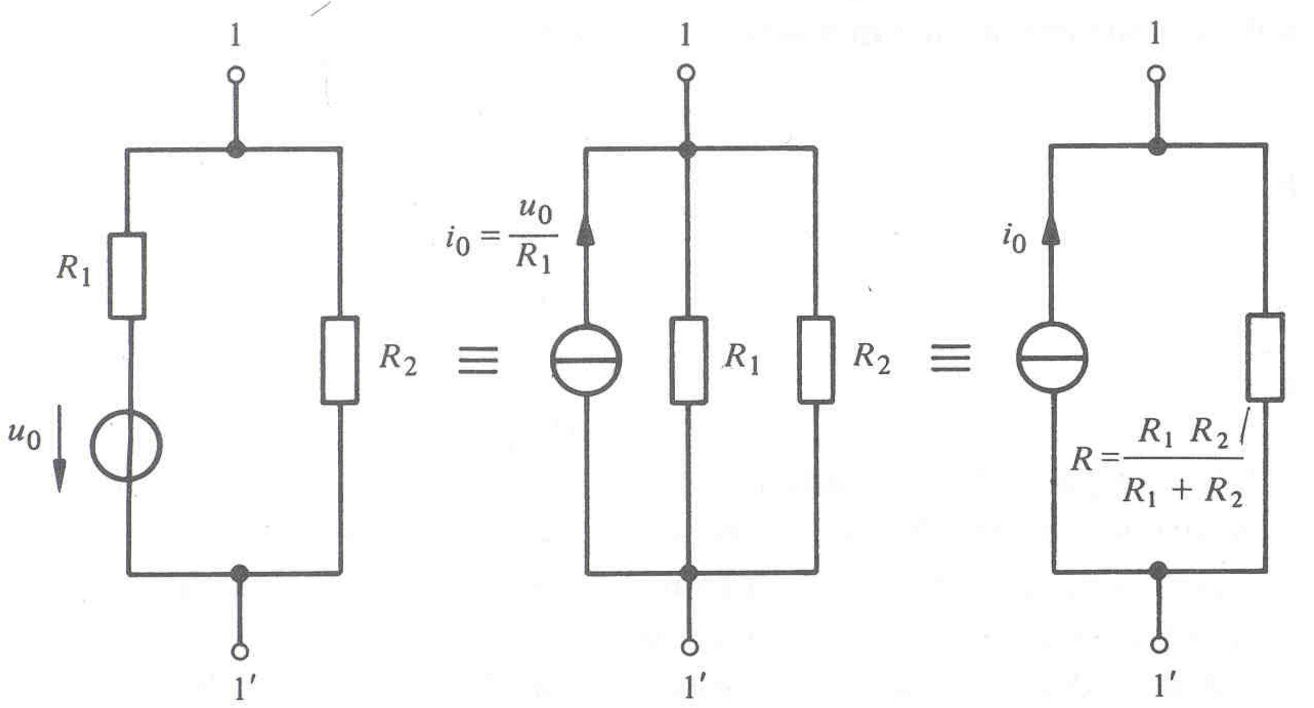
Equivalence voltage source / current source

Example:



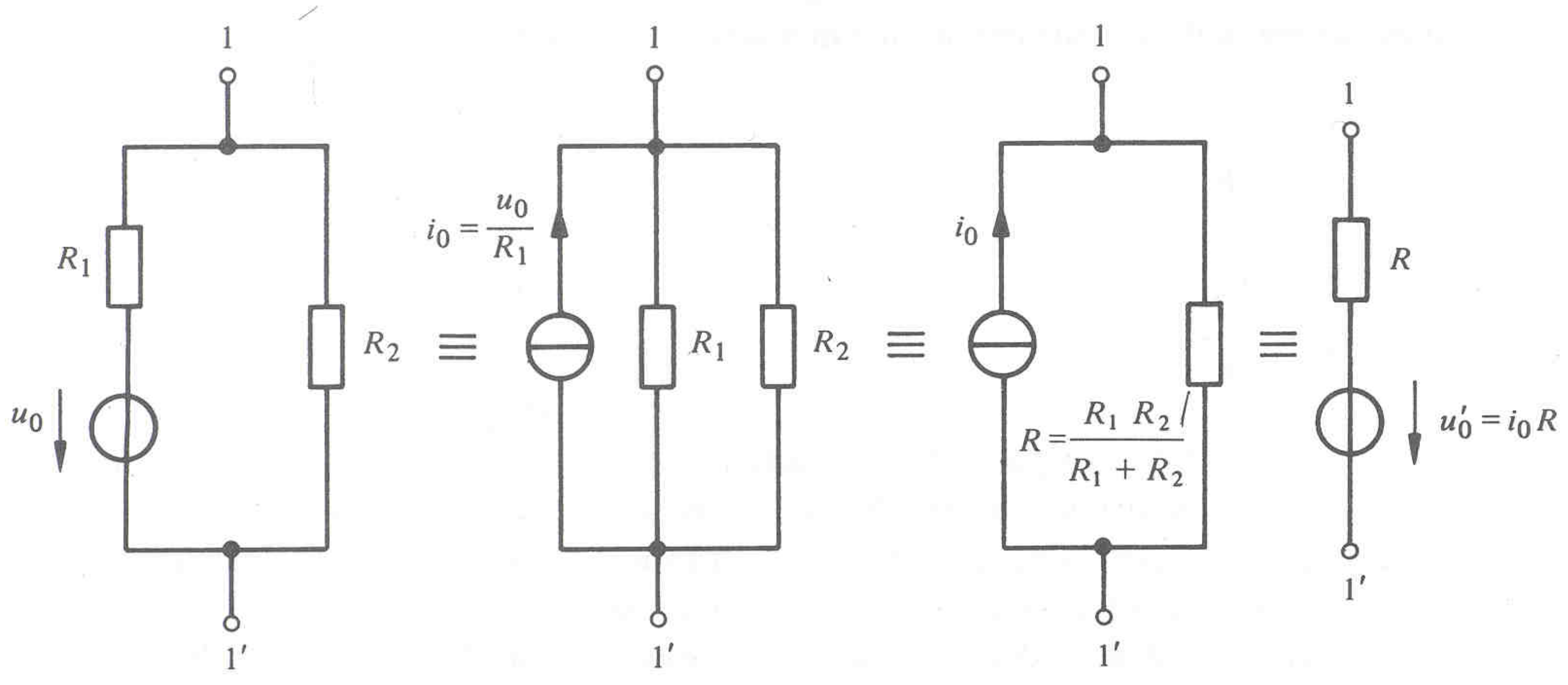
Equivalence voltage source / current source

Example:



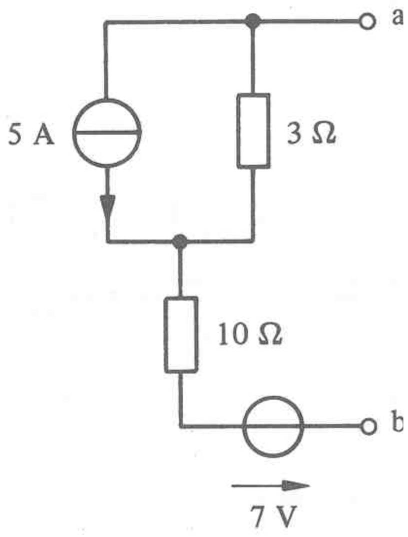
Equivalence voltage source / current source

Example:



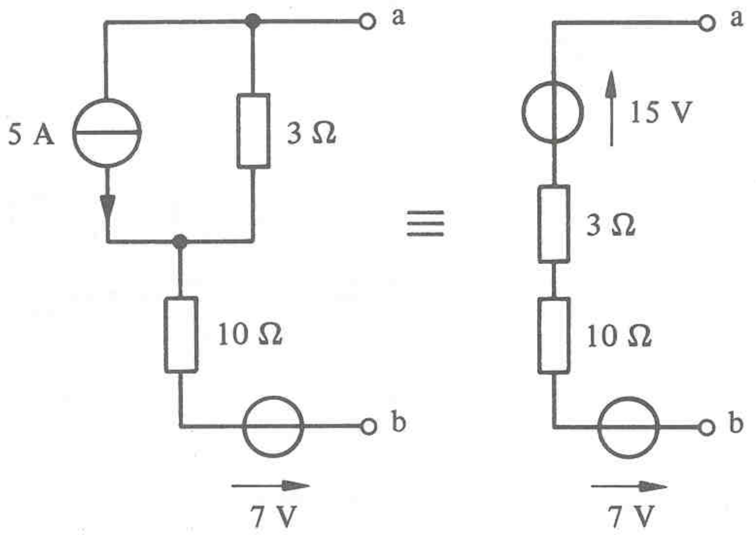
Equivalence voltage source / current source

Example:



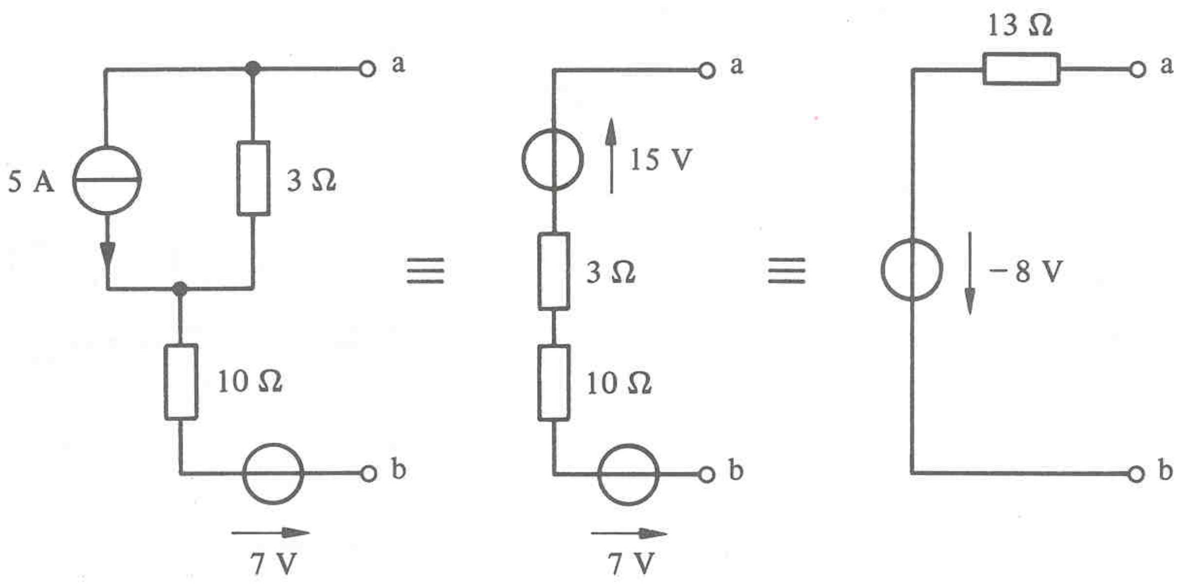
Equivalence voltage source / current source

Example:

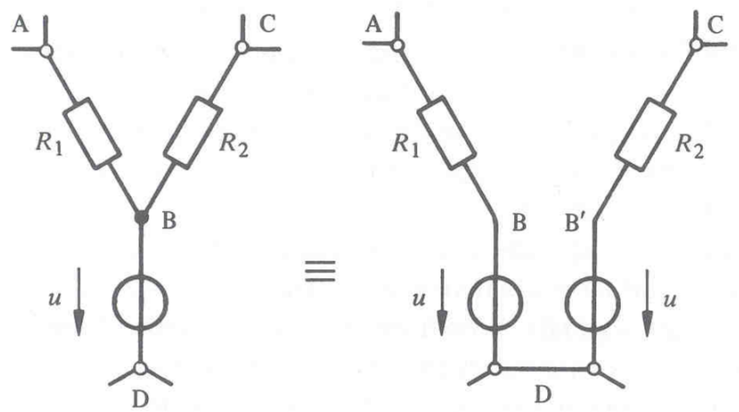


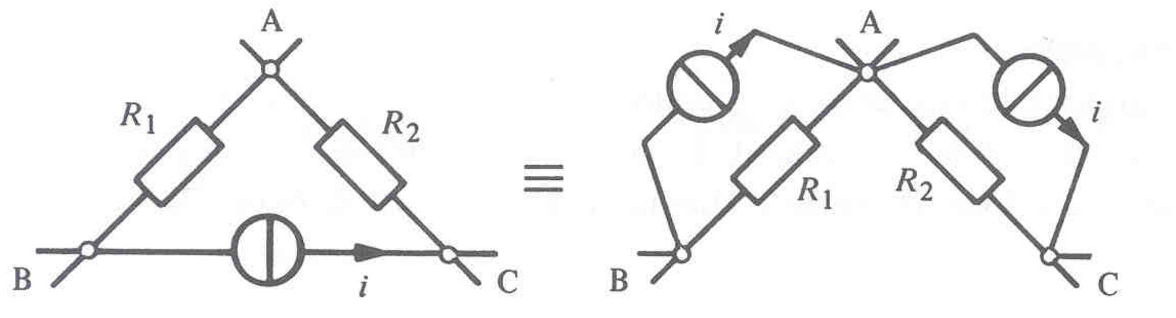
Equivalence voltage source / current source

Example:

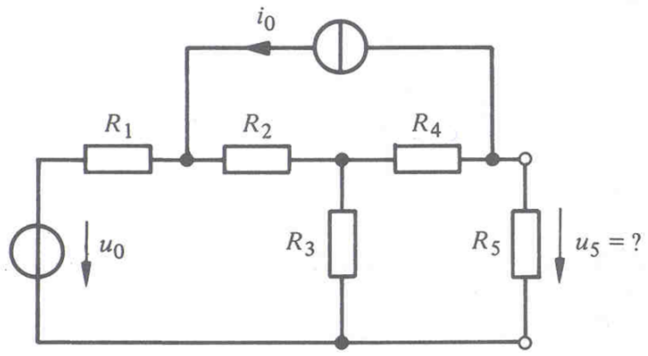


Substitution of sources

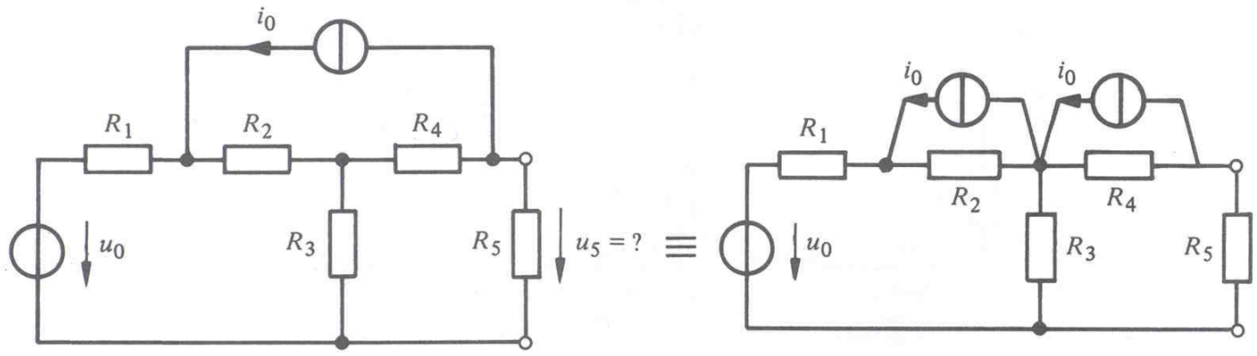




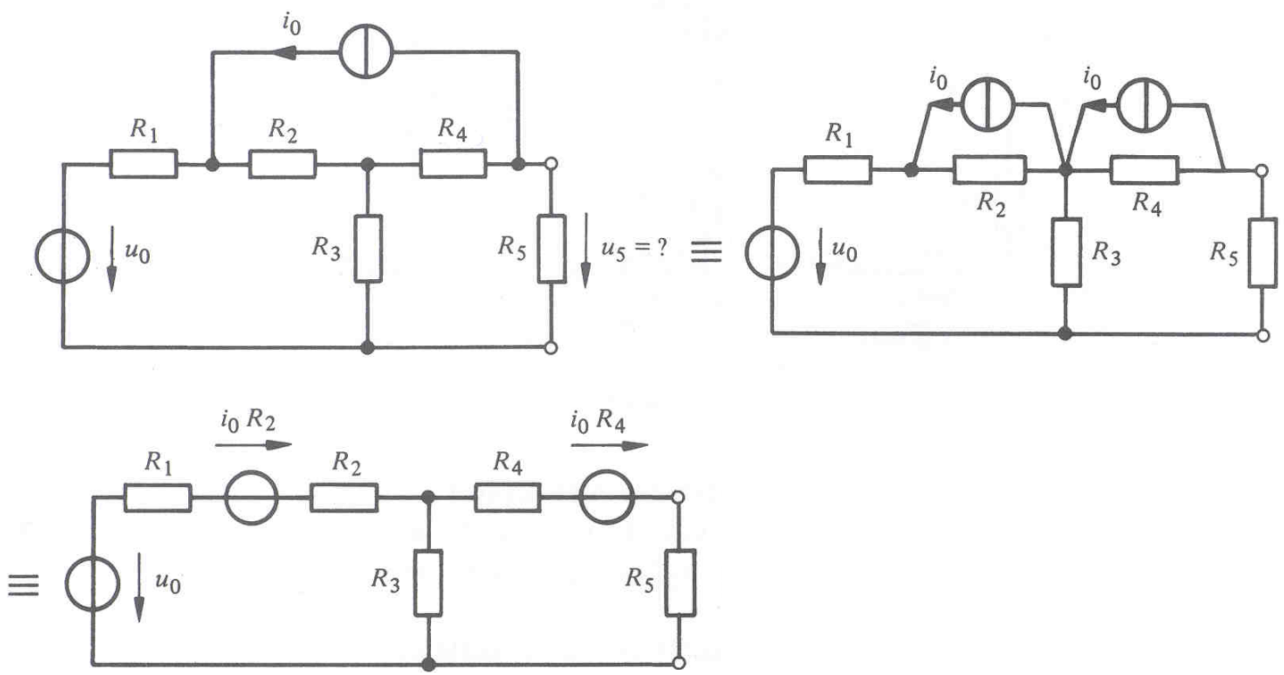
Example



Example



Example



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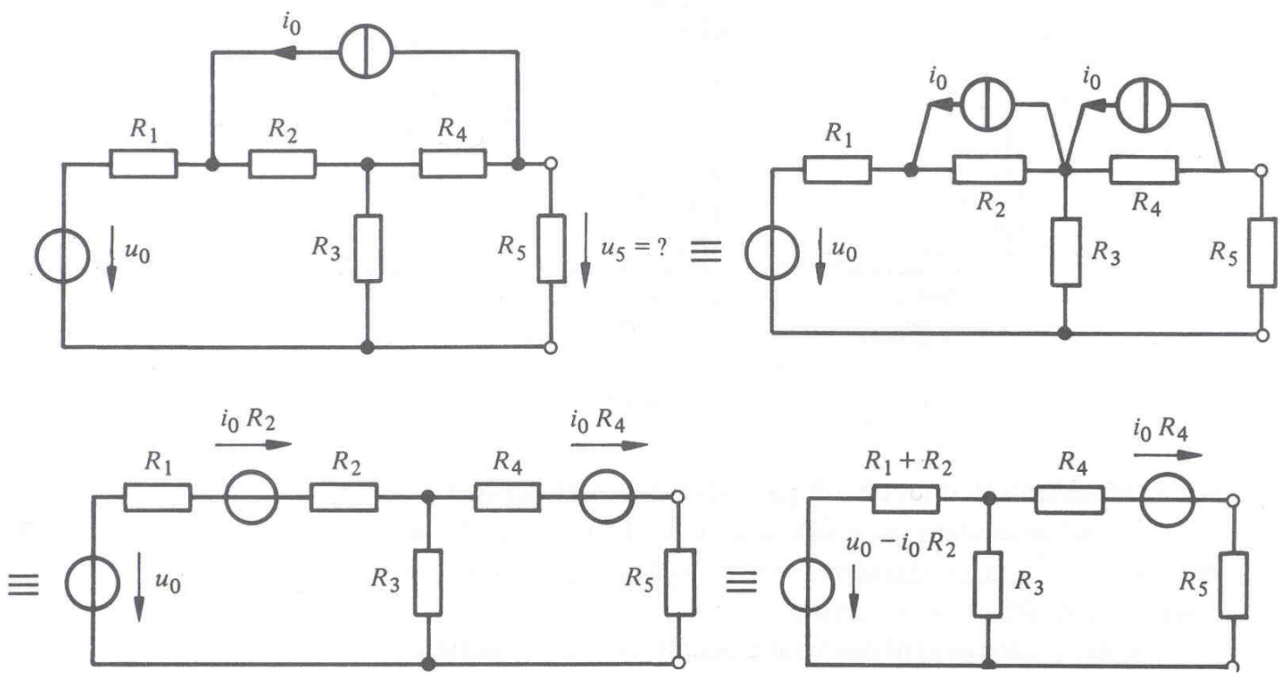
Example



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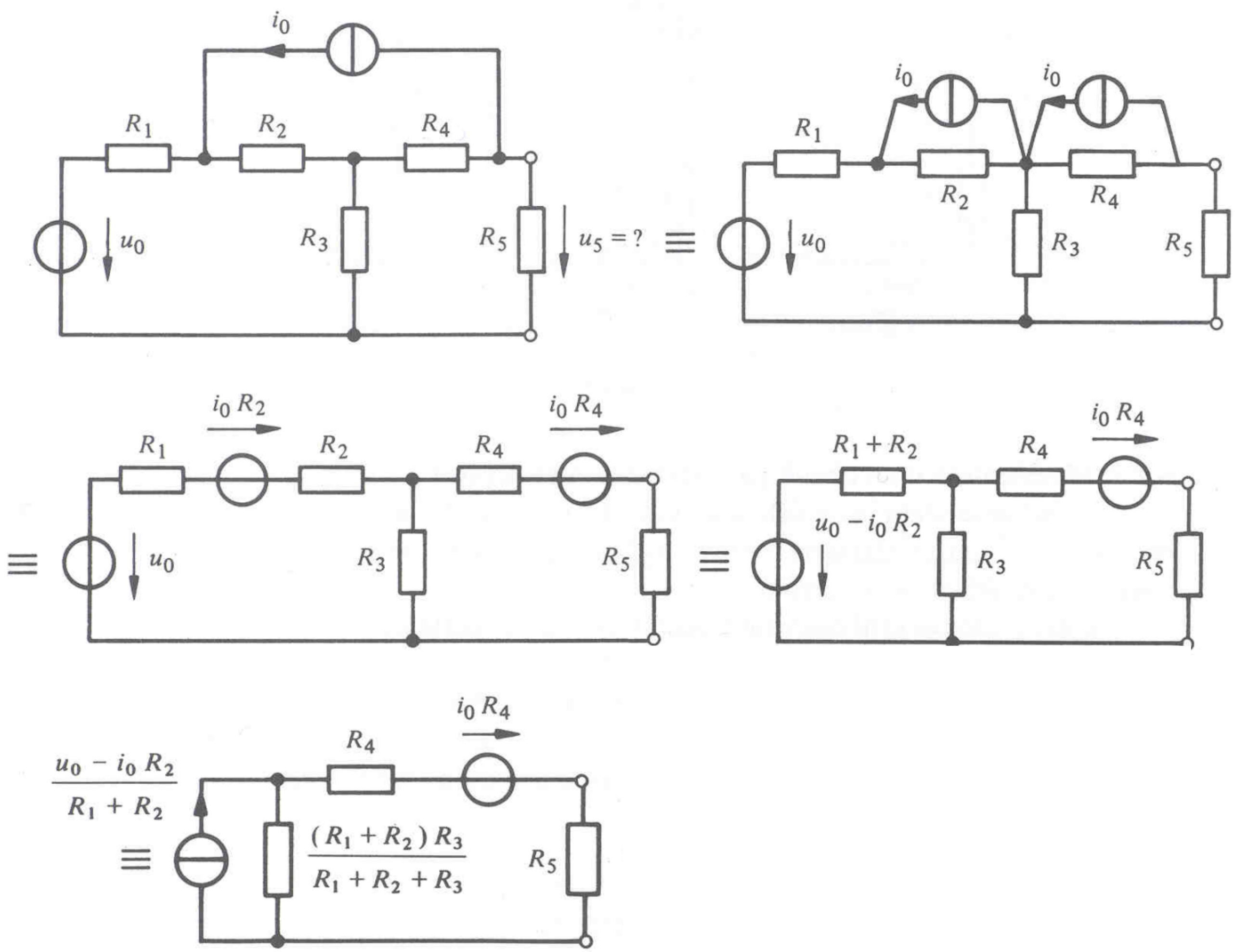
Example



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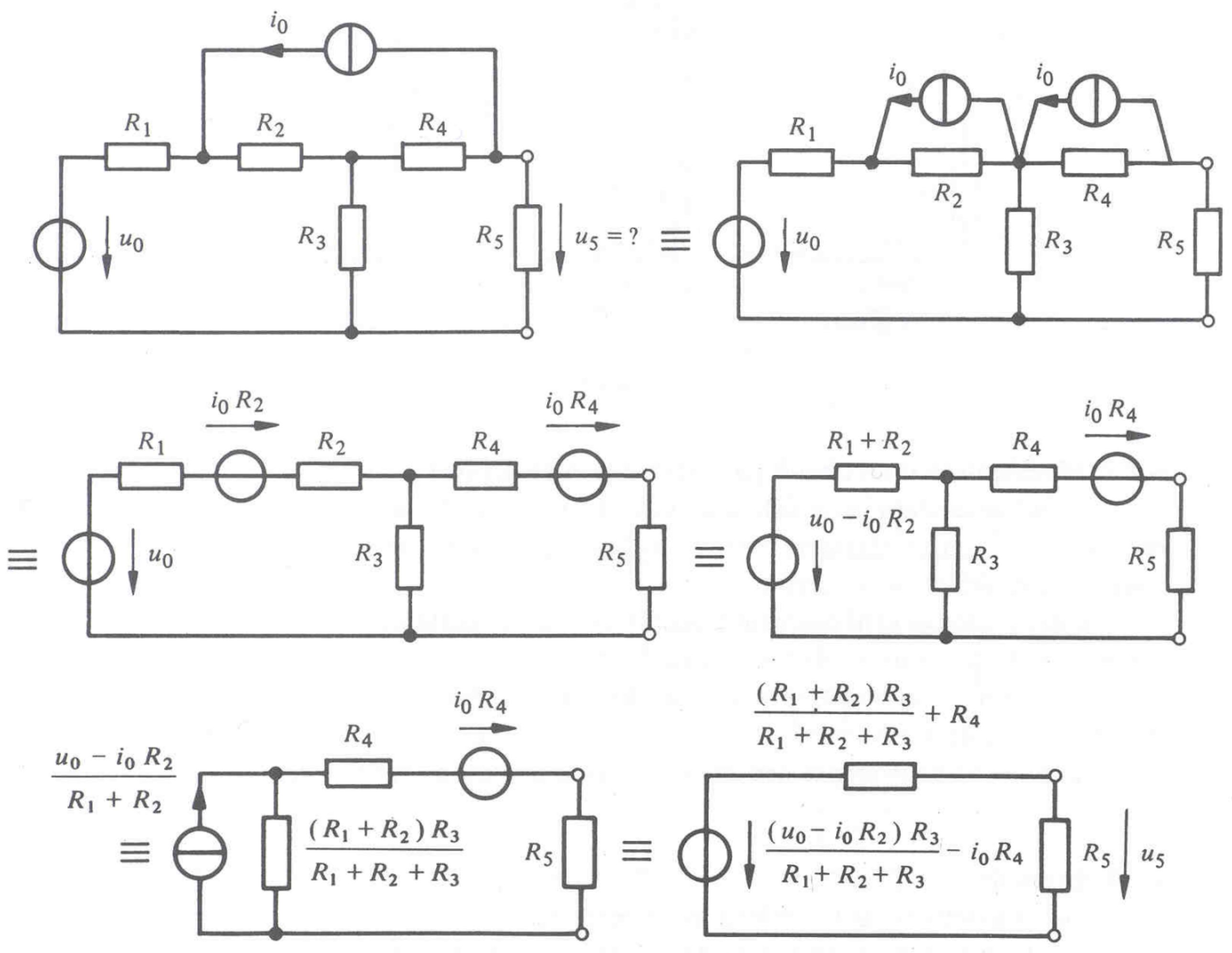
$\frac{(R_1 + R_2) R_3}{R_1 + R_2 + R_3}$
R_5

Example



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$\frac{u_0 - i_0 R_2}{R_1 + R_2}$
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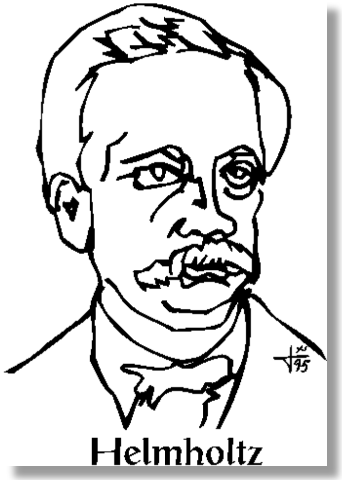
$\frac{(R_1 + R_2) R_3}{R_1 + R_2 + R_3}$
R_5
$\equiv$
$\frac{(R_1 + R_2) R_3}{R_1 + R_2 + R_3} + R_4$
$\frac{(u_0 - i_0 R_2) R_3}{R_1 + R_2 + R_3} - i_0 R_4$
R_5
u_5

Thévenin’s theorem

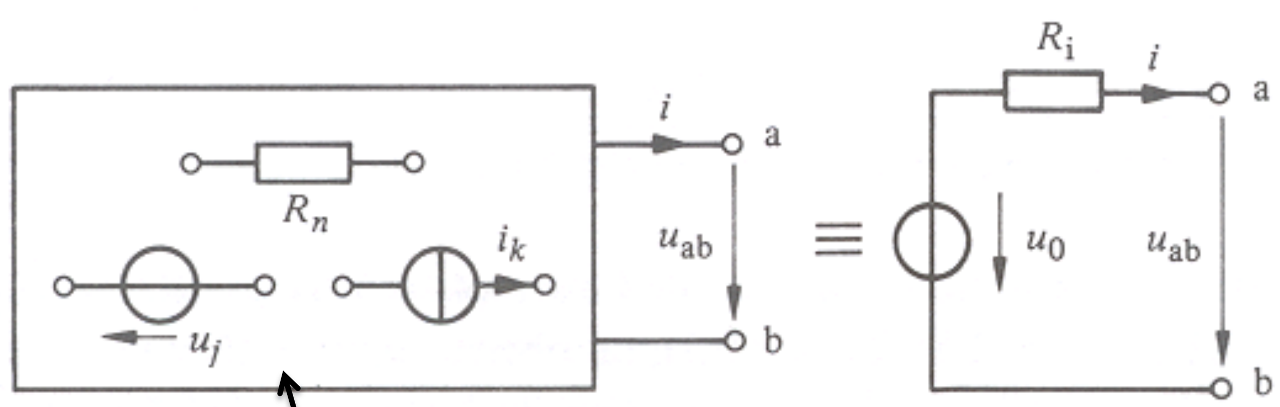
Léon **Thévenin** (1857-1926), French physicist. He published his 1883 theorem.



Hermann von **Helmoltz** (1821-1894), German physicist and physiologist. He published the theorem called Thévenin in 1863.

Thévenin's equivalence

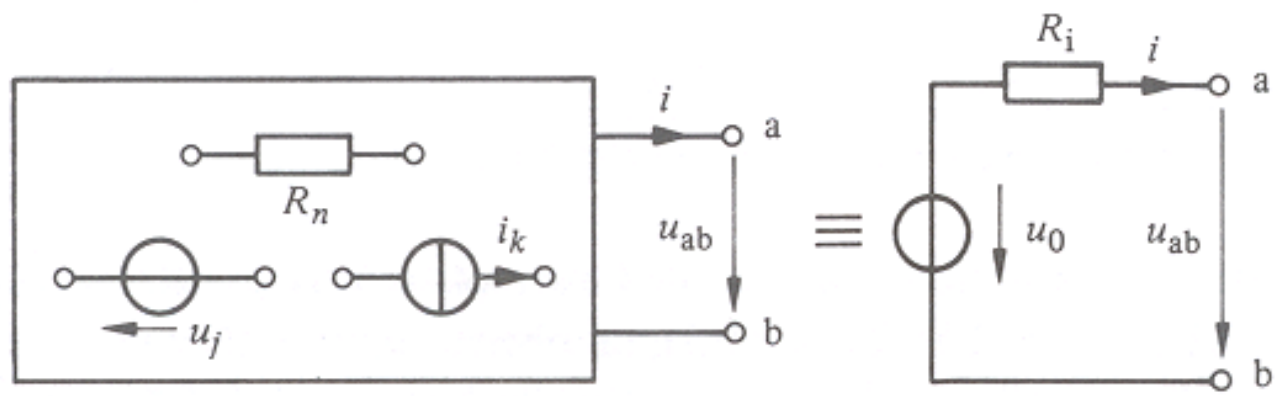
- Goal: replace a complex circuit with a circuit consisting of a voltage source and a resistor



Circuit composed of resistors, voltage sources (independent and controlled), current sources (independent and controlled)

Thévenin's equivalence

- Goal: replace a complex circuit with a circuit consisting of a voltage source and a resistor



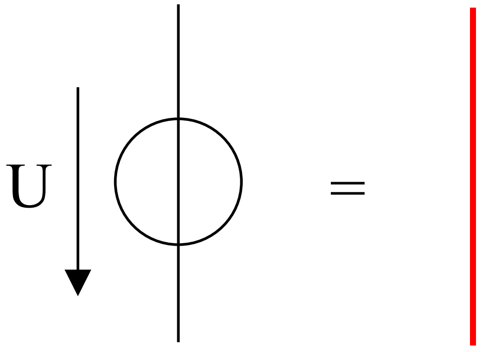
$$u_o = u_{ab}\big|_{i=0} \qquad R_i = R_{ab}\big|_{u_j=0,\, i_k=0}$$

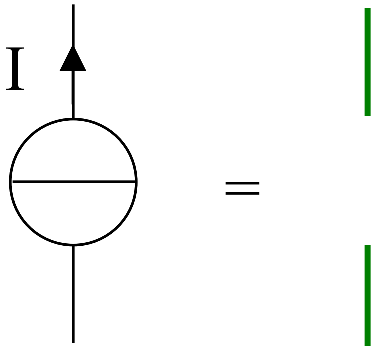
Thévenin’s equivalence: Procedure

- Identify the terminals of the circuit and the external charge
- Measure or calculate the voltage across the circuit without external charge. This is Thévenin’s voltage.
- Cancel the independent sources and determine the resistance seen from the terminals of the circuit. This is Thévenin's resistance.

Thévenin’s theorem

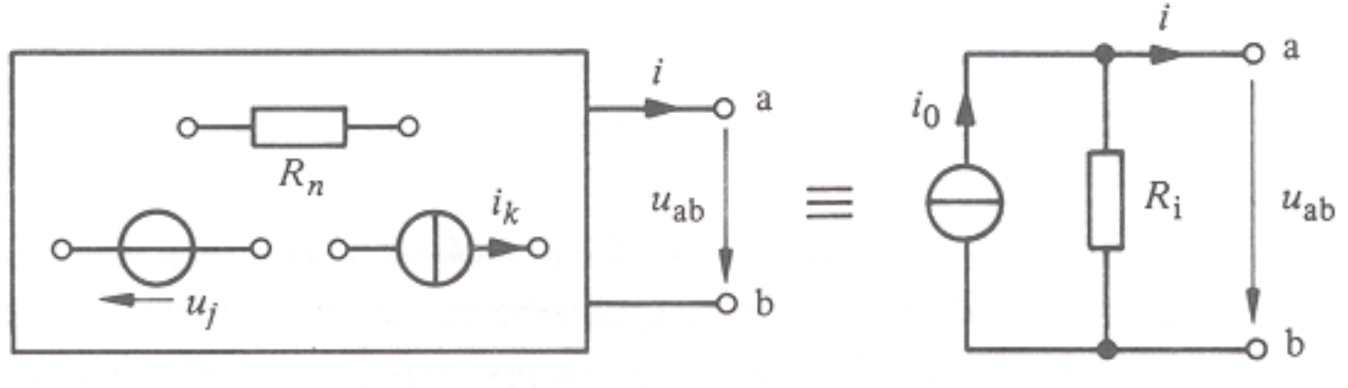
- To cancel a voltage source is to replace it with a short circuit.
- To cancel a current source is to replace it with an open circuit.





Norton's equivalence

- Corollary of Thévenin's theorem



$$i_o = i\big|_{u_{ab}=0} \qquad R_i = R_{ab}\big|_{u_j=0,\, i_k=0}$$

Norton

Edward Larry **Norton** (1898-?), researcher at Bell Telephone Labs (USA). The first mention of his theorem dates from 1926.

Norton’s equivalence: Procedure

- Identify the terminals of the circuit and the external charge
- Short-circuit the circuit terminals and theoretically or experimentally determine the intensity of the short-circuit current. This is Norton's power source.
- Cancel the independent sources and determine the resistance seen from the terminals of the circuit. This is the resistance of Norton (and Thévenin).

Question

- To establish the equivalent Thévenin and Norton diagrams of a circuit, it is necessary to measure:

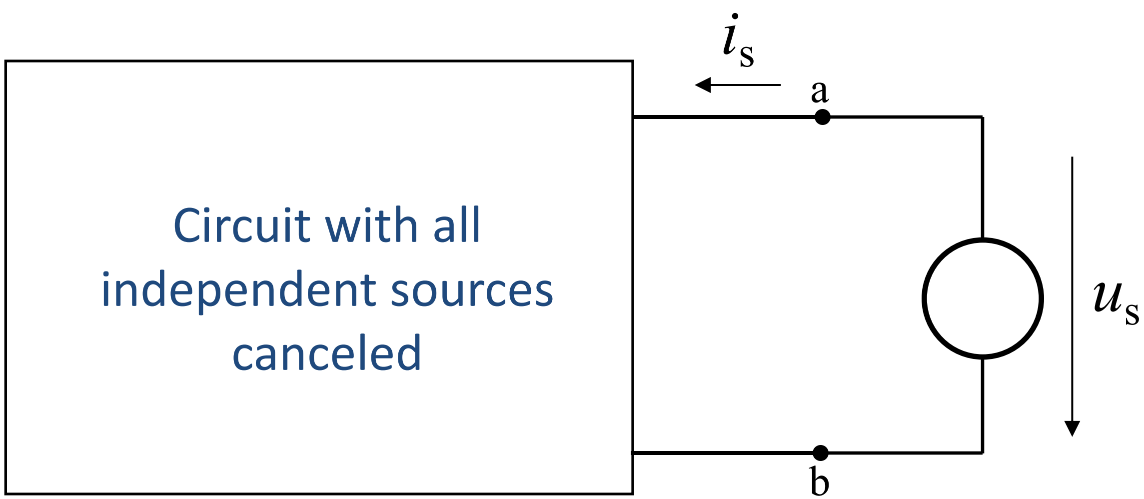
A. No-load voltage and short-circuit current
B. No-load voltage, short-circuit current and circuit resistance (with sources canceled)

Resistance of Thévenin / Norton in case of dependent sources

- If the circuit has dependent sources, the resistance of Thévenin can be determined by two methods

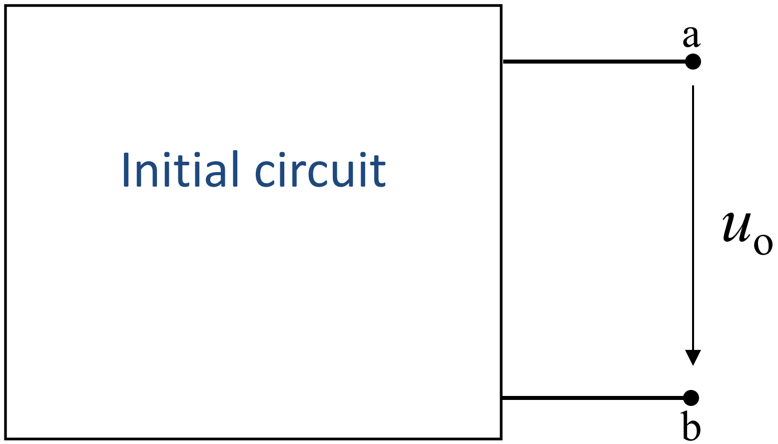
Resistance of Thévenin / Norton in case of dependent sources: 1st method

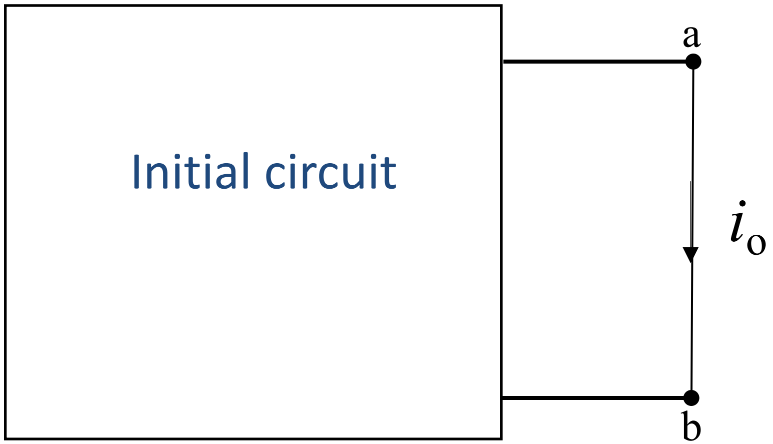
- Cancel all independent sources and apply a voltage source u_s and determine the result current i_s. Thévenin's resistance is then $R_i = u_s / i_s$.



Resistance of Thévenin / Norton in case of dependent sources: 2nd method

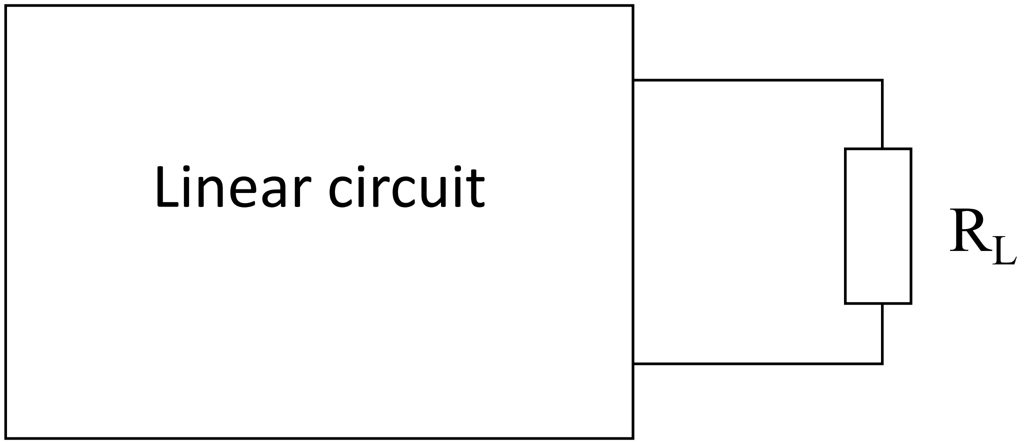
- Determine the no-load voltage u_o and the short-circuit current i_o. The resistance of Thévenin is then $R_i = u_o / i_o$.





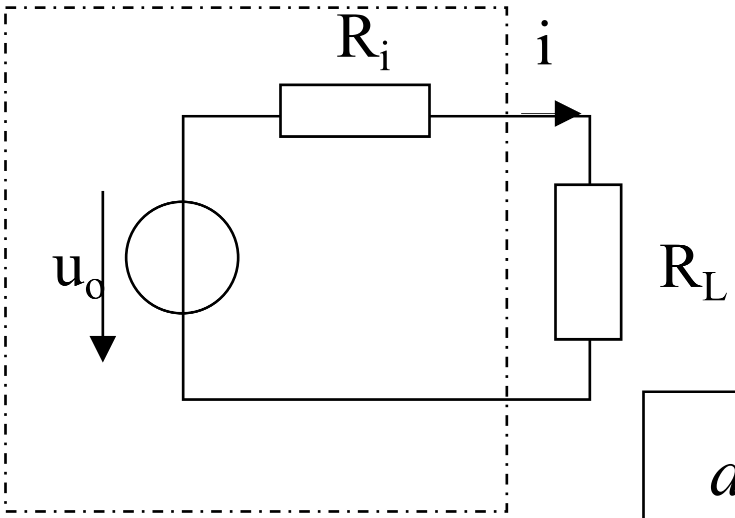
Maximum power transfer theorem

- What is the value of the load resistor R_L for which the latter will dissipate a maximum of power ?



Maximal power transfer theorem

- Answer: we replace the complex circuit with its Thévenin's equivalence :

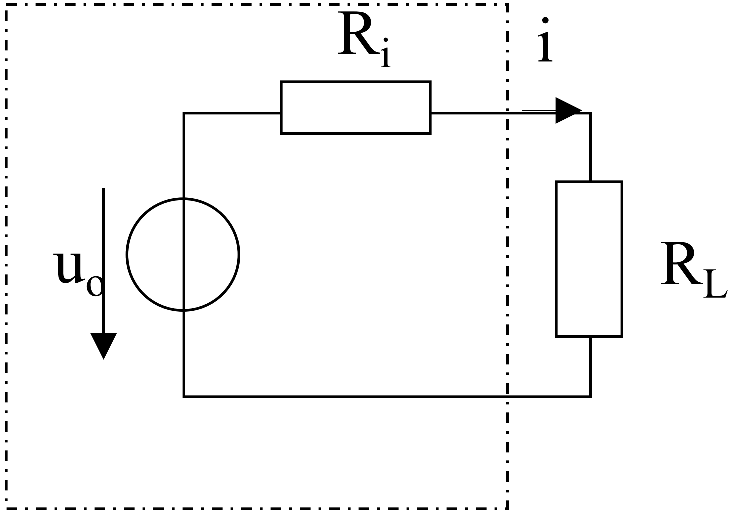


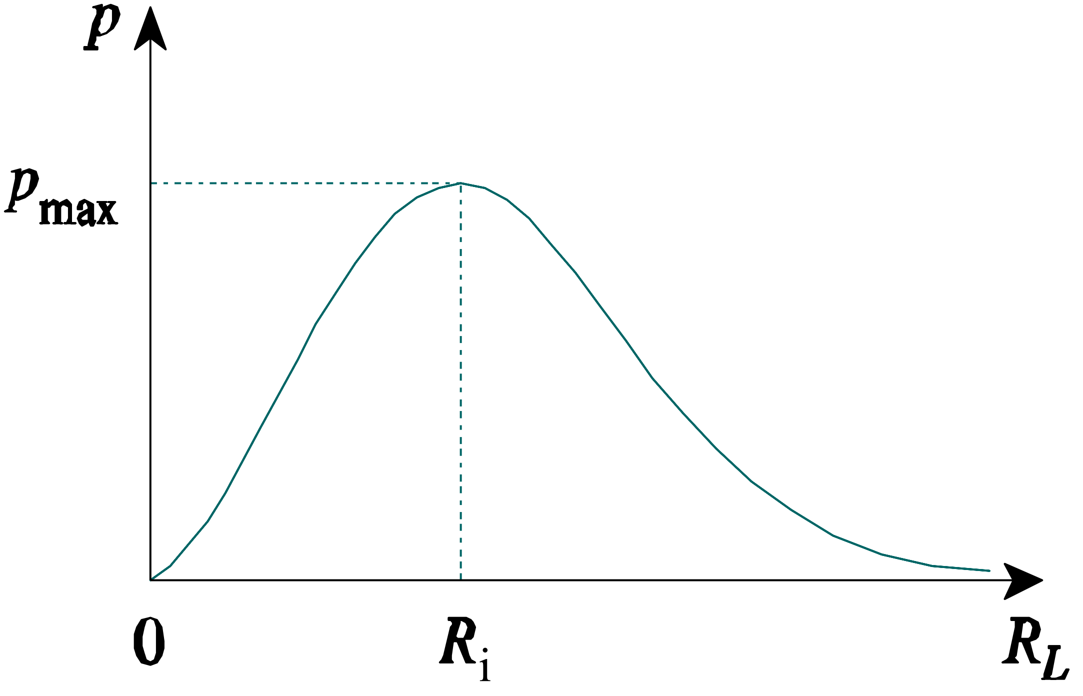
$$i = \frac{u_o}{R_i + R_L}$$

$$P = R_L i^2$$

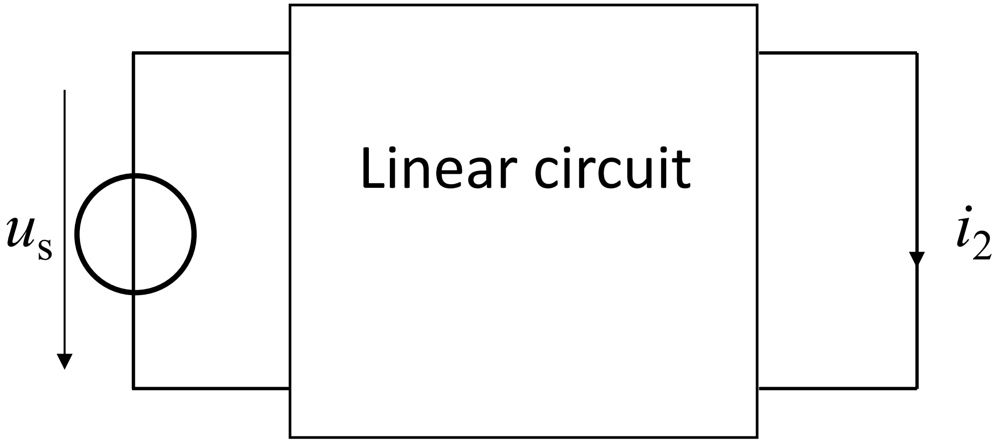
$$\boxed{\frac{dP}{dR_L} = 0 \Rightarrow R_L = R_i \,, \quad P_{\max} = \frac{u_o^2}{4R_i}}$$

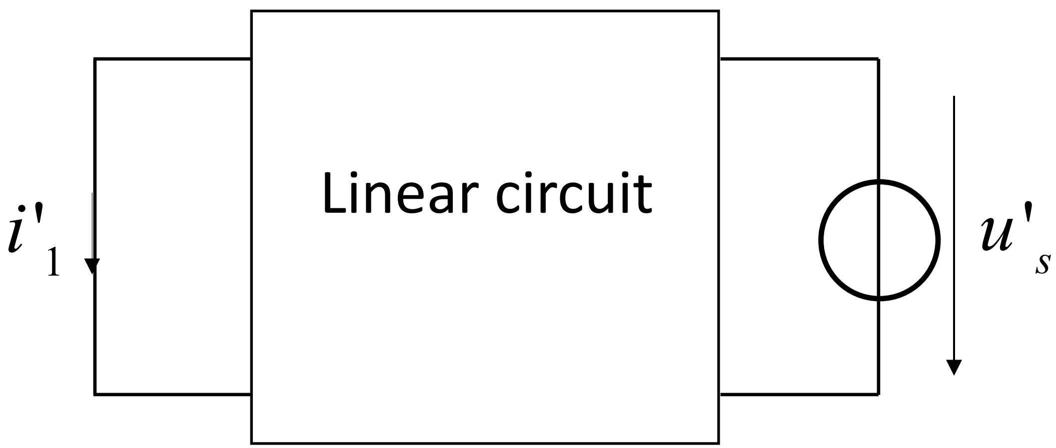
Maximal power transfer theorem





Reciprocity Theorem : 1st case

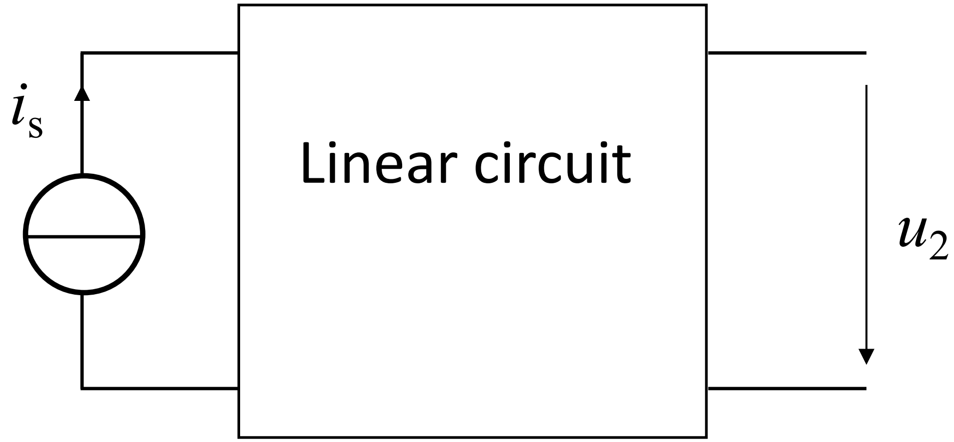


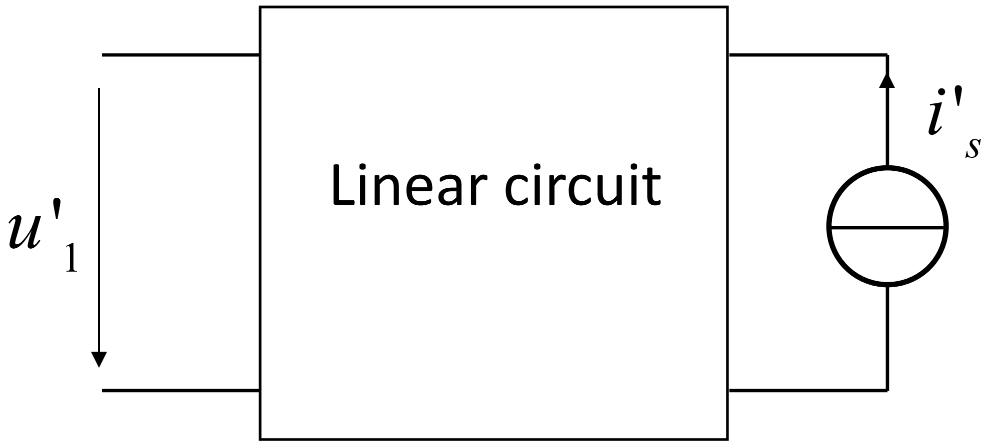


if $u_s = u'_s$ then $i'_1 = i_2$

$$\boxed{\frac{i'_1}{u'_s} = \frac{i_2}{u_s}}$$

Reciprocity Theorem : 2nd case





if $i_s = i'_s$ then $u'_1 = u_2$

$$\boxed{\frac{u'_1}{i'_s} = \frac{u_2}{i_s}}$$

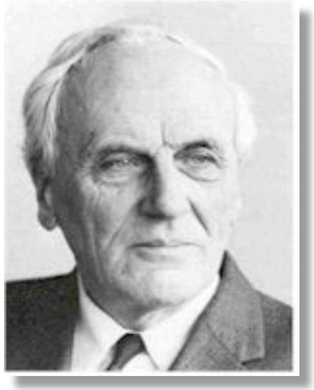
Tellegen's theorem

- If any electrical circuit has N branches, with voltages u_k and currents i_k at each branch k, then

$$\sum_{k=1}^{N} u_k i_k = 0$$

- This theorem is a consequence of Kirchoff's laws and translates the conservation of energy in an electric circuit

Bernard D.H. Tellegen (1900-1990), Dutch electrical engineer. He formulated the so-called Tellegen theorem in 1953.



Tellegen's theorem: corollary

- If two circuits have the same topology, then

$$\sum_{k=1}^{N} u_k i_k = 0 \qquad \sum_{k=1}^{N} u'_k i'_k = 0$$

- But also

$$\sum_{k=1}^{N} u_k i'_k = 0 \quad \text{and} \quad \sum_{k=1}^{N} u'_k i_k = 0$$

Tellegen's theorem: example

On the board!