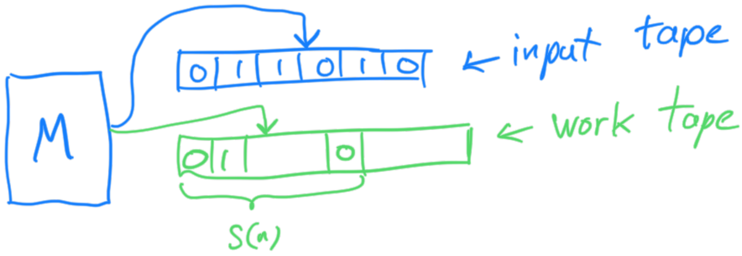
Space Complexity

Def: Space complexity of a TM M is $s: \mathbb{N} \to \mathbb{N}$

$$s(n) = \max_{x \in \{0,1\}^n} \left\{ \#\ {}^{\text{work}}\text{memory cells accessed by } M \text{ on } x \right\}$$

Note: Read-only input, separate work memory



Def: $\text{N}\,\text{SPACE}(s(n)) = \{ L : \exists\ \text{N}\,\text{TM } M \text{ deciding } L \text{ in space } O(s(n)) \}$

$$L := \text{SPACE}(\log n)$$

smallest "interesting" time bound

$$NL := \text{NSPACE}(\log n)$$

$$\text{N}\,\text{PSPAE} := \bigcup_{k \in \mathbb{N}} \text{N}\,\text{SPACE}(n^k)$$

Ex. $\text{SAT} \in \text{SPACE}(n) \implies NP \subseteq \text{PSPACE}$

Ex. Palindromes $\in L$ "saippuakivikauppias"

- Compute $n = |x|$
- for $i = 1, \ldots, n$
  Check: $x_i = x_{n-i+1}$

Things to do in L:

- Can assume $|x| = n$ is known
- Use int counters with range $0, \ldots, n^{O(1)}$

constantly many

- Basic arithmetic OK

Ex. $\text{STCONN} = \{\langle G, s, t\rangle : \exists\ s \rightsquigarrow t \text{ path in } G\}$

directed graph

BFS/DFS requires $\Omega(n)$ space!

$\text{STCONN} \in \text{NL}$: perform non-det walk from s for n steps, accept if visit t.

$\overset{?}{\in} \text{L}$

(open!)

Savitch ['70]: $\text{STCONN} \in \text{SPACE}(\log^2 n)$ ← easy

Reingold ['05]: Undirected $\text{STCONN} \in \text{L}$ ← hard!

Space vs. Time

Easy: $\text{TIME}(t(n)) \subseteq \text{SPACE}(t(n)) \Rightarrow \text{P} \subseteq \text{PSPACE}$

Claim: $\text{SPACE}(s(n)) \subseteq \text{TIME}\left(2^{O(s(n))}\right) \Rightarrow \text{L} \subseteq \text{P}$

Proof: Configuration of M on input x:

- work tape contents ← $O(s(n))$ bits
- state of TM ← $O(1)$
- work tape head pointer ← $\log s(n)$
- input tape head pointer ← $\log n$

$\Rightarrow O(s(n))$ bits

$\Rightarrow 2^{O(s(n))}$ many configs

[assume $s(n) \geq \log n$]

Note: no config repeats

$\Rightarrow$ runtime $\leq 2^{O(s(n))}$ ∎

Summary: $\text{L} \subseteq \text{P} \subseteq \text{PSPACE} \subseteq \text{EXP}$

time

space

Hierarchy thms: $\text{P} \subsetneq \text{EXP}$

$\text{L} \subsetneq \text{PSPACE}$

Savicth's Theorem

Thm: $NSPACE(s(n)) \subseteq SPACE(s(n)^2)$

$\hookrightarrow NPSPACE = PSPACE$

Configuration graph $G=(V,E)$

$V = \{0,1\}^{O(s(n))}$

$(u,v) \in E \iff$ config v can be reached from u in a single non-det step

(Easy to check given u,v)

We prove: $STCONN \in SPACE(\log^2 n)$

idea: "middle first search" $G=(V,E)$

$s \rightsquigarrow w \rightsquigarrow t$

$\underbrace{w}_{n \text{ possibilities}}$

Path(u, v, k):

"There is $u \rightsquigarrow v$ path of length $\leq k$"

if $k \leq 1$ return $(u=v)$ OR $(u,v) \in E$

else for $\forall w \in [n]$ $\leftarrow$ $\log n$ bits

  if Path$(u, w, k/2)$ AND

   Path$(w, v, k/2)$:

    return TRUE

return FALSE

Space complexity: $s(n,k) = O(\log n) + s(n, k/2)$

$\Rightarrow s(n,n) = O(\log^2 n)$

Runtime: $t(n,k) = n \cdot 2t(n, k/2)$

$\Rightarrow t(n,n) = n^{\Theta(\log n)}$

Note: Open if STCONN in simultaneoes polytime &

polytime &
polylog space

TQBF is PSPACE-complete

TQBF: $\forall x_1 \, \exists x_2 \, \forall x_3 \cdots Q_n x_n : \underbrace{\varphi(x)}_{\text{CNF}}$

Is this formula TRUE?

Claim: $\forall L \in SPACE(n^k): \; L \leq_p TQBF$

Proof: $\varphi_i(u,v) =$ "$u \mapsto v$ in $\leq 2^i$ steps in G"

u, v: $O(n^k)$-bit strings

G: config graph of M_L

$$\varphi_0(u,v) = (u=v) \vee \underbrace{(u,v) \in E}_{\text{circuit of size } O(n^k)}$$

$$\varphi_{i+1}(u,v) = \exists w: \varphi_i(u,w) \wedge \varphi_i(w,v)$$

Too Big!

Better:
$$= \forall x,y \, \exists w : \big[(x=u) \wedge (y=w)\big] \vee \big[(x=w) \wedge (y=v)\big] \Rightarrow \varphi_i(x,y)$$

$\Rightarrow \; \varphi_{n^k}(v_{start}, v_{accept})$ has size $O(n^{2k})$ $\blacksquare$