Lecture 2: NP-completeness

Mika Göös

EPFL School of Computer and Communication Sciences

Lecture 2, 19.09.2024

Recap: NP

The class **NP**

Definition: A verifier for a language A is a TM V, where

$$A = \{x \mid \exists C \text{ s.t. } V \text{ accepts } \langle x, C \rangle\}.$$

A polynomial time verifier runs in polynomial time in $|x|$.

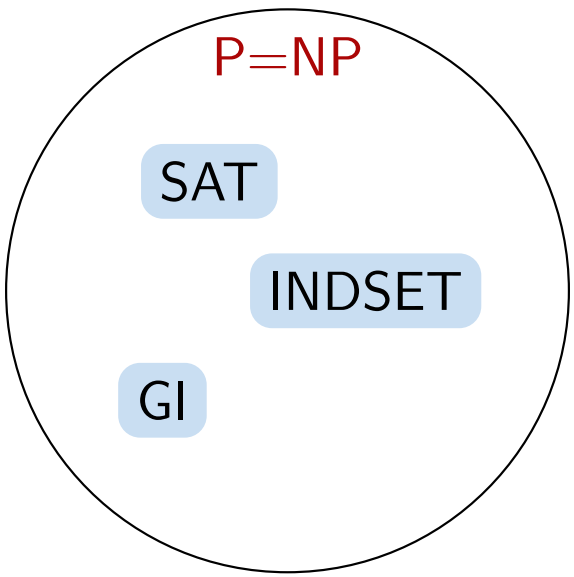
Definition: **NP** is the class of languages with polynomial time verifiers

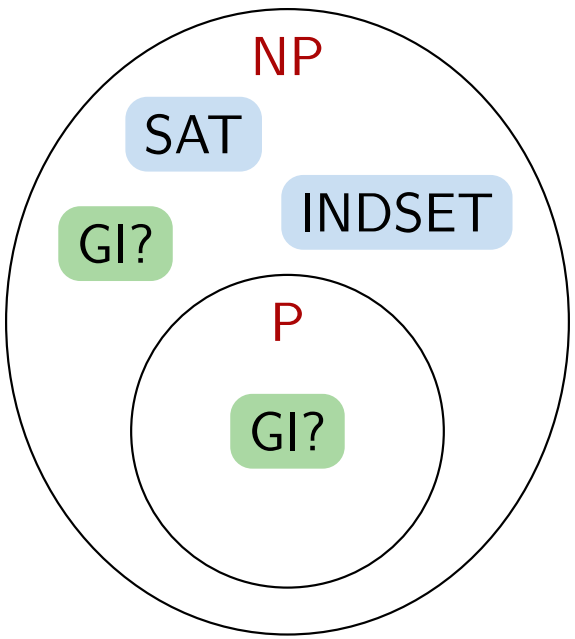
P and NP

Is $P \subseteq NP$? Yes (obviously)

Is $P = NP$? Nobody knows ...

Find the answer and win USD 1,000,000!





Cook-Levin Theorem (informal): SAT $\in$ P iff $P = NP$.

(Also INDSET $\in$ P iff $P = NP$.)

Why is it called NP?

Non-deterministic Turing Machines

Recall: In a Turing machine, $\delta : (Q \times \Gamma) \longrightarrow Q \times \Gamma \times \{L, R\}$.

In a **Nondeterministic Turing Machine (NTM)**,

$$\delta : (Q \times \Gamma) \longrightarrow \mathcal{P}(Q \times \Gamma \times \{L, R\})$$

(several possible transitions for a given state and tape symbol)

Definition: A **nondeterministic decider** for language L is an NTM N such that for each $x \in \Sigma^*$, **every computation of N on x halts**, and moreover,

- If $x \in L$, then some computation of N on x accepts.
- If $x \notin L$, then every computation of N on x rejects.

An NTM is a **polynomial time** NTM if the running time its longest computation on x is **polynomial in $|\mathbf{x}|$**.

Nondeterministic deciders $\iff$ Verifiers

Theorem: For any language $L \subseteq \Sigma^*$,

L has a nondeterministic poly-time decider $\iff$ L has a poly-time verifier.

Proof Sketch ($\Longleftarrow$):

Let V be the verifier. NTM N on input x does the following:

1. Write a certificate C nondeterministically.
2. Run V on $\langle x, C \rangle$.

Proof Sketch ($\Longrightarrow$):

Let N be the nondeterministic decider. Verifier V on $\langle x, C \rangle$ computes:

- Simulate N on x, choosing transitions given by C.

V accepts $\langle x, C \rangle$ iff C is the accepting path of N on x.

Non-deterministic Polynomial-time

Theorem: For any language $L \subseteq \Sigma^*$,

L has a nondeterministic poly-time decider $\iff$ L has a poly-time verifier.

Definition: **NP** is the class of languages which have poly-time nondeterministic deciders, or equivalently, have poly-time verifiers.

Definition:

$$\text{NTIME}(t(n)) = \{L \ : \ L \text{ has a nondeterministic } O(t(n)) \text{ time decider}\}$$

Then

$$\text{NP} = \bigcup_{k=1}^{\infty} \text{NTIME}(n^k)\,.$$

$\therefore$ SAT, GI, INDSET are all in NP.

Reductions and NP-completeness

Reductions

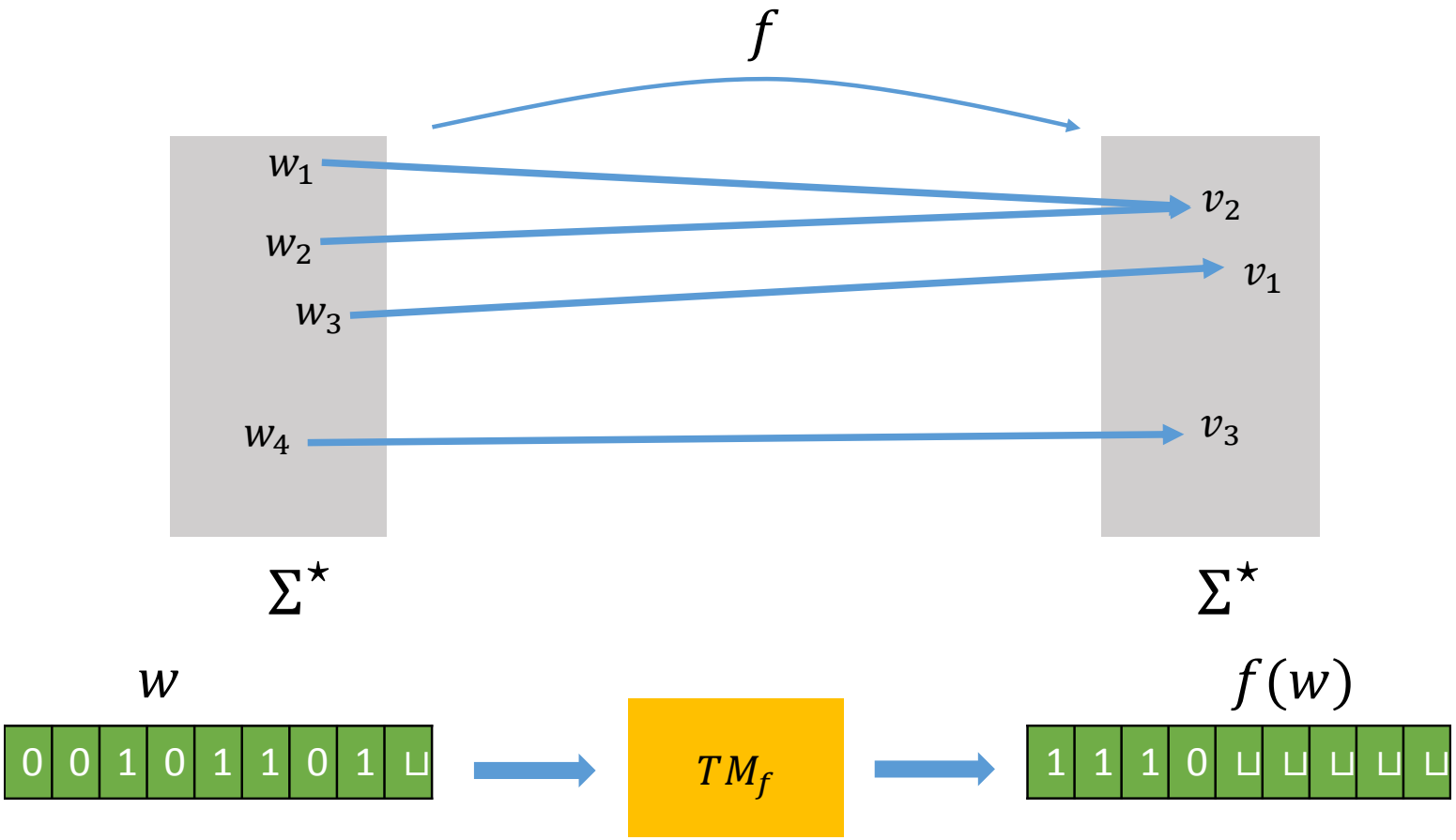
Reducibility Use knowledge about complexity of one language to reason about the complexity of other in an easy way

Reduction Way to show that to solve one problem (A), it is sufficient to solve another problem (B)

A reduces to B...

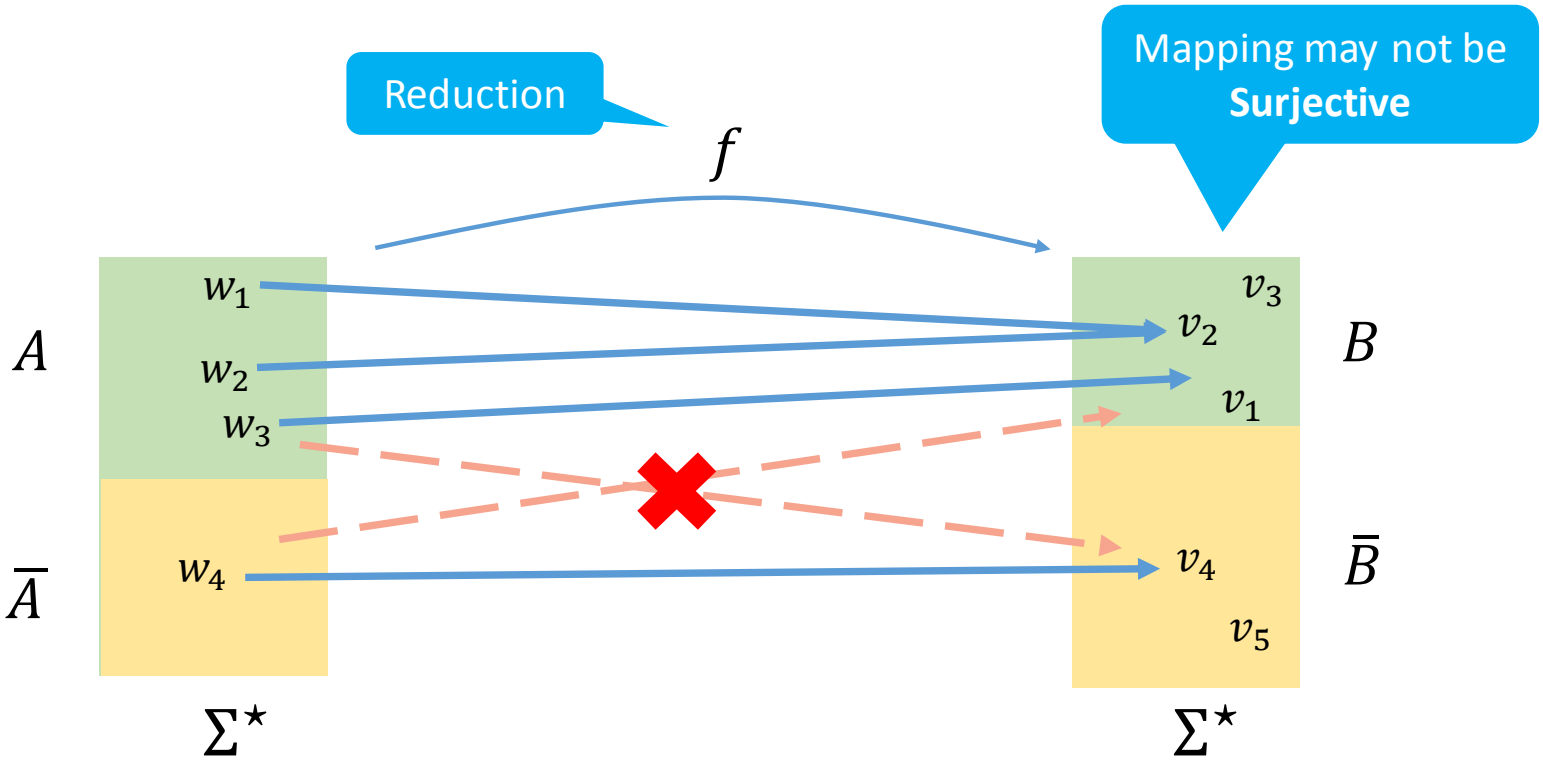
- Reductions quantify relative hardness of problems
  - If problem B is easy then problem A is easy too
  - If problem A is hard then problem B is hard too

Poly-time Reductions (1): Poly-time Computability



Definition: A function $f : \Sigma^* \to \Sigma^*$ is *polynomial time computable* if there is some poly-time TM that on every input w, outputs $f(w)$

Poly-time Reductions (2): Correctness



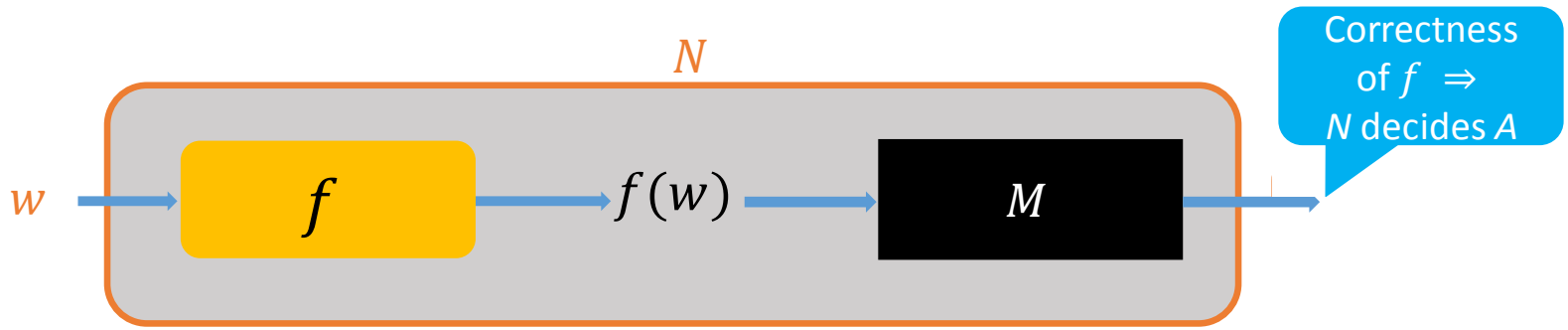
Definition: Language A is **poly-time reducible** to language B, written $A \leq_P B$, if there is a poly-time computable function $f : \Sigma^* \to \Sigma^*$, such that for every $w \in \Sigma^*$:

$$w \in A \Leftrightarrow f(w) \in B$$

Theorem: If $A \leq_P B$ and $B \in \mathbf{P}$, then $A \in \mathbf{P}$

Proof:

- Let M be a is a poly-time algorithm for B and let f be a poly-time reduction from A to B
- Define algorithm N as follows:



- $N =$ "On input w :
  1. Compute $f(w)$
  2. Run M on input $f(w)$ and output whatever M outputs"

Corollary: If $A \leq_P B$ and $A \notin \mathbf{P}$, then $B \notin \mathbf{P}$

NP-completeness

Definition: A language L is said to be **NP-complete** if

- L is in **NP**.
- For every language L' in **NP**, $L' \leq_P L$.

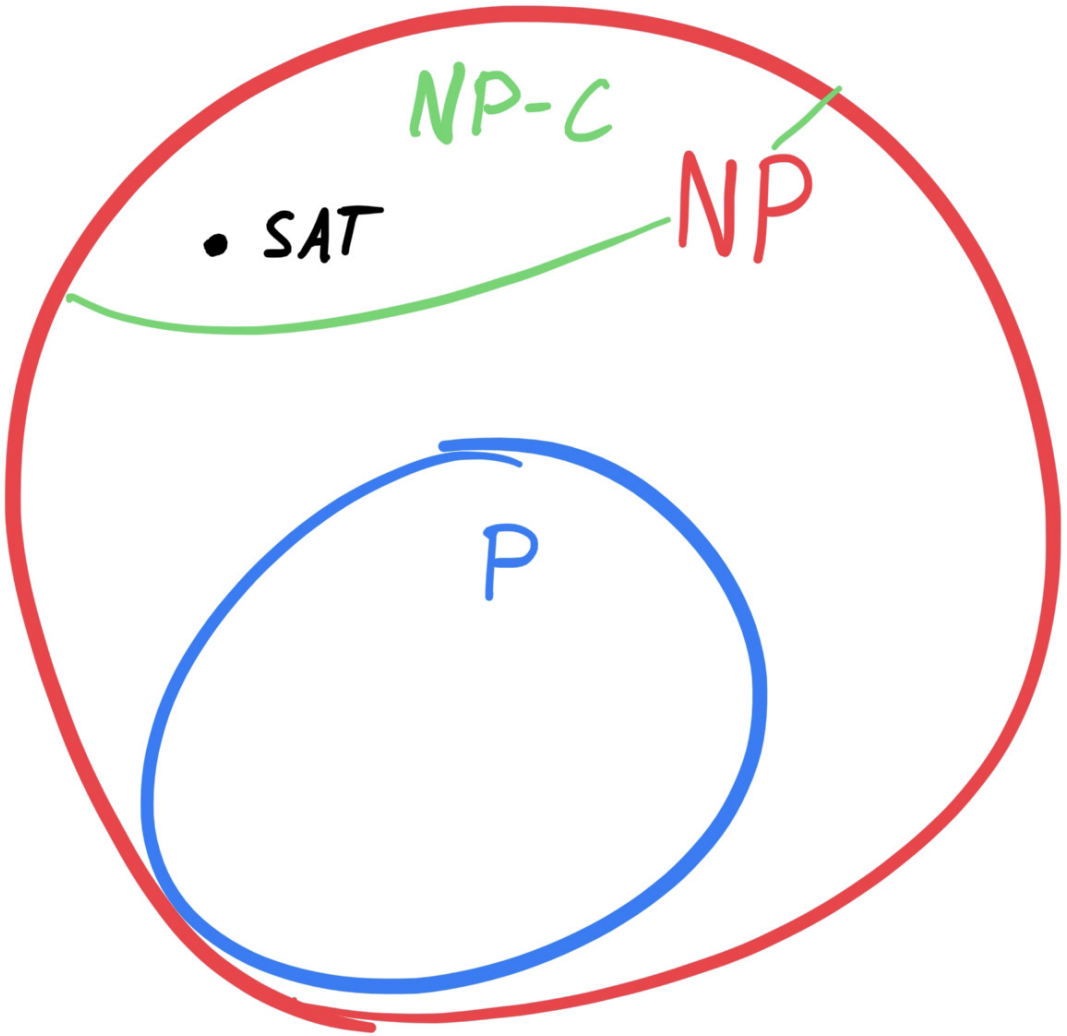
Observe: If **one NP**-complete language has a polynomial time decider, then **every** language in **NP** has a polynomial time decider, i.e. **P** = **NP**.

The Cook–Levin Theorem: SAT is **NP**-complete.

To show L is **NP**-complete:

- **[NP membership]** Give a poly-time verifier for L.
- **[NP hardness]** Show $C \leq_P L$ for some **NP**-complete language C (not the other way around)

Big picture



Takehome message

"I can't find an efficient algorithm, but neither can all these famous people."

3SAT is NP-complete

$k\text{SAT} = \{\langle\varphi\rangle \ : \ \varphi \text{ is satisfiable and each clause of } \varphi \text{ contains } \leq k \text{ literals}\}$

Verifier for 3SAT: Just use the verifier for SAT.

Claim: $\text{SAT} \leq_P \text{3SAT}$

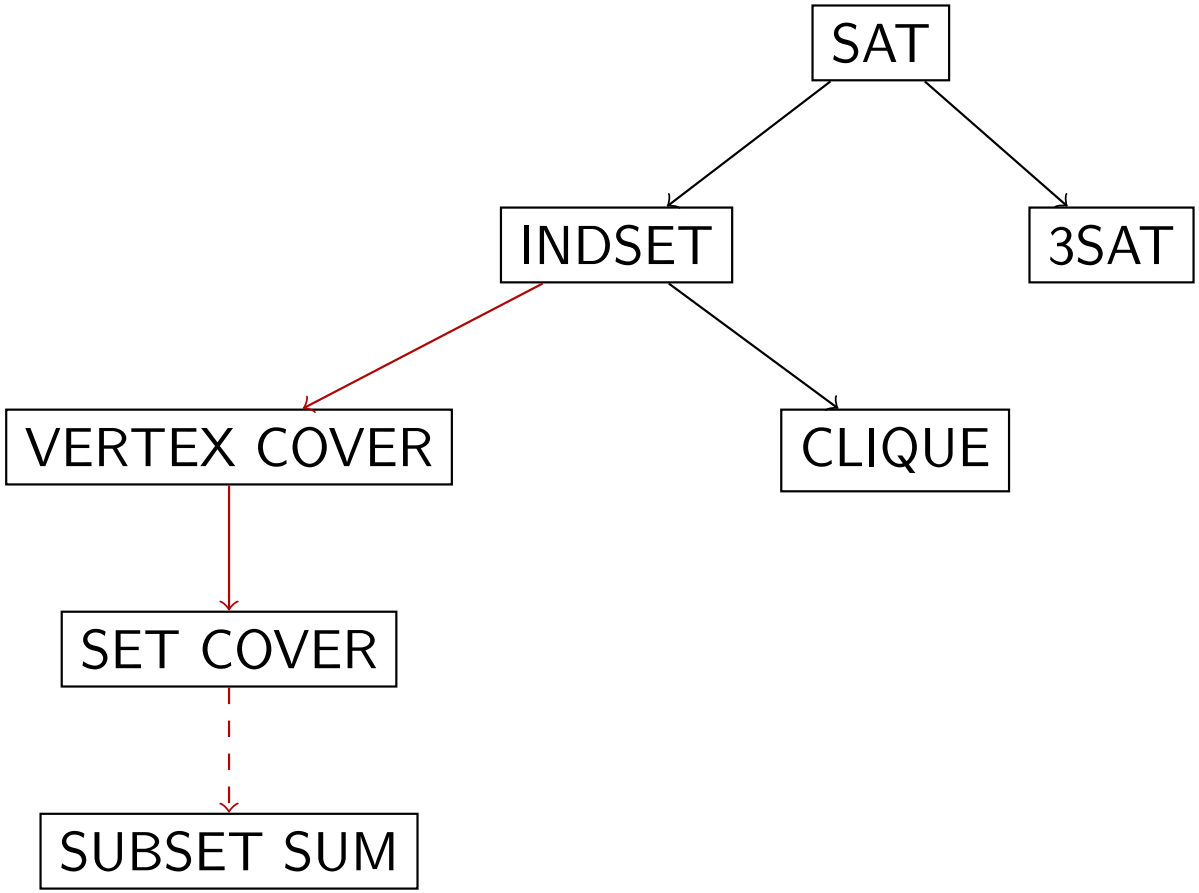
Reduction: Given φ,

- While φ contains a clause $K = (\ell_1 \vee \ell_2 \vee \ell_3 \vee \cdots \vee \ell_m)$ with > 3 literals

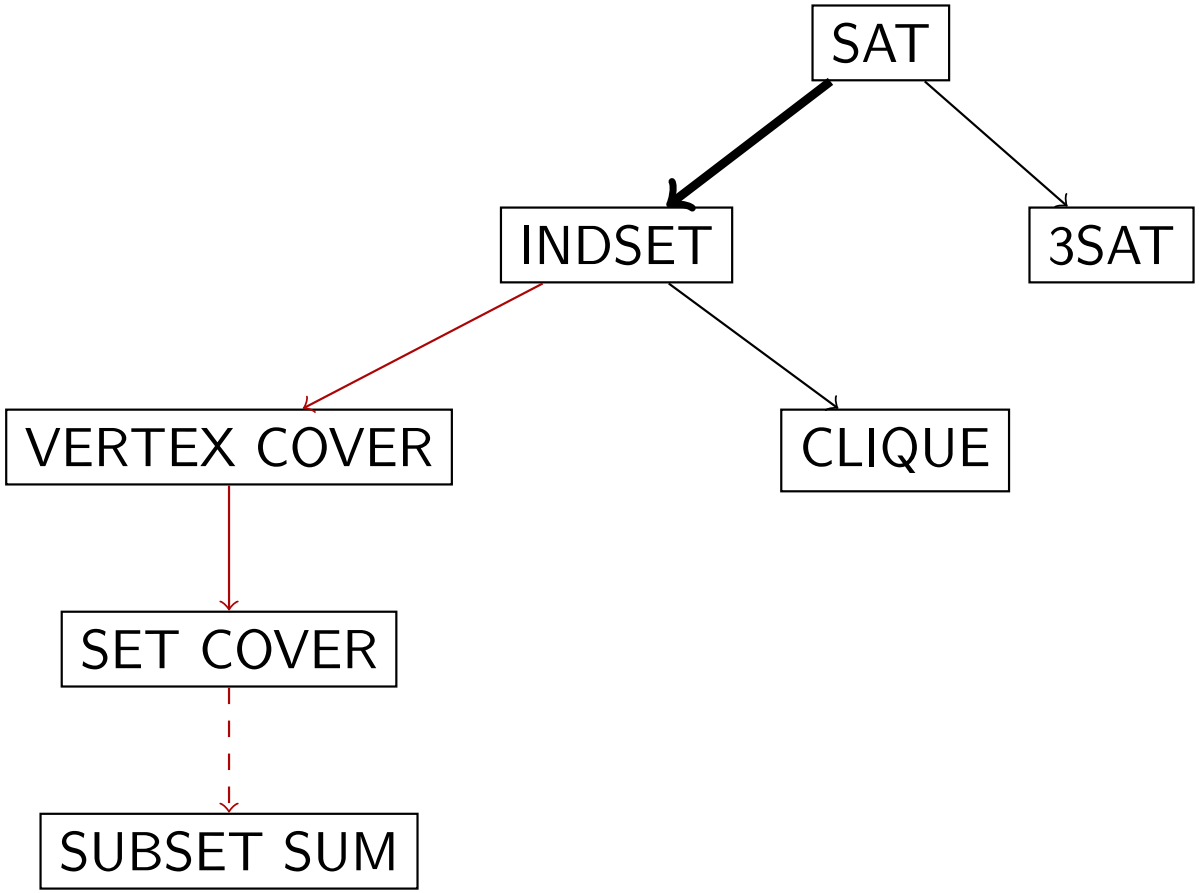
  Replace K with the following two clauses
  $$K_1 = (\ell_1 \vee \ell_2 \vee z)$$
  $$K_2 = (\overline{z} \vee \ell_3 \vee \cdots \vee \ell_m)$$
  } Preserves satisfiability (check!)

What is the runtime?

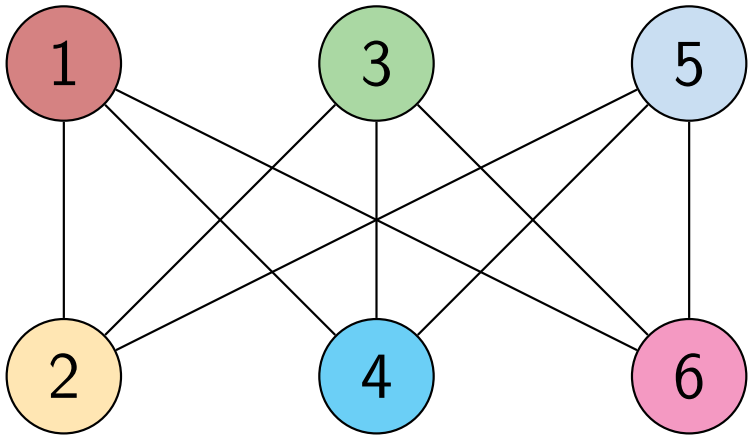
Does $\text{SAT} \leq_P \text{2SAT}$ analogously?



and many more . . .


SAT
INDSET
3SAT
VERTEX COVER
CLIQUE
SET COVER
SUBSET SUM

INDSET is NP-complete



We have already seen that

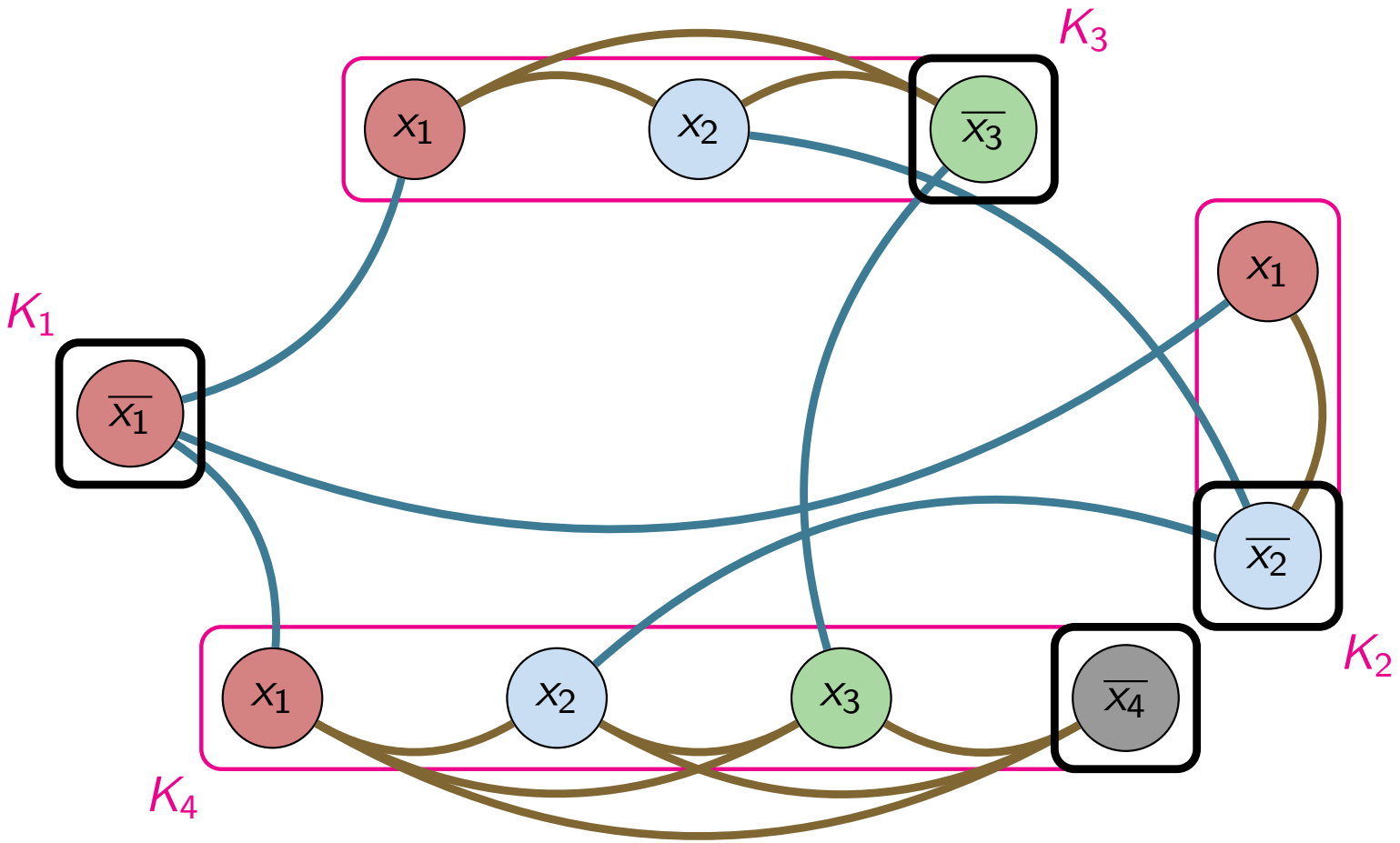
$$\text{INDSET} = \{\langle G, k\rangle : G \text{ has an independet set of size } k\}$$

is in **NP** and so it remains to give a poly-time reduction from an **NP**-complete language

INDSET is NP-complete

Claim: SAT $\leq_P$ INDSET

$$\varphi = \underbrace{\overline{x_1}}_{K_1} \wedge \underbrace{(x_1 \vee \overline{x_2})}_{K_2} \wedge \underbrace{(x_1 \vee x_2 \vee \overline{x_3})}_{K_3} \wedge \underbrace{(x_1 \vee x_2 \vee x_3 \vee \overline{x_4})}_{K_4}$$



INDSET is NP-complete

Claim: SAT $\leq_P$ INDSET

Reduction f: On input φ,

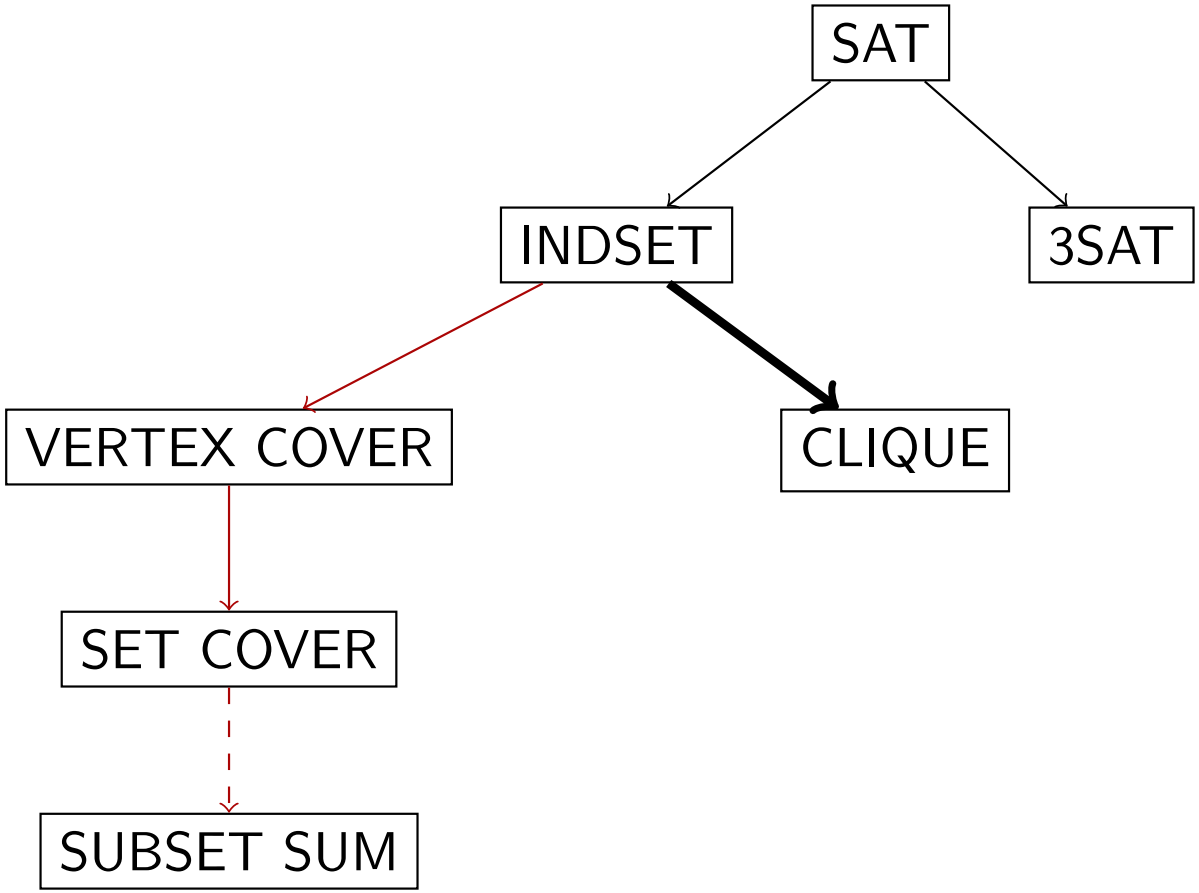
1. Let G be the graph generated as follows.
   1. Take a vertex for each literal of each clause.
   2. Add edges for pairs of conflicting literals.
   3. Add edges for pairs of literals from the same clause.
2. Let m be the number of clauses in φ.
3. Output (G, m).

Claim: $\varphi \in \text{SAT} \implies f(\varphi) \in \text{INDSET}$

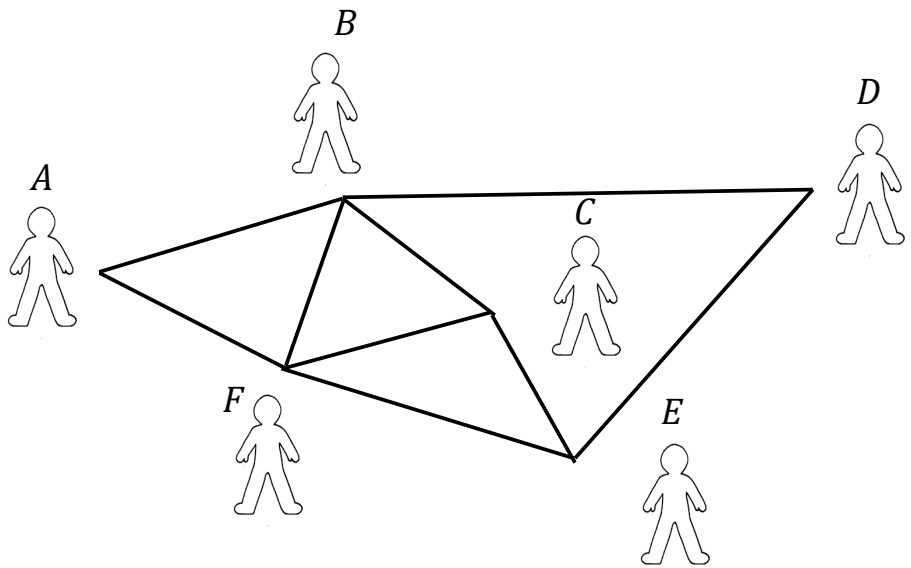
Proof: C: satisfying assignment of φ. Pick one `true` literal from each clause. The corresponding vertices form a independent set.

Claim: $f(\varphi) \in \text{INDSET} \implies \varphi \in \text{SAT}$

Proof: C: independent set in G, $|C| = m$. C contains one vertex from each group. Set the corresponding literals to `true` to get a satisfying assignment.


SAT
INDSET
3SAT
VERTEX COVER
CLIQUE
SET COVER
SUBSET SUM

CLIQUE



A,B,F is a 3-clique

Definition: A k-clique is a subset of k pairwise connected vertices

$$\text{CLIQUE} = \{\langle G, k\rangle \mid G \text{ has a clique of size } k\}$$

$\langle G, 3\rangle \in \mathit{CLIQUE}$? Yes

$\langle G, 4\rangle \in \mathit{CLIQUE}$? No

CLIQUE is **NP**-Complete

Theorem: INDSET $\leq_p$ CLIQUE

Corollary: CLIQUE is **NP**-complete

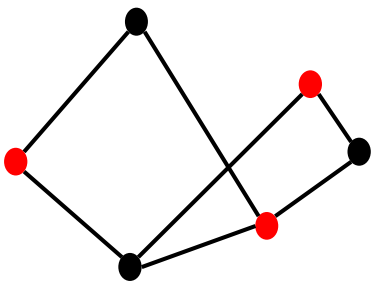
Goal find a poly-time reduction from INDSET to CLIQUE

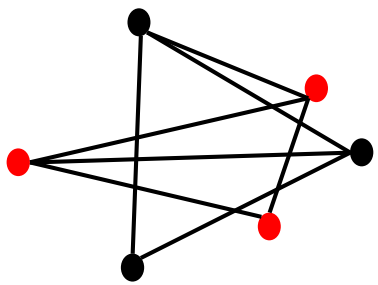
INDSET vs CLIQUE

For a graph G:

Independent set: A subset of pairwise **non-adjacent** vertices

Clique: A subset of pairwise **adjacent** vertices





Def (Complement): The *complement* of a graph $G = (V, E)$ is a graph $\bar{G} = (V, \bar{E})$ with same vertex set and edge set $\bar{E}$ s.t. $uv \in \bar{E}$ iff $uv \notin E$

Observation: If G is a graph and $\bar{G}$ its complement, then a subset S of the vertices of G is an independent set iff S is a clique of $\bar{G}$

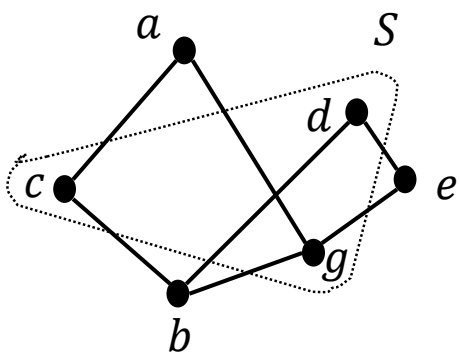
INDSET $\leq_p$ CLIQUE

Reduction: $f(\langle G = (V, E), k\rangle) := \langle \bar{G} = (V, \bar{E}), k\rangle$

Efficiency: The reduction is polynomial time

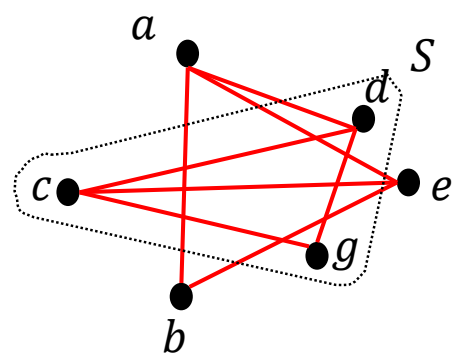
Correctness: $\langle G, k\rangle \in \text{INDSET} \Leftrightarrow \langle \bar{G}, k\rangle \in \text{CLIQUE}$

Proof of correctness:

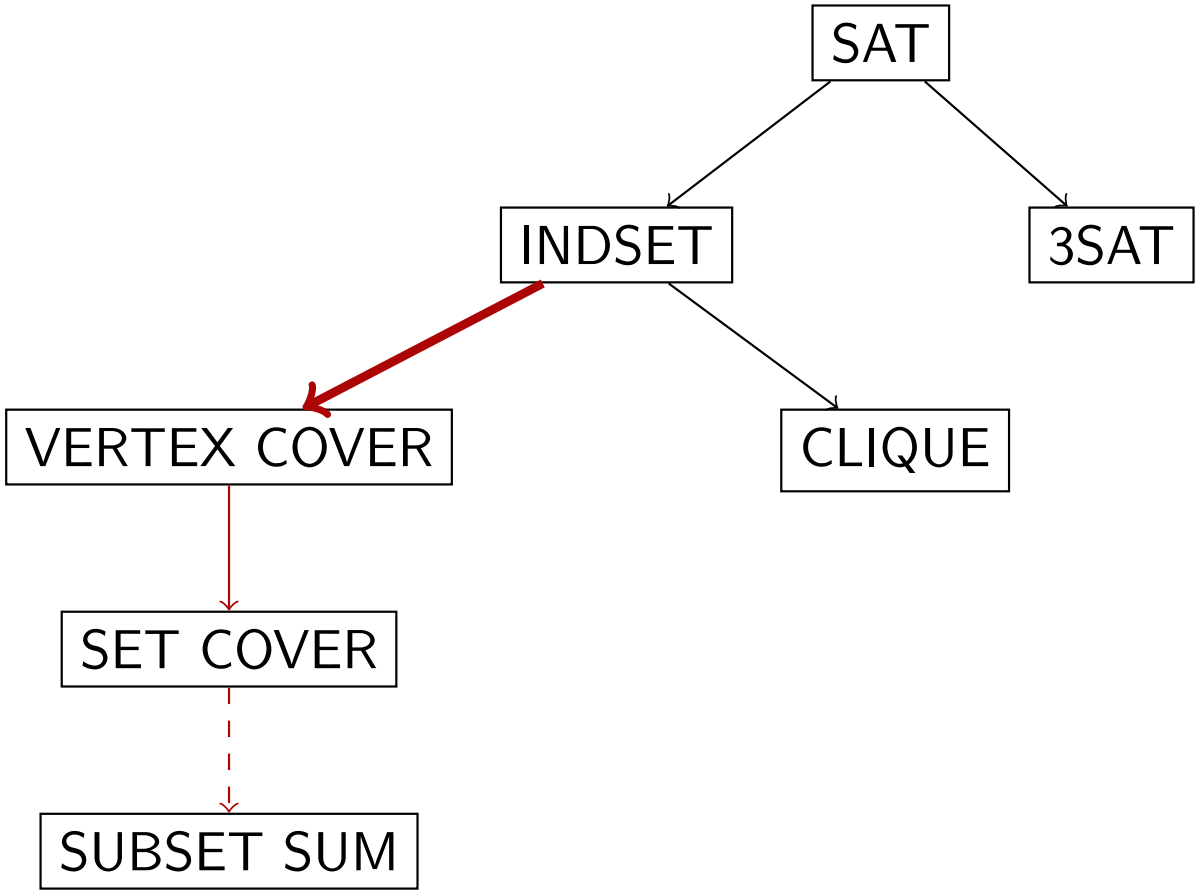


$\langle G, k\rangle \in INDSET, |S| = k$

$\Leftrightarrow$



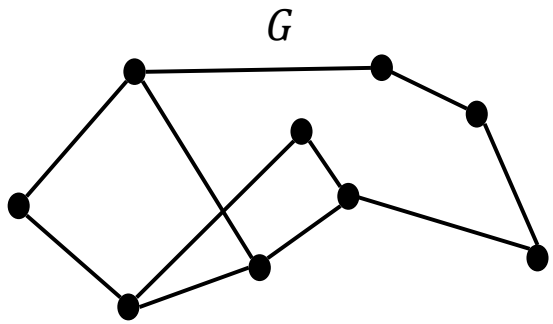
$\langle \bar{G}, k\rangle \in CLIQUE$


SAT
INDSET
3SAT
VERTEX COVER
CLIQUE
SET COVER
SUBSET SUM

Vertex Cover

Definition: For a graph $G = (V, E)$, a *vertex cover* is a subset S of V such that every edge of G is incident to at least one vertex in S

Example



Definition:

VERTEX COVER $= \{\langle G, k\rangle : G$ is a graph that has a vertex cover of size $k\}$

- $\langle G, 4\rangle \in$ VERTEX COVER? Yes
- $\langle G, 3\rangle \in$ VERTEX COVER? No

VERTEX COVER is **NP**-Complete

STEP 1: VERTEX COVER $\in$ **NP**

TM V: "On input $\langle G, k, C \rangle$

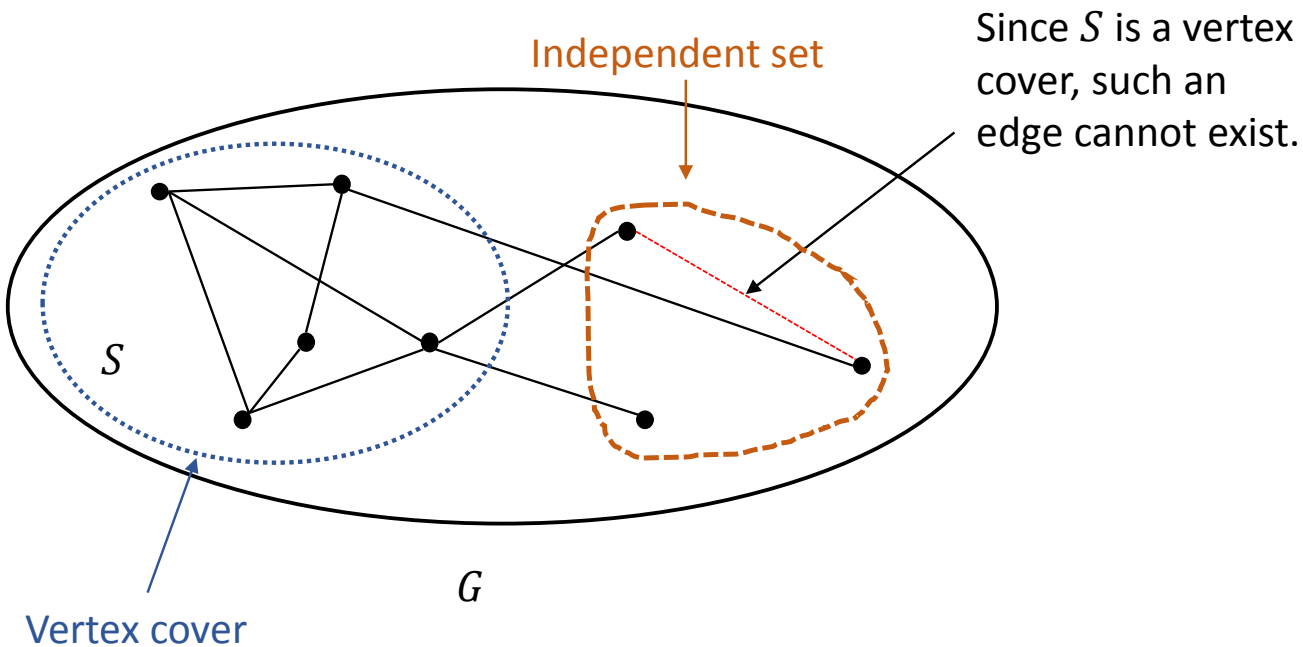
1. IF $|C| \neq k$, THEN REJECT
2. FOR every pair $u \neq v$ of vertices DO
   - IF uv is an edge AND $u \notin C$ AND $v \notin C$ THEN REJECT
3. ACCEPT"

STEP 2: INDSET $\leq_p$ VERTEX COVER

How to reduce INDSET to VERTEX COVER?

INDSET vs VERTEX COVER

Lemma: For every graph G, a subset S of the vertices is a vertex cover if and only if $\bar{S}$ is an independent set where $\bar{S} = V \setminus S$



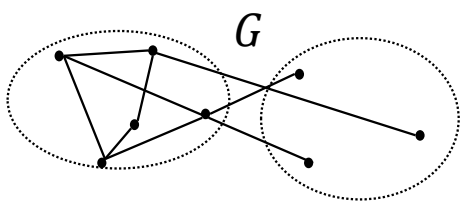
Corollary: For every graph G and a positive integer k, G has an independent set of size k iff G has a vertex cover of size $n - k$

INDSET $\leq_p$ VERTEX COVER

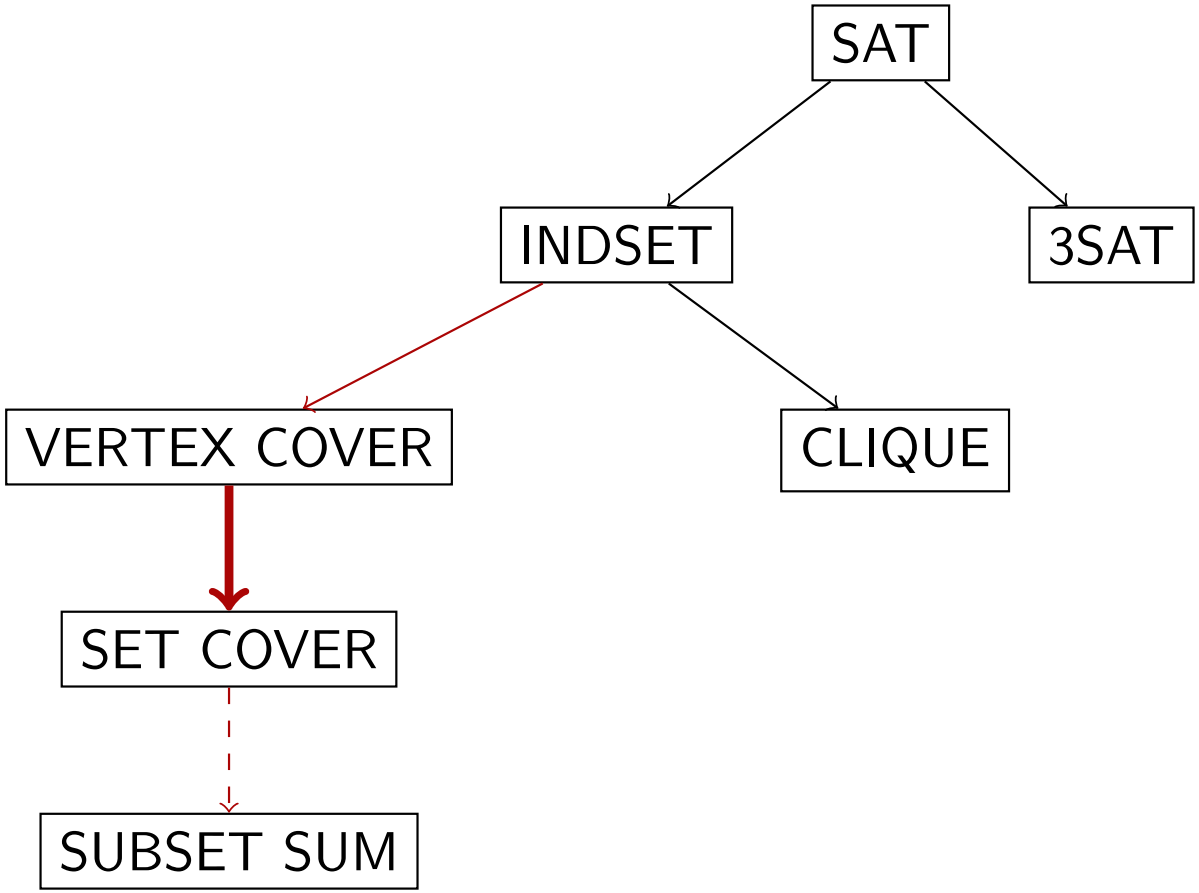
Reduction: $f(\langle G, k\rangle) := \langle G, n-k\rangle$, where n is the number of vertices of G

Efficiency: The reduction f is polynomial time

Correctness: Lemma/Corollary on previous slide!

$\langle G, k\rangle \in INDSET \iff$ 

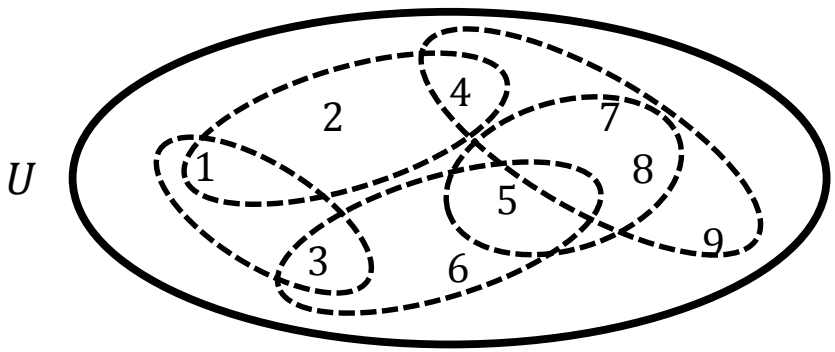
$\iff \langle G, n-k\rangle \in VERTEX\ COVER$


SAT
INDSET
3SAT
VERTEX COVER
CLIQUE
SET COVER
SUBSET SUM

Set Cover

Definition: Let $U = \{1, \ldots, n\}$ and $\mathcal{F} = \{T_1, \ldots, T_m\}$ be a family of subsets $\forall i, T_i \subseteq U$

A subset $\{T_{i_1}, \ldots, T_{i_k}\} \subseteq \mathcal{F}$ is called a set cover of size k if $\bigcup_{j=1}^{k} T_{i_j} = U$



$$\text{SET COVER} = \{\langle U, \mathcal{F}, k\rangle \mid \mathcal{F} \text{ contains a set cover of size } k\}$$

Example:

- $\langle U, \mathcal{F}, 3\rangle \in$ SET COVER? Yes $\{\{1, 2, 4\}, \{3, 6, 4\}, \{4, 7, 8, 9\}\}$
- $\langle U, \mathcal{F}, 2\rangle \in$ SET COVER? No

SET COVER is **NP**-Complete

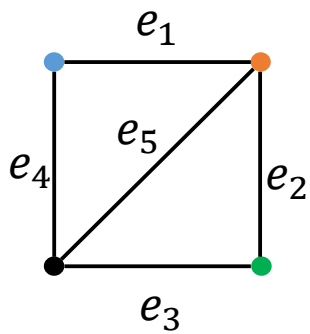
Theorem: VERTEX COVER $\leq_p$ SET COVER

Corollary: SET COVER is **NP**-complete

VERTEX COVER $\leq_p$ SET COVER

Key idea: VERTEX COVER is a *special case* of SET COVER

Example:



$k = 2$

$$U = \{e_1, e_2, e_3, e_4, e_5\}$$

$$\mathcal{F} = \{\{e_1, e_4\}, \{e_1, e_2, e_5\}, \{e_2, e_3\}, \{e_3, e_4, e_5\}\}$$

$$k' = k = 2$$

Reduction:

- Let $\langle G = (V, E), k \rangle$ be an instance of VERTEX COVER
- Set $U := E$
- For every vertex $v \in V$ create a set S_v

$$S_v := \{e \in E \mid e \text{ is incident to } v\}$$

- Let $\mathcal{F} := \{S_v \mid v \in V\}$ and set $k' = k$

Efficiency: Obvious from construction

Correctness ($\Rightarrow$): If G has a vertex cover of cardinality k, then U can be covered by k sets

Proof:

- Suppose $C \subseteq V$ is a vertex cover of G and $|C| = k$
- Every edge e_i is adjacent to at least one vertex in C

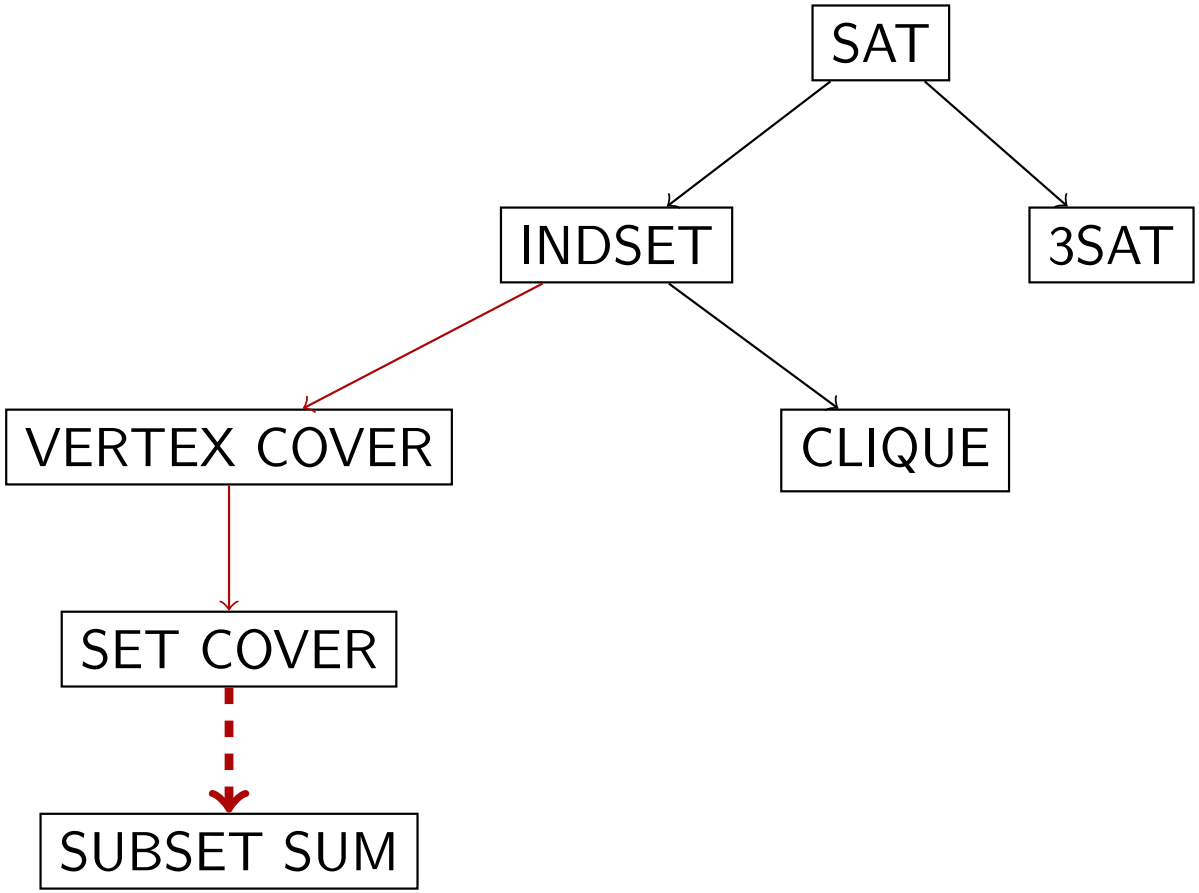
$$\bigcup_{v \in C} S_v = E$$

- Hence U can be covered by k sets

Correctness ($\Leftarrow$): If U can be covered by k sets, then G has a vertex cover of cardinality k

Proof:

- Let $S_{v_1}, S_{v_2}, \ldots, S_{v_k}$ be a collection of sets which cover $U = E$
- We claim that $C = \{v_1, v_2, \ldots, v_k\}$ is a vertex cover of G
- Indeed, every edge e in G belongs to S_{v_i} for some $i \in \{1, 2, \ldots, k\}$
- Hence, every edge e in G is incident to some vertex $v_i \in C$


SAT
INDSET
3SAT
VERTEX COVER
CLIQUE
SET COVER
SUBSET SUM

SUBSET-SUM

Let X denote a (multi) set of positive integers

Definition: SUBSET-SUM
$= \{\langle X, s\rangle : X \text{ contains a subset whose elements sum to } s\}$

Example: $X = \{1, 3, 4, 6, 13, 13\}$

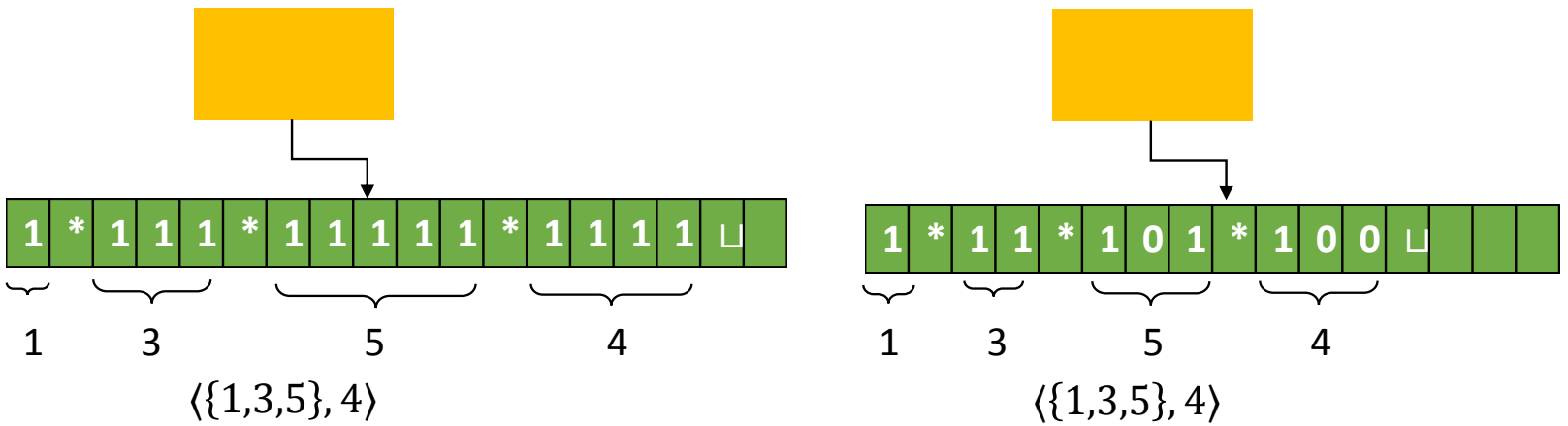
- $\langle X, 8\rangle \in$ SUBSET-SUM? Yes $T = \{1, 3, 4\}$
- $\langle X, 12\rangle \in$ SUBSET-SUM? No

Question: SUBSET-SUM $\in P$?

It depends on the input length

In $SUBSET\text{-}SUM$ we use binary encoding

Unary vs Binary



Input length: $x_1 + x_2 + \cdots + x_m + s$ $\gg$ Input length: $\lceil \log(x_1) \rceil + \cdots + \lceil \log(x_m) \rceil + \lceil \log(s) \rceil$

SUBSET-SUM: An Algorithm

- Let $X = \{x_1, x_2, \dots, x_m\}$, $sum := \sum_{j=1}^{m} x_j$
- FOR $i = 1 \,..\, m$ DO
- FOR $j = 0 \,..\, sum$ DO
- $A[i,j] := False$
- FOR $i = 0$ to m DO
- $A[i,0] := True$
- FOR $i = 1 \,..\, m$ DO
- FOR $j = 1 \,..\, sum$ DO
- IF $A[i-1, j-x_i] = True$ OR $A[i-1,j] = True$ THEN $A[i,j] := True$
- **IF $A[m,s] = True$, ACCEPT; ELSE, REJECT**

SUBSET-SUM is **NP**-Complete

Theorem: SUBSET-SUM is **NP**-Complete

Proof: Later in the course