Introduction to P vs. NP

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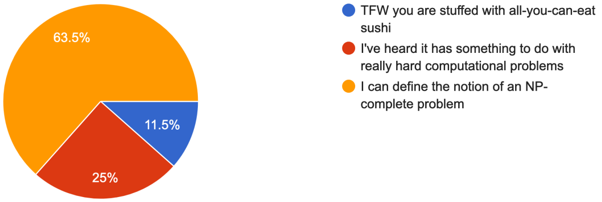
What is NP-completeness?

52 responses

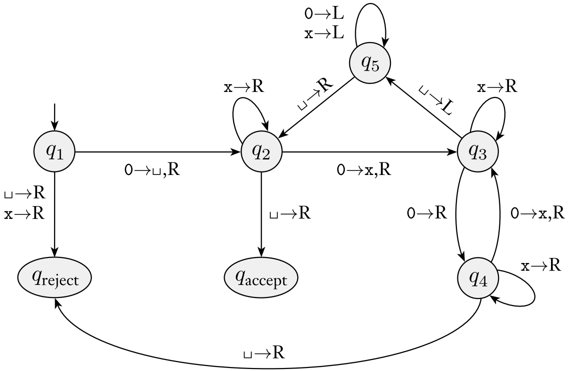


Cook–Levin theorem?

52 responses

Model of computation
(and why it doesn't matter)

Remember Turing machines?



Church-Turing Thesis

Intuitive notion of algorithms	equals	*Turing machine algorithms*

- All algorithms we know of can be executed on TMs
- Anything you write in C, Java, Scala, Python and so on
- The definition is also robust to variations: if we allow for many tapes instead of one, then nothing changes

This course: Suffices to describe algorithms in a high level language

Time Complexity

Running time of a TM

Definition: Let M be a TM that halts on all inputs (decider). The running time or time complexity of M is the function $t : \mathbb{N} \to \mathbb{N}$ where

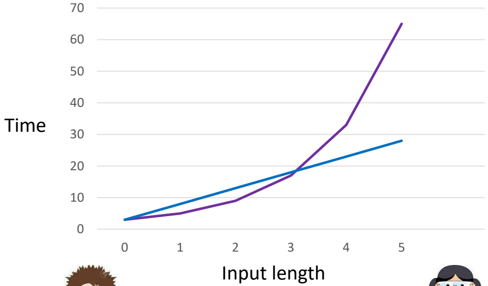
$$t(n) = \max_{w \in \{0,1\}^n} \text{number of steps } M \text{ takes on } w$$

- M runs in time $t(n)$
- n represents the input length

$$t(0) = 2$$
$$t(1) = 10$$
$$t(2) = 45$$
$$t(3) = 85$$

ε	2	2
0	9	10
1	10	
00	21	45
01	30	
10	45	
11	33	
000	73	85
001	77	
010	85	
011	80	
$\vdots$		

$$t_1(n) = 2^{n+1} + 1 \qquad \text{vs} \qquad t_2(n) = 5n + 3$$



How to compare running time functions?

Big-O and Small-o notation

Definition (Big-O): Let $f, g : \mathbb{N} \to \mathbb{R}_{\geq 0}$. We say $f(n) = O(g(n))$ if

$$\exists C > 0, n_0 \in \mathbb{N} \text{ s.t.} \qquad \forall n \geq n_0 \;\; f(n) \leq C \cdot g(n)$$

Examples:

- $5n^3 + 1 = O(2^n)$? YES
- $5n^3 + 1 = O(20n + 5)$? NO

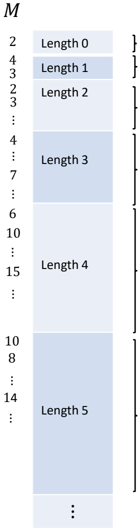
Definition (Small-o): Let $f, g : \mathbb{N} \to \mathbb{R}_{\geq 0}$. We say $f(n) = o(g(n))$ if

$$\forall c > 0, \exists n_0 \in \mathbb{N} \text{ s.t.} \qquad \forall n \geq n_0 \;\; f(n) < c \cdot g(n)$$

Examples:

- $\sqrt{n} = o(n)$? YES
- $f(n) = o(f(n))$? NO

To summarize...



$t(0) = 2$

$t(1) = 4$

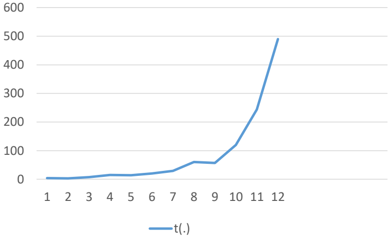
$t(2) = 3$

$t(3) = 7$

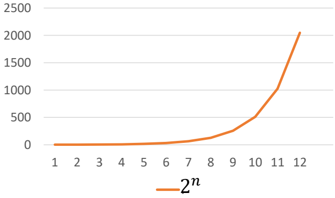
$t(4) = 15$

$t(5) = 14$

⋮



$t(n) = O(2^n)$



Decision problems

Definition: Let Σ be a finite input alphabet (typically $\Sigma = \{0, 1\}$)
Denote by Σ^* the set of all finite words over Σ

Example: $\{0,1\}^* = \{\epsilon, 0, 1, 00, 01, 10, 10, \ldots\}$

Definition: A *language* (or *decision problem*) is a subset $L \subseteq \Sigma^*$.

- Input: $x \in \Sigma^*$
- Output:
  - 1/YES/True/Accept if $x \in L$
  - 0/NO/False/Reject if $x \notin L$

Time Complexity

Definition: Time complexity class

$$TIME(t(n)) = \{L \subseteq \Sigma^* \mid L \text{ is decided by a TM in time } O(t(n))\}$$

Example:

$TIME(n) \subseteq TIME(n^2) \subseteq \cdots \subseteq TIME(2^{\sqrt{n}}) \subseteq TIME(2^n) \subseteq TIME(2^{2^n}) \ldots$

The complexity class P and efficiency

Definition: **P** is the class of languages that are decidable in polynomial time on a (deterministic) Turing machine. In other words,

$$\mathbf{P} = \bigcup_{k=1}^{\infty} TIME(n^k)\,.$$

Some languages in P:

- $\{\langle A \rangle \;:\; A \text{ is a sorted array of integers}\}$
- $\{\langle G, s, t \rangle \;:\; s \text{ and } t \text{ are vertices connected in graph } G\}$
  (Breadth-First Search)
- $\{\langle G \rangle \;:\; G \text{ is a connected graph}\}$

NP: Verification vs. Search

SAT-verify and SAT

Conjunctive Normal Form (CNF) Formula:

$$\varphi_1 = (\overline{x} \vee \overline{y} \vee z_0) \wedge (x \vee \overline{y} \vee z_1) \wedge (\overline{x} \vee y \vee z_2) \wedge (x \vee y \vee z_3)$$

$$\varphi_2 = \overline{x_1} \wedge (x_1 \vee \overline{x_2}) \wedge (x_1 \vee x_2 \vee \overline{x_3}) \wedge (x_1 \vee x_2 \vee x_3 \vee \overline{x_4})$$

$$\varphi_3 = \overline{x_1} \wedge (x_1 \vee \overline{x_2}) \wedge (x_1 \vee x_2)$$

- CNF Formula: AND of Clauses
- Clause: OR of Literals
- Literal: variable or its negation

Satisfying assignment: Boolean assignment to variables which makes the formula TRUE.

Check: φ_1 has 32 satisfying assignments, φ_2 as only one, φ_3 has zero.

A formula is **satisfiable** if it has at least one satisfying assignment.

SAT-verify and SAT

$$\text{SAT-verify} = \{\langle \varphi, C \rangle \ : \ C \text{ is a satisfying assignment of } \varphi\}$$

Is SAT-verify in P? Yes!

1. Substitute for literals according to C.
2. Check that every clause has at least one TRUE literal.

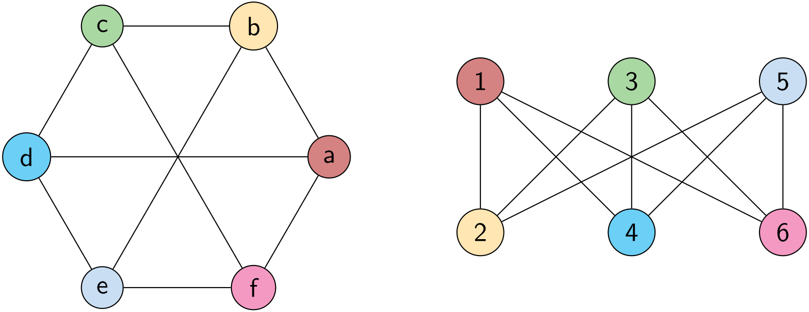
$$\begin{aligned} \text{SAT} &= \{\langle \varphi \rangle \ : \ \varphi \text{ is satisfiable}\} \\ &= \{\langle \varphi \rangle \ : \ \exists C \text{ such that } \langle \varphi, C \rangle \in \text{SAT-verify}\} \end{aligned}$$

Is SAT in P?

Decider for SAT:

1. For each assignment C:
   - If $\langle \varphi, C \rangle \in$ SAT-verify, ACCEPT φ.
2. REJECT φ.

GI-verify and GI



Graph Isomorphism: Bijection $f\colon V(G_1) \longrightarrow V(G_2)$ which preserves adjacency: $\{u, v\} \in E(G_1) \Leftrightarrow \{f(u), f(v)\} \in E(G_2)$

Eg. $a \to 1$ $b \to 2$ $c \to 3$ $d \to 4$ $e \to 5$ $f \to 6$ in the graphs above.

Two graphs are **isomorphic** if they have at least one graph isomorphism.

GI-verify and GI

$$\text{GI-verify} = \{\langle G_1, G_2, C\rangle \ : \ C : V(G_1) \longrightarrow V(G_2) \text{ is a graph isomorphism}\}$$

Is GI-verify in P? Yes!

1. Check that C is a bijection: For each $u, v \in V(G_1)$:
   - Check $\{u, v\} \in E(G_1) \Leftrightarrow \{C(u), C(v)\} \in E(G_2)$.

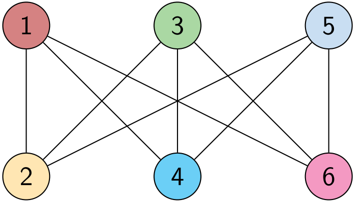
$$\begin{aligned} \text{GI} &= \{\langle G_1, G_2\rangle \ : \ G_1 \text{ and } G_2 \text{ are isomorphic}\} \\ &= \{\langle G_1, G_2\rangle \ : \ \exists C \text{ such that } \langle G_1, G_2, C\rangle \in \text{GI-verify}\} \end{aligned}$$

Is GI in P?

Decider for GI:

1. For each function $C : V(G_1) \longrightarrow V(G_2)$:
   - If $\langle G_1, G_2, C\rangle \in$ GI-verify, ACCEPT $\langle G_1, G_2\rangle$.
2. REJECT $\langle G_1, G_2\rangle$.

INDSET-verify and INDSET



Independent Set: Subset $S \subseteq V(G)$ such that no two vertices in S are adjacent in G.

Eg. $\{1, 3, 5\}$, $\{2, 4\}$, $\{6\}$, $\emptyset$, etc. in the graph above.

INDSET-verify and INDSET

INDSET-verify $= \{\langle G, k, C\rangle \ : \ C$ is an independent set of size k in $G\}$

Is INDSET-verify in P? Yes!

1. Check that $|C| = k$.
2. For each $u, v \in C$:
   - Check $\{u, v\} \notin E(G)$.

$$\begin{aligned} \text{INDSET} &= \{\langle G, k\rangle \ : \ G \text{ has an independent of size } k\} \\ &= \{\langle G, k\rangle \ : \ \exists C \text{ such that } \langle G, k, C\rangle \in \text{INDSET-verify}\} \end{aligned}$$

Is INDSET in P?

Decider for INDSET:

1. For each subset $C \subseteq V(G)$:
   - If $\langle G, k, C\rangle \in$ INDSET-verify, ACCEPT $\langle G, k\rangle$.
2. REJECT $\langle G, k\rangle$.

Verifiers and the class NP

Recall: A **decider** for language L is a TM M such that for each $x \in \Sigma^*$

- If $x \in L$, then M accepts x.
- If $x \notin L$, then M rejects x.

Definition:
A **verifier** for language L is a TM M such that for each $x \in \Sigma^*$

- If $x \in L$, then there exists C such that M accepts $\langle x, C \rangle$.
- If $x \notin L$, then for every C, M rejects $\langle x, C \rangle$.

(C is called a *certificate* or *witness*)

A verifier is a **polynomial time** verifier if its running time on any $\langle x, C \rangle$ is **polynomial in** $|\mathbf{x}|$. (Thus $|C|$ is polynomial in $|x|$)

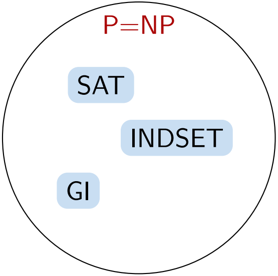
Definition: NP is the class of languages that have poly-time verifiers.

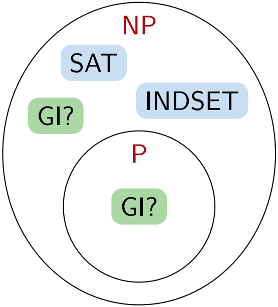
P and NP

Is P $\subseteq$ NP? Yes (obviously)

Is P $=$ NP? Nobody knows . . .

Find the answer and win USD 1,000,000!





Cook-Levin Theorem (informal): SAT $\in$ P iff P $=$ NP.

(Also INDSET $\in$ P iff P $=$ NP.)

Why is it called NP?

Detour: Non-deterministic Turing Machines

Recall: In a Turing machine, $\delta : (Q \times \Gamma) \longrightarrow Q \times \Gamma \times \{L, R\}$.

In a **Nondeterministic Turing Machine (NTM)**,

$$\delta : (Q \times \Gamma) \longrightarrow \mathcal{P}(Q \times \Gamma \times \{L, R\})$$

(several possible transitions for a given state and tape symbol)

Definition: A **nondeterministic decider** for language L is an NTM N such that for each $x \in \Sigma^*$, **every computation of N on x halts**, and moreover,

- If $x \in L$, then some computation of N on x accepts.
- If $x \notin L$, then every computation of N on x rejects.

An NTM is a **polynomial time** NTM if the running time of its longest computation on x is **polynomial in $|\mathbf{x}|$**.

Nondeterministic deciders $\Longleftrightarrow$ Verifiers

Theorem: For any language $L \subseteq \Sigma^*$,

L has a nondeterministic poly-time decider $\Longleftrightarrow$ L has a poly-time verifier.

Proof Sketch ($\Longleftarrow$):

Let M be the verifier. NTM N on input x does the following:

1. Write a certificate C nondeterministically.
2. Run M on $\langle x, C \rangle$.

Proof Sketch ($\Longrightarrow$):

Let N be the nondeterministic decider. Verifier M on $\langle x, C \rangle$ computes:

- Simulate N on x, choosing transitions given by C.

M accepts $\langle x, C \rangle$ iff $x \in L$ and C is an accepting path of N on x.

Non-deterministic Polynomial-time

Theorem: For any language $L \subseteq \Sigma^*$,

L has a nondeterministic poly-time decider $\iff$ L has a poly-time verifier.

Definition: **NP** is the class of languages which have poly-time nondeterministic deciders, or equivalently, have poly-time verifiers.

Definition:

$$\mathrm{NTIME}(t(n)) = \{L \ : \ L \text{ has a nondeterministic } O(t(n)) \text{ time decider}\}$$

Then

$$\mathrm{NP} = \bigcup_{k=1}^{\infty} \mathrm{NTIME}(n^k)\,.$$