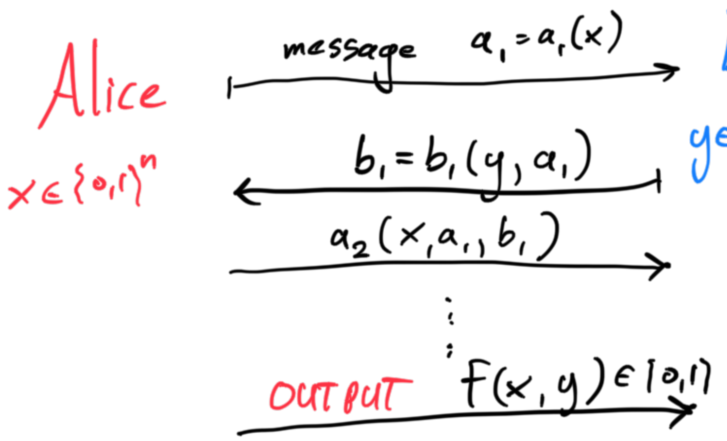
Communication Complexity

Two-party function $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$



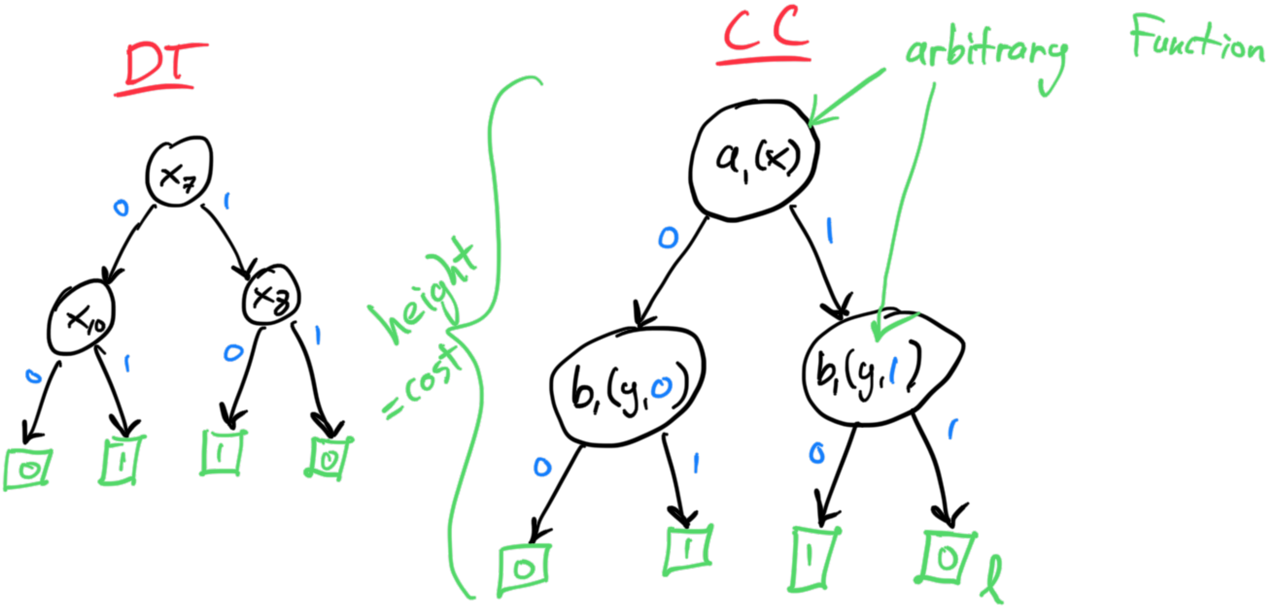
cost:
\#bits sent
(on worst-case input)

$D^{cc}(f)$ = least cost of protocol for f.

Ex $D^{cc}(XOR_{2n}) = 2$

$D^{dt}(XOR_{2n}) = 2n$

Fact: $\forall f : D^{cc}(f) \leq D^{dt}(f)$



Ex. $D^{cc}(Maj_{2n}) \leq \log n + 1$ $\quad Maj_{2n}(xy) = 1$

iff $|xy|_1 \geq n$

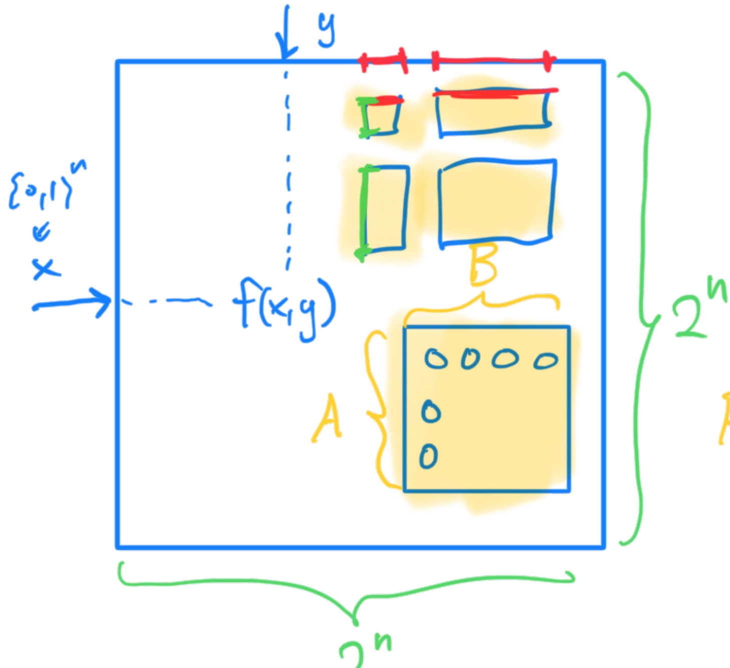
Ex. "Equality"

$$EQ_n(\overset{\{0,1\}^n}{x, y}) = \begin{cases} 1, & x=y \\ 0, & x\neq y \end{cases}$$

$$D^{cc}(EQ_n) = n+1$$

↑ We'll see this

Communication Matrix



$M_F \in \{0,1\}^{2^n \times 2^n}$

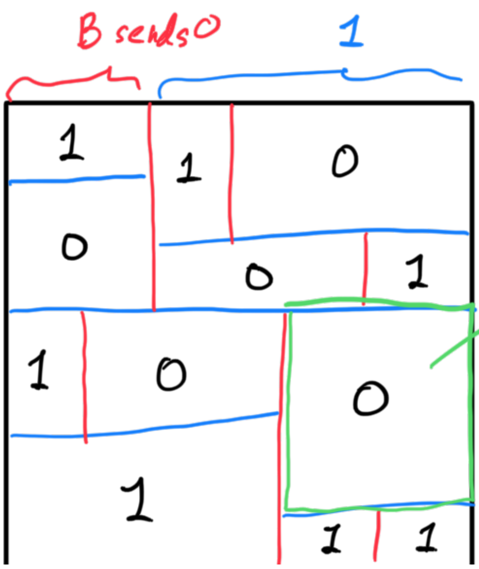
$(M_F)_{x,y} = f(x,y)$

$R = A \times B$

Fact Protocol of cost c partitions M_F into 2^c many $\underbrace{\text{monochromatic}}$ $\underbrace{\text{rectangles}}$

f is constant on R

$R = A \times B$
$A \subseteq \{0,1\}^n$
$B \subseteq \{0,1\}^n$



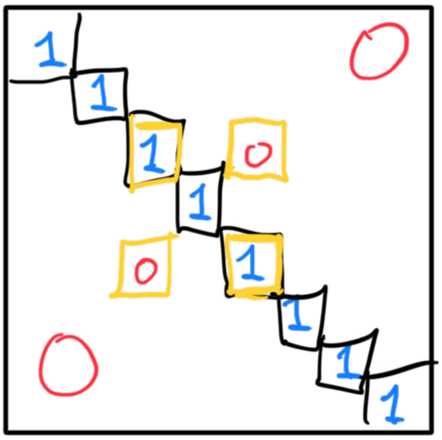
"leaf rectangle"
set of all x,y that reach leaf ℓ

Def Partition number $\text{part}(f)$ = least # of rectangles in a partition of M_F into monochromatic rectangles

Fact $\forall f$: $D^{cc}(f) \geq \log \text{part}(f)$

Lemma $\forall f$: $D^{cc}(f) \leq \left(\log \text{part}(f)\right)^2$

Ex $\text{part}(EQ) \geq 2^n$



⇒ Every 1-chromatic rectangle is 1×1

⇒ Need 2^n rects to cover diagonal

Nondeterministic protocols

Def N.D. protocol on input (x,y)

1. Guess "certificate" string $w \in \{0,1\}^{\overset{\substack{\text{cost}\\ \downarrow}}{s}}$
2. Verify: Accept iff Alice accepts (x,w) & Bob accepts (y,w)

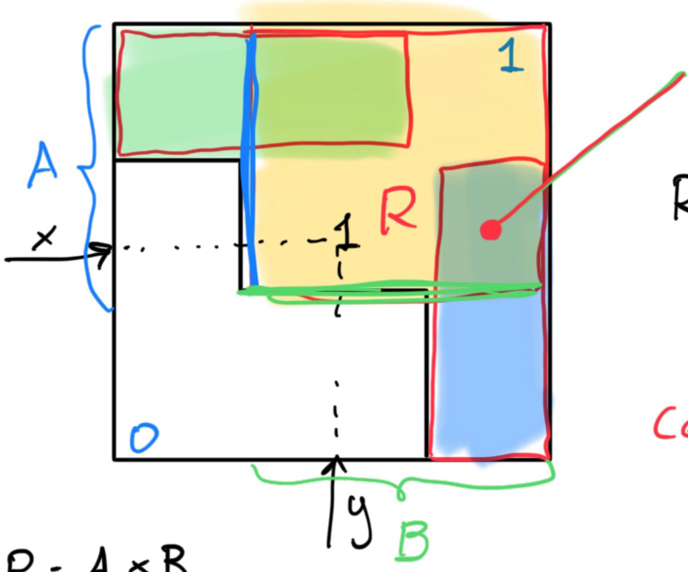
$N^{cc}_{\underset{0}{1}}(f)$ = least cost of ND protocol accepting $f^{-1}(\underset{0}{1}) = \{(x,y) : f(x,y) = \underset{0}{1}\}$

Ex $N_0(EQ) \underset{?}{=} \log n + 1$ certificate: $\begin{cases} i \in [n] \\ x_i \in \{0,1\} \end{cases}$

$$N_1(EQ) \doteq n$$

["NP ≠ coNP"]["P ≠ NP"]

Lemma $N_1(f) = \log\left[\text{rectangle cover \# of } f^{-1}(1)\right]$



Rectangles in cover

$\updownarrow$ 1-to-1

Certificates $w \in \{0,1\}^s$

$R = A \times B$

$\begin{cases} x \in A \\ y \in B \end{cases}$

$$w \longmapsto R_w = A_w \times B_w$$

$$A_w = \{x : \text{Alice accepts } (x,w)\}$$

$$B_w = \{y : \text{Bob accepts } (y,w)\}$$

Lemma $N_1(EQ_n) \geq n$

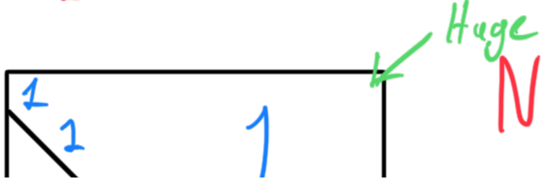
$\left(\text{rec. cover \# } EQ_n^{-1}(1) \geq 2^n\right)$

Ex Greater Than

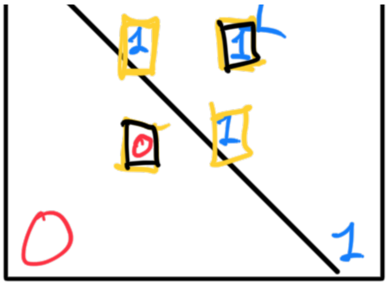
n-bit number

$$GT_n(x,y) = \begin{cases} 1, & x \geq y \\ 0, & x < y \end{cases}$$

$N_1(GT) \overset{?}{=}$ $\qquad$ $N_0(GT) \overset{?}{=}$



$N_1(GT) \geq n$



"Fooling set argument"

<u>Lemma</u> $D^{dt}(f) \leq C_0(f) \cdot C_1(f)$

$D^{cc}(f) \leq N_1(f) \cdot N_0(f)$