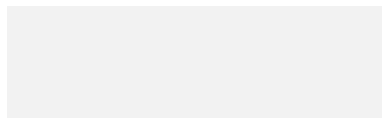




Student Two

SCIPER: **222222**

Signature: 

Final Exam, Computational Complexity 2021

- Books, notes, communication, calculators, cell phones, computers, etc... are not allowed.
- Your explanations and proofs should be clear enough and in sufficient detail so that they are easy to understand and have no ambiguities.
- You are allowed to refer to material covered in lectures and exercises including theorems without reproving them.
- Please answer inside the space indicated (the exams will be scanned for grading purposes).
- **Do not touch until the start of the exam.**

Good luck and have fun!

Problem 1	Problem 2	Problem 3	Problem 4	Problem 5
/ 15 points	/ 25 points	/ 20 points	/ 20 points	/ 20 points

Total / 100



1. **Quick-fire round.** (15 pts) Consider the following statements.

1. $3\text{-SAT} \leq_p 2\text{-SAT}$.
2. $\text{TIME}(n) \neq \text{TIME}(n^{2021})$.
3. If $A \leq_p B$, then $A \in \mathsf{P}^B$.
4. If $\mathsf{P} = \mathsf{NP}$, then $\mathsf{P}^A = \mathsf{NP}^A$ for every oracle A .
5. There exists an oracle A such that $\mathsf{P}^A = \mathsf{EXP}^A$.
6. Directed s - t connectivity is in L .
7. PH has a complete problem.
8. If $\mathsf{NP} \subseteq \mathsf{P/poly}$, then PH collapses to its first level, i.e., $\mathsf{PH} = \mathsf{NP} = \mathsf{coNP}$.
9. $\mathsf{NSPACE}(n) \subseteq \mathsf{SPACE}(n^2)$.
10. If $\mathsf{NP} \subseteq \mathsf{BPP}$, then $\mathsf{NP} \subseteq \mathsf{RP}$.
11. Graph isomorphism is in coAM .
12. Every n -bit function computed by a $\text{poly}(n)$ -sized circuit can also be computed with a $\text{poly}(n)$ -sized CNF.
13. Every n -bit function can be computed by a depth- $O(n)$ circuit with fan-in 2 gates.
14. If a boolean function can be written both as k -DNF and k -CNF, then it admits a depth- $O(k)$ decision tree.
15. Certificate complexity $\mathsf{C}(f)$ and sensitivity $\mathsf{s}(f)$ satisfy $\mathsf{C}(f) \leq \mathsf{s}(f)$ for all functions f .

For each box below, write one of the following symbols:

- **T** if the statement is known to be true.
- **F** if the statement is false *or not known to be true* (e.g., both $\mathsf{P} = \mathsf{NP}$ and $\mathsf{P} \neq \mathsf{NP}$ should be marked **F**).
- or leave the box empty.

A correct **T/F** answer is worth +1 point, an incorrect answer is worth −1 point, and an empty answer is worth 0 points.

Your answers:

1.	$\mathcal{F}$
2.	$\mathcal{T}$
3.	$\mathcal{T}$
4.	$\mathcal{F}$
5.	$\mathcal{F}$

6.	$\mathcal{F}$
7.	$\mathcal{F}$
8.	$\mathcal{F}$
9.	$\mathcal{T}$
10.	$\mathcal{T}$

11.	$\mathcal{T}$
12.	$\mathcal{F}$
13.	$\mathcal{T}$
14.	$\mathcal{F}$
15.	$\mathcal{F}$

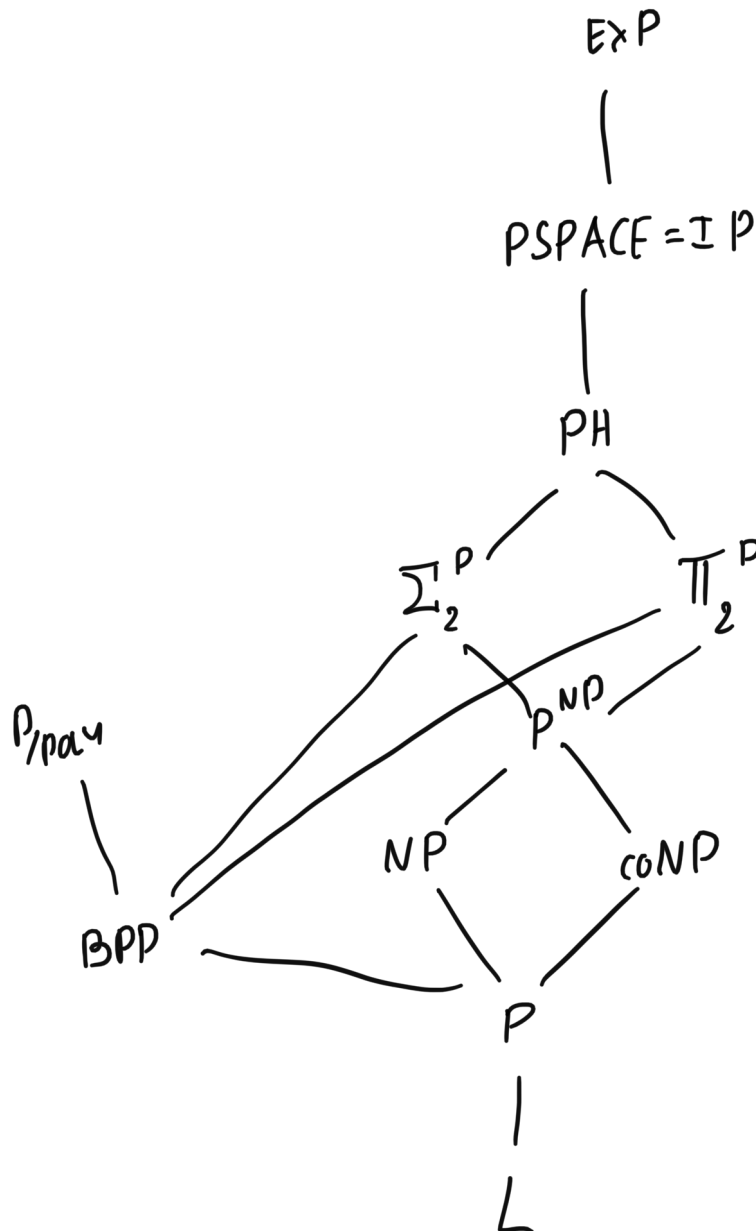


2. **Complexity Zoo.** (25 pts) Consider the following list of complexity classes:

P , NP , $coNP$, EXP , $PSPACE$, L , PH , Σ_2P , Π_2P , P^{NP} , BPP , IP , $P/poly$.

Draw a **class inclusion diagram** for the above classes that has a directed arrow $A \rightarrow B$ between classes A and B whenever $A \subseteq B$. You do not have to draw *transitive arrows*, i.e., if $A \rightarrow B$ and $B \rightarrow C$, you can leave out $A \rightarrow C$. For ease of readability, please draw the smallest class at the bottom, and the largest class at the top of the page.

Please answer inside the box





3. Problem Garden. (15 pts)

(a) For each problem below, classify it into the smallest possible complexity class (as seen in lectures/exercises). If you use a class not mentioned in Problem 2, then relate your class to those in Problem 2 by inclusions. (You do not need to justify your answers.)

1. **Input:** DNF formula φ . **Output:** YES iff $\varphi(x) = 1$ for some x .
2. **Input:** DNF formula φ . **Output:** YES iff $\varphi(x) = 1$ for all x .
3. **Input:** Graph G and number k . **Output:** YES iff the smallest vertex cover in G is of size exactly k .
4. **Input:** Boolean circuit C . **Output:** YES iff C is the smallest circuit that computes the same function as C .
5. **Input:** Arithmetic circuit C over integers $\mathbb{Z}$. **Output:** YES iff $C(x) = 0$ for all $x \in \mathbb{Z}^n$.

Please answer inside the box

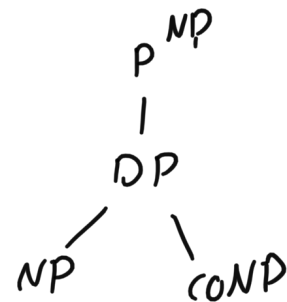
1: L

2: coNP

3: DP

4: Π_2^P

5: coRP



(b) Give an example (including a precise definition) of a PSPACE-complete problem.

$TQBF = \{ \varphi : \varphi \text{ is a QBF which evaluates to true} \}$

a QBF φ is of the form

$\varphi = Q_1 x_1 Q_2 x_2 \dots Q_n x_n \phi[x_1, \dots, x_n]$

closed $\uparrow$ $\uparrow$ $\uparrow$ $\uparrow$

$\in \{ \forall, \exists \}$



4. NP-completeness. (20 pts) Let $G = (V, E)$ be an undirected graph. A set of vertices $D \subseteq V$ is *dominating* if every vertex $v \in V \setminus D$ has a neighbour in D (that is, $\{v, u\} \in E$ for some $u \in D$). Define

$$\text{DOMINATINGSET} = \{ \langle G, k \rangle : G \text{ has a dominating set of size at most } k \}.$$

Prove that DOMINATINGSET is NP-complete. In your proof, you may assume the NP-completeness of any of the problems discussed in class (SAT, Independent Set, Clique, Subset Sum, Vertex Cover, Set Cover, etc.).

Please answer inside the box

DS $\in$ NP as a certificate is simply $D \subseteq V$ of size $\leq k$
(checking that D is DS is $\in \mathcal{P}$).

Vertex Cover $\in$ DS (sketch)

1) remove isolated vertices

2) replace edge by Δ $u \text{ --- } v \Rightarrow u \text{ --- } w \text{ --- } v$

e.g: $G =$  $\Rightarrow \varphi(G) =$ 

(Claim $\varphi(G)$ has a DS of size $\leq k \Leftrightarrow G$ has a VC of size $\leq k$)





5. Communication complexity. (20 pts)

(a) In the INDEX_n problem, Alice holds $x \in \{0,1\}^n$, Bob holds $i \in [n]$, and their goal is to output $\text{INDEX}(x,i) = x_i \in \{0,1\}$ (i -th bit of x). Show that any deterministic *one-way communication protocol* (Alice sends a single message to Bob who then outputs the value of the function) for INDEX_n requires $\Omega(n)$ bits of communication.

Please answer inside the box

Suppose toward contradiction Alice can do $n-1$ bits only.

Let $A: \{0,1\}^n \rightarrow \{0,1\}^{n-1}$ be her encoding function.

by the pigeonhole principle, $\exists x, y \in \{0,1\}^n: \begin{cases} x \neq y \\ A(x) = A(y) \end{cases}$

Since $x \neq y$, $\exists i \in [n]$ with $x_i \neq y_i$ so if Bob receives i , it will err on x on y .



(b) Define the *Set-Intersection* problem and then use it to show that the following two-party communication problem requires $\Omega(n)$ bits of communication:

- Alice holds a graph $G_A = ([n], E_A)$;
- Bob holds a graph $G_B = ([n], E_B)$;
- Decide whether the union graph $G_A \cup G_B = ([n], E_A \cup E_B)$ contains a perfect matching.

Please answer inside the box

$$SI_n(x, y) = 1 \iff \exists i \in [n]: x_i = y_i = 1$$

The reduction works as follows: ^(SKETCH) graph on $2n$ vertices
put an edge iff value is zero

Ex uph: Alice has 011 : $\begin{matrix} \bullet & \bullet & \bullet \\ | & \cdot & \cdot \\ \bullet & \cdot & \cdot \end{matrix}$

Bob does the same

Full ex uph:

	0	1	0
Alice	$\begin{matrix} \bullet \\ \\ \bullet \end{matrix}$	$\begin{matrix} \bullet \\ \cdot \end{matrix}$	$\begin{matrix} \bullet \\ \\ \bullet \end{matrix}$
Bob	$\begin{matrix} \bullet \\ \\ \bullet \end{matrix}$	$\begin{matrix} \bullet \\ \cdot \end{matrix}$	$\begin{matrix} \bullet \\ \\ \bullet \end{matrix}$
	0	1	1

(Claim: $G_A \cup G_B$ has a perfect matching $\iff$ sets do not intersect)

$$S: R(PM_{2n}) \geq R(\overline{SI_n}) = R(SI_n) \geq \Omega(n)$$

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