Exercise VI, Computational Complexity 2024

These exercises are for your own benefit. Feel free to collaborate and share your answers with other students. Solve as many problems as you can and ask for help if you get stuck for too long. Problems marked * are more difficult but also more fun :).

Polynomial Hierarchy

1 Recall from Exercise III(5) that there are CNFs φ such that the smallest DNF equivalent to φ is exponentially larger than φ. Inspired by this, consider the following problem:

$$\text{DNF-Size} = \{\langle \varphi, 1^k \rangle : \varphi \text{ is a CNF and there is a DNF equivalent to } \varphi \text{ with } k \text{ terms}\}.$$

Show that DNF-Size $\in \Sigma_2\mathsf{P}$.

2 Show that PH does not have a complete problem, unless PH collapses.

(*Hint: Assume some problem $L \in \mathsf{PH}$ is complete. Then it lies in some finite level $L \in \Sigma_i\mathsf{P}$. Proceed to show that $\Pi_i\mathsf{P} = \Sigma_i\mathsf{P}$ and apply a lemma from lecture.*)

3 Define the *difference polynomial-time* class DP as consisting of those languages L such that $L = L_1 \cap L_2$ for some $L_1 \in \mathsf{NP}$ and $L_2 \in \mathsf{coNP}$. (Do not confuse DP with $\mathsf{NP} \cap \mathsf{coNP}$!)

(a) Show that $\mathsf{DP} \subseteq \mathsf{P}^{\mathsf{NP}}$.

(b) Show that the following problem is in DP

$$\text{UniqueSat} = \{\langle \varphi \rangle : \varphi \text{ is a CNF and it has a unique satisfying assignment}\}.$$

(c) Show that the following problem is DP-complete:

$$\text{Sat-Unsat} = \{\langle \varphi, \varphi' \rangle : \varphi \text{ is a satisfiable CNF and } \varphi' \text{ is an unsatisfiable CNF}\}.$$

(d) True or false: If L is NP-complete and L' is coNP-complete, then $L \cap L'$ is DP-complete?

4 Prove that if $\mathsf{NP} \subseteq \mathsf{TIME}(n^{\log n})$ then $\mathsf{PH} \subseteq \bigcup_{k\in\mathbb{N}} \mathsf{TIME}(n^{\log^k n})$.

5 (*) Denote by $\mathsf{P}^{\mathsf{NP}[\log n]}$ the class of problems soluble by a poly-time TM that makes at most $O(\log n)$ queries to a Sat oracle. Denote by $\mathsf{P}^{\|\mathsf{NP}}$ the class of problems soluble by a poly-time TM that queries a Sat oracle *in parallel*, that is, the TM first computes deterministically a list of $m = n^{O(1)}$ many Sat-instances, $\varphi_1, \ldots, \varphi_m$, then queries the oracle with all the φ_i at once, receives some string of answers $a \in \{0,1\}^m$, and then produces an output depending on a.

Show that

$$\mathsf{P}^{\mathsf{NP}[\log n]} = \mathsf{P}^{\|\mathsf{NP}}.$$

(*Hint: The inclusion $\mathsf{P}^{\mathsf{NP}[\log n]} \subseteq \mathsf{P}^{\|\mathsf{NP}}$ is the easier one to prove. For the harder inclusion, namely $\mathsf{P}^{\mathsf{NP}[\log n]} \supseteq \mathsf{P}^{\|\mathsf{NP}}$, consider the answer string $a \in \{0,1\}^m$ and first find its Hamming weight (i.e., number of $i \in [m]$ with $a_i = 1$). Then, knowing the Hamming weight, make one more clever NP-oracle query.*)