Exercise X, Computational Complexity 2024

These exercises are for your own benefit. Feel free to collaborate and share your answers with other students. Solve as many problems as you can and ask for help if you get stuck for too long. Problems marked * are more difficult but also more fun :).

Communication complexity

1 Suppose Alice has a set $A \subseteq [n]$ of size $|A| \geq n/2$ and Bob has a set $B \subseteq [n]$ of size $|B| < n/2$. Find an $O(\log^2 n)$-bit protocol that outputs some element $i \in [n]$ such that $i \in A \setminus B$.

2 Define the *Set-Intersection* function $\mathrm{SI}\colon \{0,1\}^{2n} \to \{0,1\}$ by $\mathrm{SI}(x,y) = \bigvee_{i\in[n]}(x_i \wedge y_i)$. (If we think of x and y as subsets of $[n]$, then $\mathrm{SI}(x,y) = 1$ iff $x \cap y \neq \emptyset$.)

(a) Show that $\mathrm{N}_1^{\mathrm{cc}}(\mathrm{SI}) \leq O(\log n)$.

(b) Show that $|\mathrm{SI}^{-1}(0)| = 3^n$

(c) Show that every 0-chromatic rectangle for SI is of size at most 2^n.

(d) Use (b) and (c) to conclude a lower bound on $\mathrm{N}_0^{\mathrm{cc}}(\mathrm{SI})$.

3 We saw in the lecture that $\mathrm{D}^{\mathrm{cc}}(\textsc{Maj}_{2n}) \leq O(\log n)$. Prove a matching lower bound by a reduction from the *Greater-Than* function.

4 The *Clique vs. Independent Set* (CIS) problem is defined relative to an n-vertex graph $G = ([n], E)$ as follows: Alice holds a clique $C \subseteq [n]$, Bob holds an independent set $I \subseteq [n]$, and their goal is to output $\mathrm{CIS}_G(C, I) = |C \cap I|$. (Note that $|C \cap I| \in \{0,1\}$.)

(a) Show that $\mathrm{D}^{\mathrm{cc}}(\mathrm{CIS}_G) \leq O(\log^2 n)$ for every G.
(Hint: If Alice's C contains a vertex v of degree $\leq n/2$, Alice can send the name of v to Bob and they can wlog restrict G to the neighbourhood of v by discarding half the vertices. What is the analogous property for Bob? How can you use this idea recursively?)

(b) Find a graph $G = ([n], E)$ such that every *one-way* (Alice sends one message to Bob, and Bob outputs the answer) deterministic protocol for CIS_G requires $\Omega(n)$ bits.

5 Let $f\colon \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ and define a language $L_f = f^{-1}(1) = \{xy : f(x,y) = 1\}$.

(a) Show that if L_f is accepted by a deterministic (resp. nondet.) finite automaton with s states, then f has deterministic (resp. nondet.) communication complexity $O(\log s)$.

(b) Use the above connection and the equality function $f = \textsc{Eq}_n$ to construct a language L that is accepted by a nondeterministic automaton with $n^{O(1)}$ states but such that every nondeterministic automaton for the complement language $\overline{L}$ requires $2^{\Omega(n)}$ states.