CS457 Geometric Computing

10A - Differential Geometry of Curves

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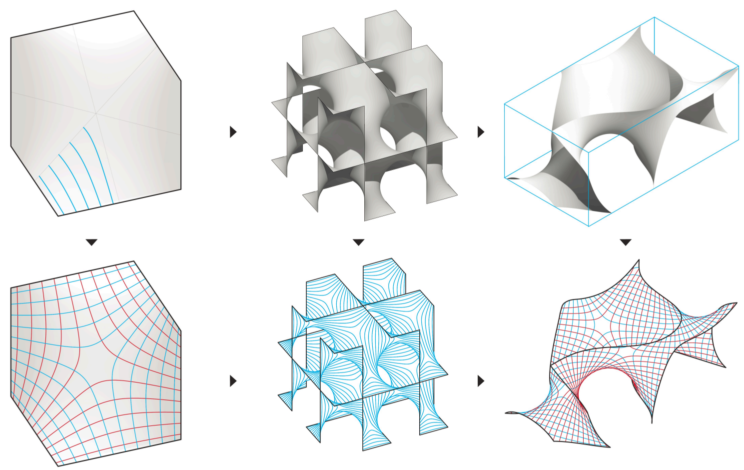
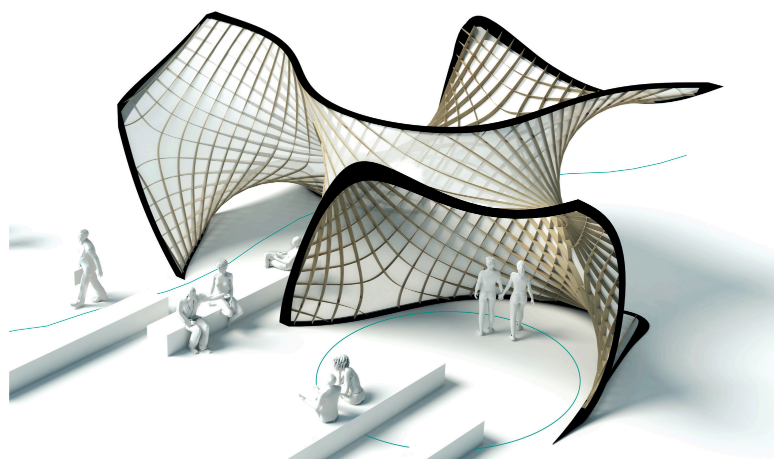
Challenge III: Asymptotic Gridshell

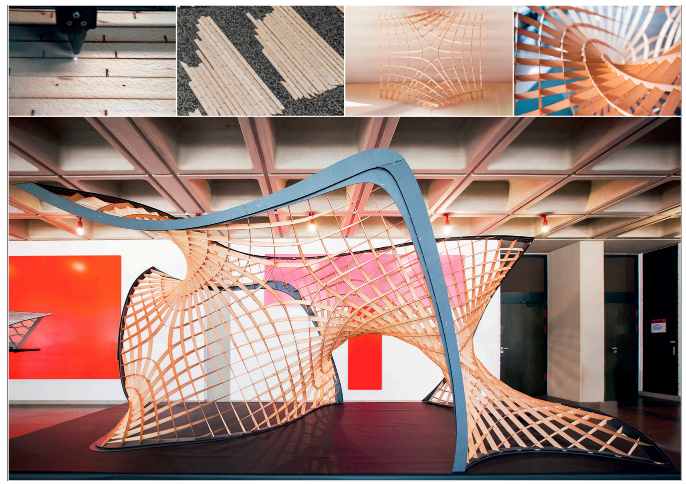
- Design and fabricate a complex curved surface from straight flexible strips.
  - Specifically: Vertically intersecting strips, intersections at (approximately) right angles.
  - Is this even possible? If so, for which surfaces?



Geometry



Computational Design



Fabrication

Source: Eike Schling

Challenge III: Asymptotic Gridshell

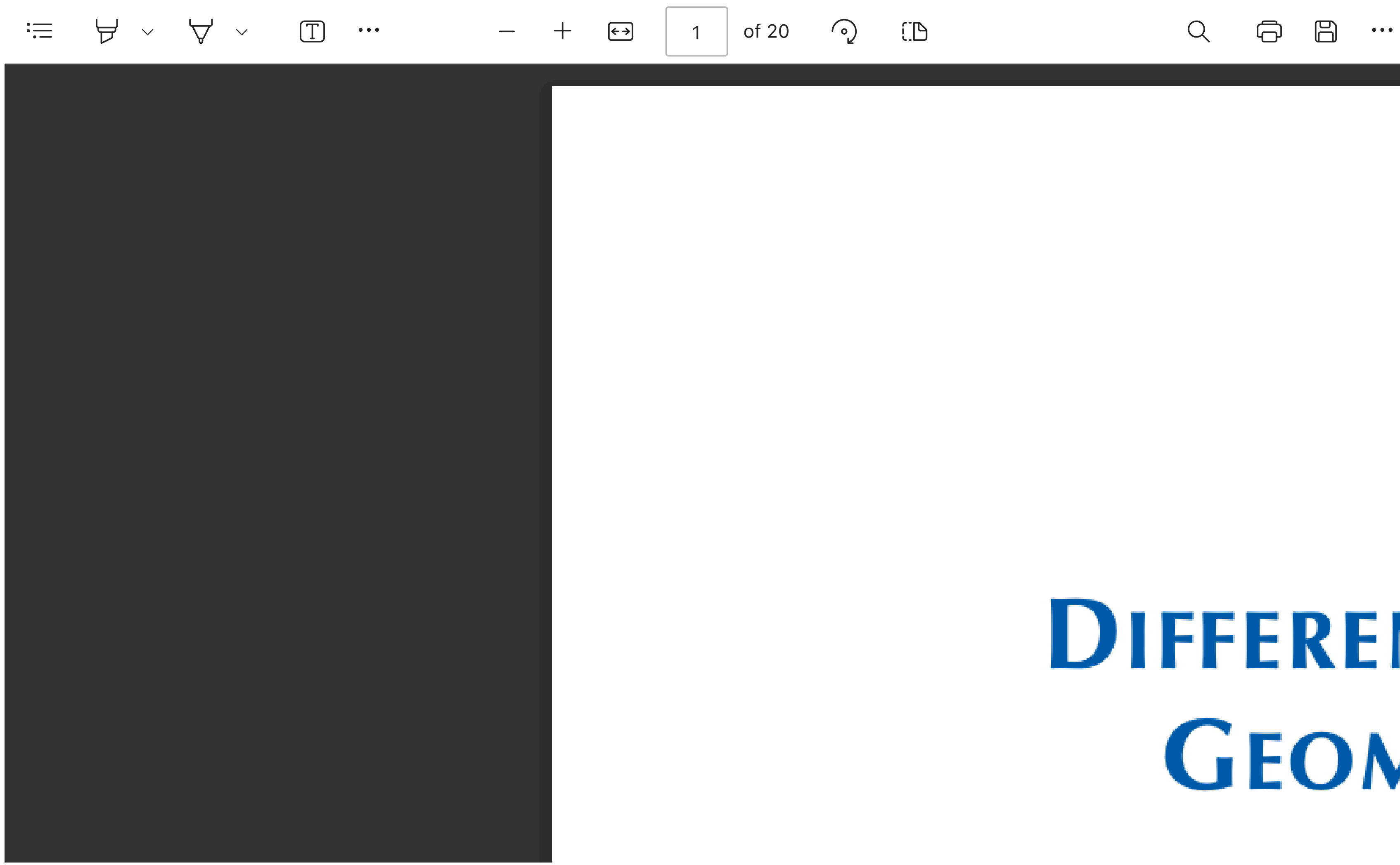
- How can we model curves and surfaces?
  - Which surfaces allow an asymptotic gridshell?
  - How can we efficiently compute such surfaces?
- How can we design the gridshell layout?
  - How can we extract the grid lines?
  - How can we explore design alternatives?

- We will look at:
  - (Discrete) differential geometry
  - Optimization of minimal surfaces
- We will look at:
  - Curve tracing on surfaces
  - Interactive modeling

Overview for Today

- Differential Geometry of parametric curves
  - Tangent vector, normal vector
  - Arc length parameterization
  - Curvature, osculating circle, discrete curvature
  - A first application: Curve smoothing
- Differential Geometry of parametric surfaces
  - tangent space, metric, first fundamental form
  - normal curvature
  - principal curvatures
  - Mean & Gaussian curvature

Reading



Polygon Mesh Processing, Chapter 3

Geometry Representations

- How can we define a unit circle centered at the origin?
- Implicit Representation
  - Kernel of function $F : \mathbb{R}^2 \mapsto \mathbb{R}$, i.e. $\{\mathbf{x} \in \mathbb{R}^2 \mid F(\mathbf{x}) = 0\}$
  - Unit Circle: $F(x, y) = \sqrt{x^2 + y^2} - 1$ or $F(x, y) = x^2 + y^2 - 1$
- Explicit (or Parametric)
  - Range of function $\mathbf{x} : [a, b] \subset \mathbb{R} \mapsto \mathbb{R}^2$, i.e. $\{\mathbf{x}(t) \mid t \in [a, b]\}$
  - Unit Circle: $\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}, t \in [0, 2\pi]$

Parametric Representation

- Parametric representation $\mathbf{x} : [a, b] \subset \mathbb{R} \mapsto \mathbb{R}^2$

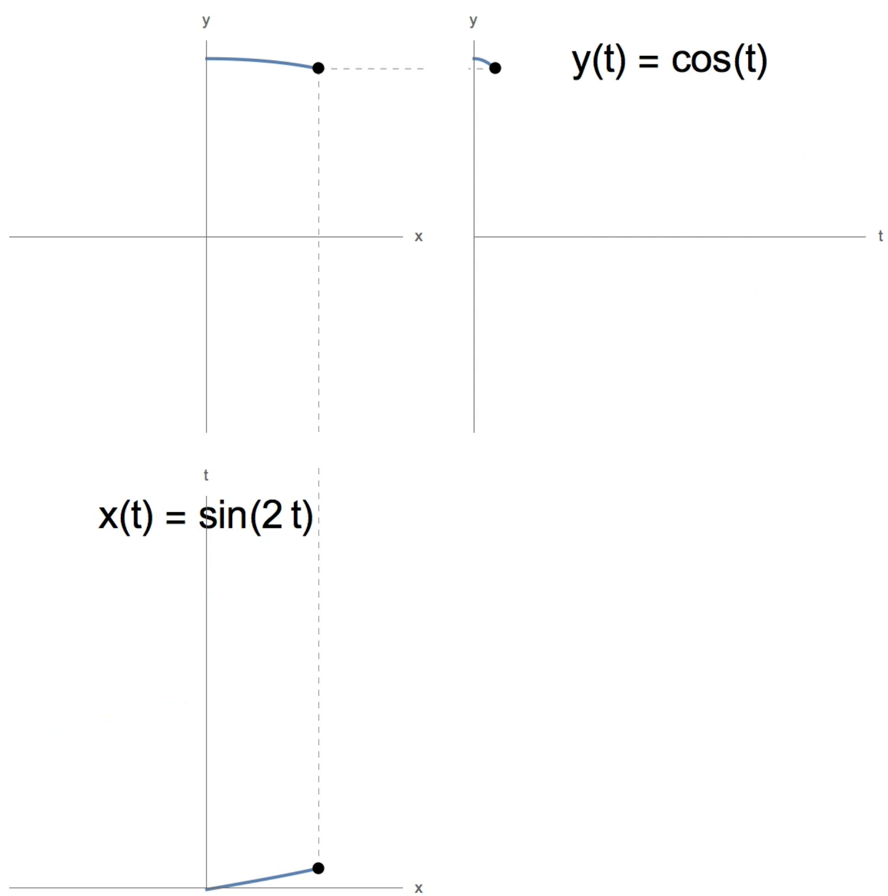
$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Curve is defined as *image* of interval $[a, b]$ under parameterization function $\mathbf{x}$.
- What shape does this curve have?

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \sin(2t) \\ \cos(t) \end{bmatrix}, \quad t \in [0, 2\pi]$$

Parametric Representation

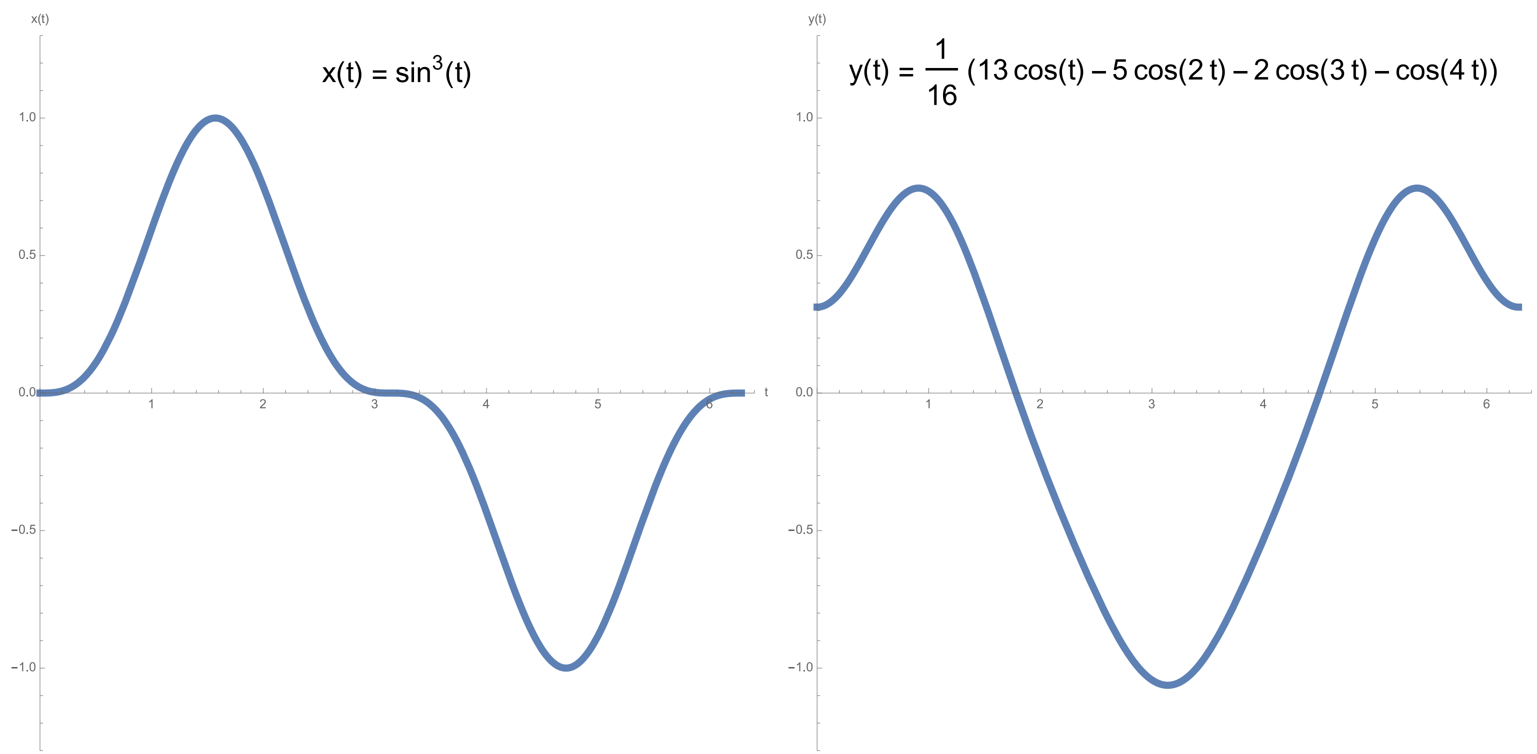
- Parametric representation $\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \sin(2t) \\ \cos(t) \end{bmatrix}, \quad t \in [0, 2\pi]$



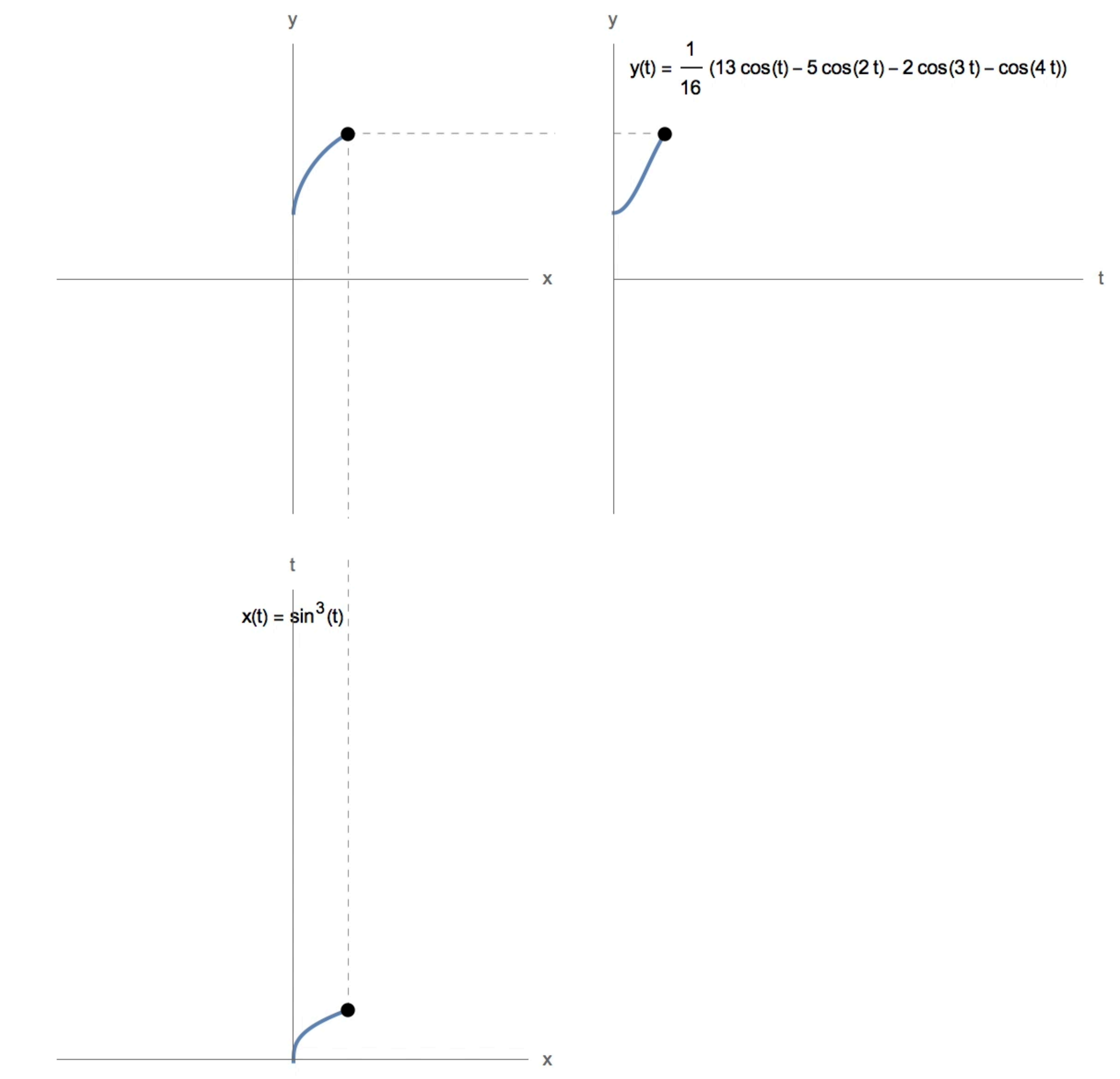
Parametric Representation

- Guess the shape of the curve!

$$\mathbf{x}(t) = \begin{bmatrix} \sin^3(t) \\ \frac{1}{16}(13\cos(t) - 5\cos(2t) - 2\cos(3t) - \cos(4t)) \end{bmatrix}$$

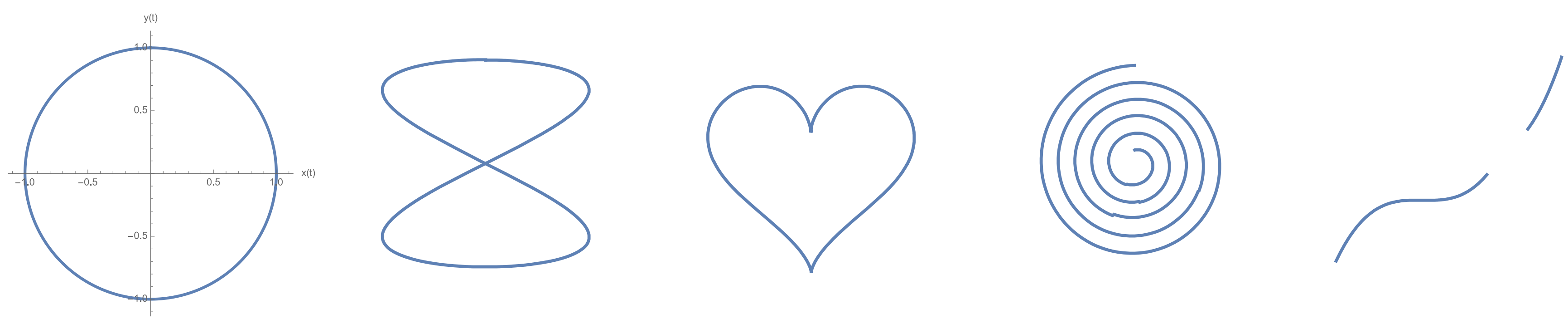


Parametric Representation



Parametric Curve Properties

- A parametric curve $\mathbf{x}(t)$ is
  - *simple*: $\mathbf{x}(t)$ is injective (no self-intersections)
  - *differentiable*: $\mathbf{x}'(t)$ is defined for all $t \in [a, b]$
  - *regular*: $\mathbf{x}'(t) \neq \mathbf{0}$ for all $t \in [a, b]$
- Which of the following are simple, differentiable, regular?



Re-parameterization

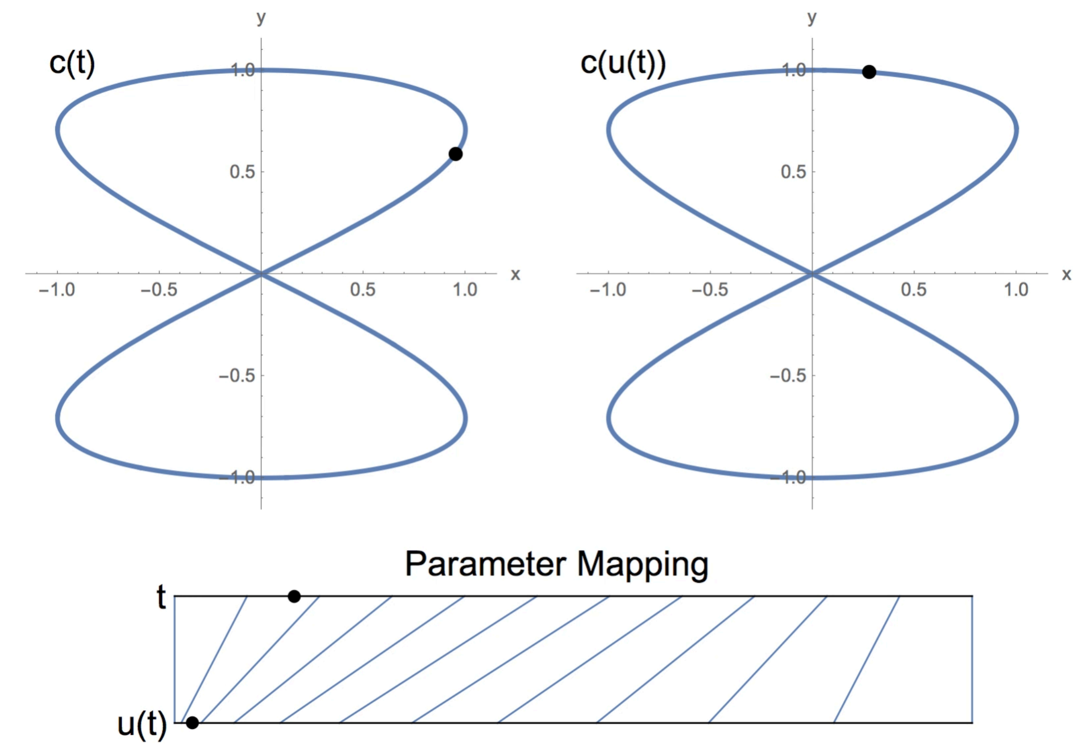
- We can represent the same geometry with different parameter functions.
- For example, the same curve is defined for $t \in [0,1]$ by the functions

$$\mathbf{x}_1(t) = \begin{bmatrix} \sin(4\pi t) \\ \cos(2\pi t) \end{bmatrix} \text{ and } \mathbf{x}_2(t) = \begin{bmatrix} \sin(4\pi\sqrt{t}) \\ \cos(2\pi\sqrt{t}) \end{bmatrix}.$$

- In other words, the image of $[0,1]$ under $\mathbf{x}_1$ and $\mathbf{x}_2$ is equivalent.
- However: $\mathbf{x}_1(t) \neq \mathbf{x}_2(t)$!

Re-parameterization

- We can map from $\mathbf{x}_1$ to $\mathbf{x}_2$ using a re-parameterization function u.
  - In our example, we have u: $[0, 1] \to [0, 1]$ with $u(t) = \sqrt{t}$.
  - If $\mathbf{x}_1(t) = \mathbf{c}(t)$, then $\mathbf{x}_2(t) = \mathbf{c}(u(t))$.



Re-parameterization

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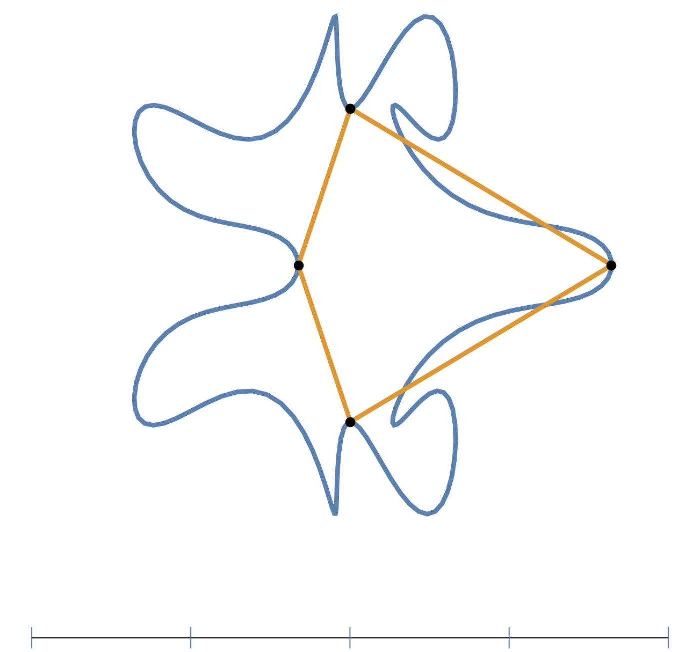
  - In our example, we have u: $[0,1] \to [0,1]$ with $u(t) = \sqrt{t}$.
  - If $\mathbf{x}_1(t) = \mathbf{c}(t)$, then $\mathbf{x}_2(t) = \mathbf{c}(u(t))$.
- Parameter intervals do not need to be identical.
  - For example, if $\mathbf{x}_1$: $[a,b] \to \mathbb{R}^2$ and $\mathbf{x}_2$: $[c,d] \to \mathbb{R}^2$ define the same curve, we can define a re-parameterization function u: $[a,b] \to [c,d]$ such that $\mathbf{x}_1(t) = \mathbf{x}_2(u(t))$.

Discrete Explicit Representation

- Sample the parameter interval $[a, b]$, e.g., at parameters $t_i = a + i\frac{b-a}{n},\ i = 0, \ldots, n$.
- Then the polyline through the points $\mathbf{x}(t_i)$ is a piecewise linear approximation of the curve $\mathbf{x}$.
- With increasing n, the polyline converges to the curve.



Length of a Curve

- How can we measure length of a continuous curve?
- Example: What is the length of a parabola $y = x^2$, $x \in [0, 1]$?
- We know how to measure the length of a polyline!
- Let $t_i = a + i\Delta t$ and $\mathbf{x}_i = \mathbf{x}(t_i)$
- Polyline *chord length*

$$L_P = \sum_i \|\Delta \mathbf{x}_i\| \qquad \Delta \mathbf{x}_i := \|\mathbf{x}_{i+1} - \mathbf{x}_i\|$$

Length of a Curve

- Polyline *chord length*

$$L_P = \sum_i \|\Delta \mathbf{x}_i\| = \sum_i \left\| \frac{\Delta \mathbf{x}_i}{\Delta t} \right\| \Delta t \qquad \Delta \mathbf{x}_i := \|\mathbf{x}_{i+1} - \mathbf{x}_i\|$$

- Curve *arc length* ($\Delta t \to 0$)

$$L = \int_a^b \|\mathbf{x}'\| \mathrm{d}t = \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \mathrm{d}t$$

Length of a Curve

- Example: Length of Circle

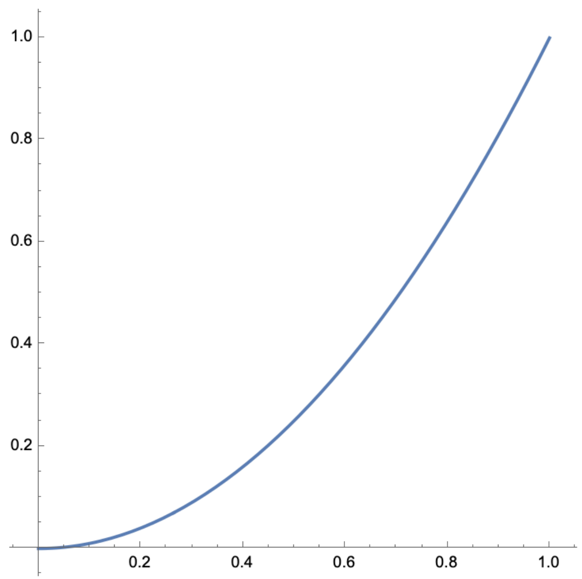
$$\mathbf{x}(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} \qquad \mathbf{x}'(t) = \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}$$

$$L = \int_a^b \|\mathbf{x}'(t)\| \mathrm{d}t = \int_a^b \sqrt{\left(\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y(t)}{\mathrm{d}t}\right)^2} \mathrm{d}t$$

$$L = \int_0^{2\pi} \sqrt{\cos^2(t) + \sin^2(t)} \mathrm{d}t = \int_0^{2\pi} 1 \mathrm{d}t = 2\pi$$

Length of a Curve

- Example: Length of Parabola



$$\mathbf{x}(t) = \begin{bmatrix} t \\ t^2 \end{bmatrix}$$

$$\mathbf{x}'(t) = \begin{bmatrix} 1 \\ 2t \end{bmatrix}$$

$$L = \int_0^1 \sqrt{1+4t^2}\mathrm{d}t$$

$$L = \frac{1}{4}(2\sqrt{5} + \sinh^{-1}(2)) \approx 1.47894$$

Tangent & Normal

- Parametric representation of planar curve $\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$.
- First derivative defines a *tangent vector*.

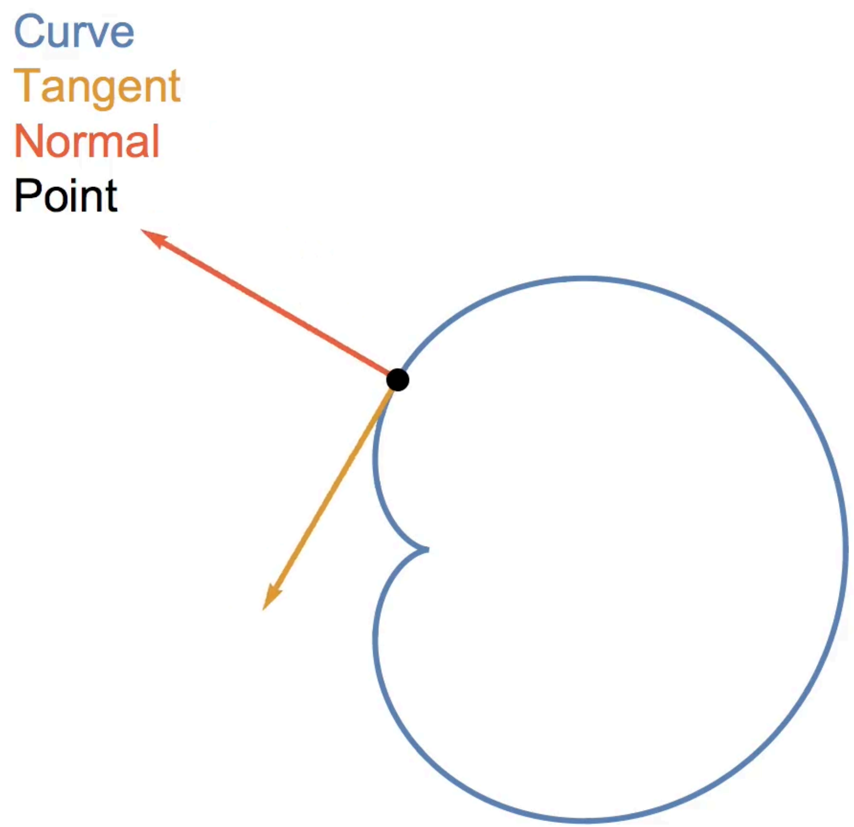
$$\mathbf{t} = \mathbf{x}'(t) := \frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \begin{bmatrix} \mathrm{d}x(t)/\mathrm{d}t \\ \mathrm{d}y(t)/\mathrm{d}t \end{bmatrix}$$

- The curve *normal vector* is

$$\mathbf{n} = \text{Rot}(90)\frac{\mathbf{t}}{\|\mathbf{t}\|}$$

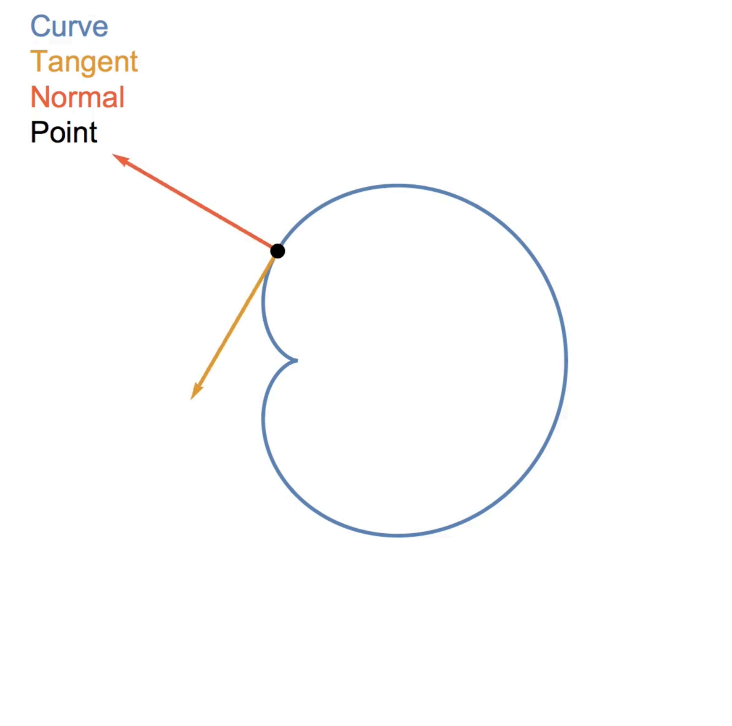
Tangent & Normal

- Example: $\mathbf{x}(t) = (1 + \cos(t)) \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$.


Curve
Tangent
Normal
Point

Curve as Particle Trajectory

- Curve parameter t is time.
- $\mathbf{x}(t)$ defines the position of particle at time t.
- Tangent $\mathbf{x}'(t)$ defines the velocity vector at time t.
- Length (magnitude) of tangent vector is particle speed.
- Second derivative $\mathbf{x}''(t)$ is acceleration.



Curve as Particle Trajectory: Example

- For $t \in [0, 1]$, the two curves

$$\mathbf{x}_1(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{bmatrix} \cos\left(t^2\right) \\ \sin\left(t^2\right) \end{bmatrix}$$

  define the same particle path.

- However, particles travel with different speed!
- $\|\mathbf{x}_1'(t)\| = 1$
- $\|\mathbf{x}_2'(t)\| = \sqrt{(-2t\sin(t^2))^2 + (2t\cos(t^2))^2} = 2|t|$

Arc Length Parameterization

- We see that a curve can be parameterized in different ways. But is there a unique, canonical way to parameterize a curve?
- Yes! It's called *arc length parameterization*.
- Parameterize curve $\mathbf{x}(s)$ over $[0, L]$ such that length from $\mathbf{x}(0)$ to $\mathbf{x}(s)$ is equal to s.

$$\int_0^s \|\mathbf{x}'(t)\| \mathrm{d}t \;=\; s$$

Arc Length Parameterization

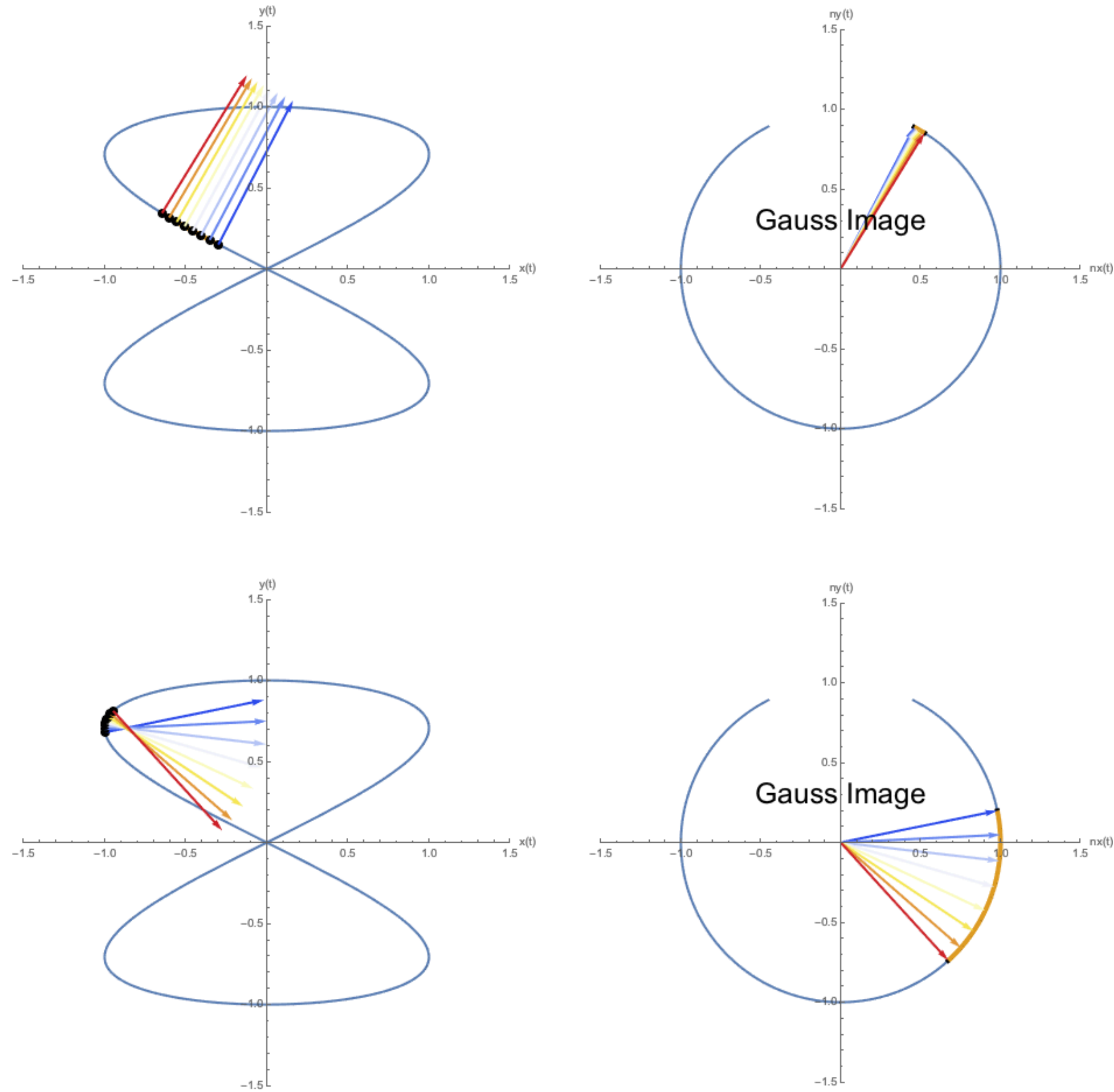
- Intuitively, think about a rope of length $L = \int \|\mathbf{x}'\| \mathrm{d}t$ that is bend (but not stretched or compressed!) to assume the shape of the curve.

- Curves parameterized with respect to arc length have some useful properties.
  - Unit speed: $\|\mathbf{x}'(s)\| = 1$
  - Orthogonality: $\mathbf{x}'(s) \cdot \mathbf{x}''(s) = 0$

Curvature

- Curvature is a measure of how much the curve deviates from a straight line.
- This can be quantified by looking at how much the curve normal varies as we traverse the curve.
- The curve *normal vector* is $\mathbf{n} = \text{Rot}(90)\frac{\mathbf{t}}{\|\mathbf{t}\|}$.
- The *Gauss map* of the curve $\mathbf{x}(t)$ maps the parameter interval $[a, b]$ to the unit circle: $\mathbf{n} : [a, b] \mapsto S^1$.
- This means that for every $t \in [a, b]$ we obtain a point on the unit circle defined by the curve normal $\mathbf{n}(t)$ at point $\mathbf{x}(t)$.

Gauss Map



Curvature

- Let
  - $\Omega = [t-\epsilon, t+\epsilon]$ be a small interval around parameter t,
  - $l_{\mathbf{x}}(\epsilon)$ be the length of the curve segment $\mathbf{x}(\Omega)$,
  - $l_{\mathbf{n}}(\epsilon)$ be the length of the corresponding segment of the Gauss map.
- Then the magnitude of the *curvature* at point $\mathbf{x}(t)$ is defined as

$$|\kappa(t)| = \lim_{\epsilon \to 0} \frac{l_{\mathbf{n}}(\epsilon)}{l_{\mathbf{x}}(\epsilon)}$$

- If a curve is parameterized by arc length, then curvature is simply the magnitude of the second derivative

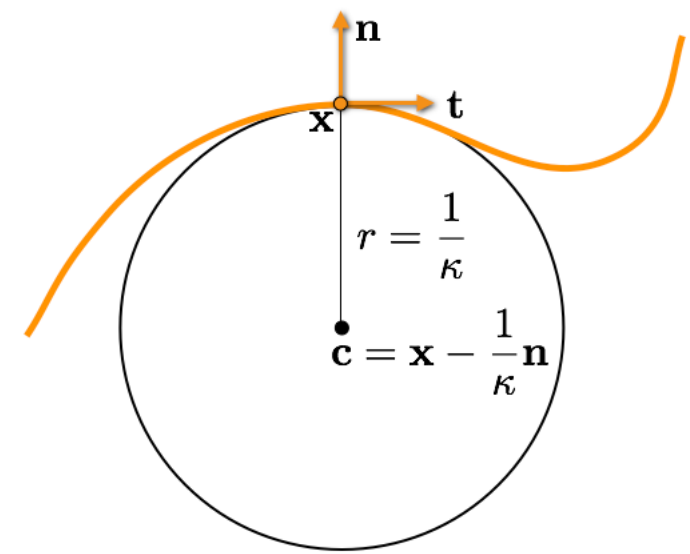
$$\mathbf{t}'(s) = \mathbf{x}''(s) = \kappa(s)\mathbf{n}(s) \;\rightarrow\; |\kappa(s)| = |\mathbf{x}''(s)|$$

- For general parametrizations $\kappa(t) = \frac{||\mathbf{x}'(t) \times \mathbf{x}''(t)||}{||\mathbf{x}'(t)||^3}$

Curvature

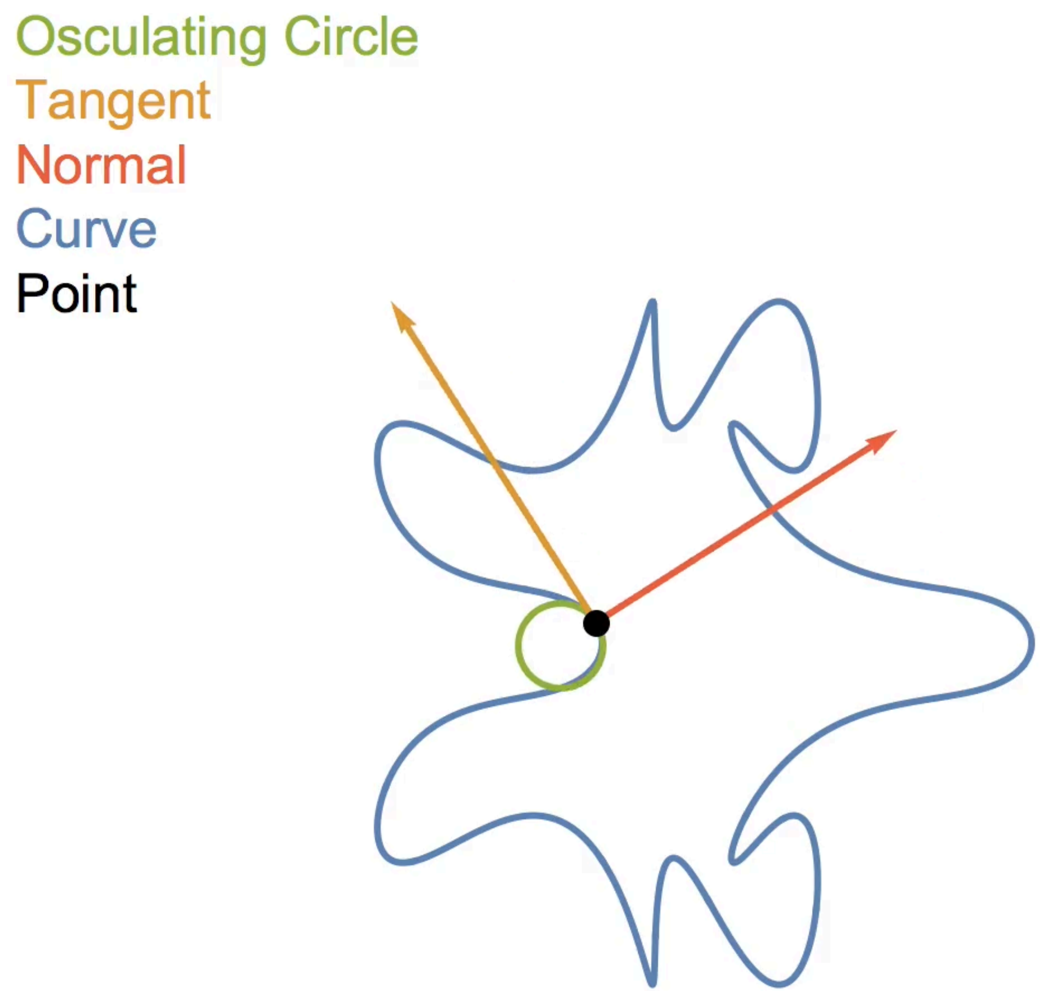
- The *osculating circle* at point $\mathbf{x}$ is the circle tangent to the curve at $\mathbf{x}$ that best approximates the curve locally.
- Its center is given as $\mathbf{x} - \frac{1}{\kappa}\mathbf{n}$, where κ is the *signed* curvature and $\mathbf{n}$ is the normal at $\mathbf{x}$.
  - Orientation of normal is important!
- Its radius is the inverse of the absolute curvature:

$$r = \frac{1}{|\kappa|}$$

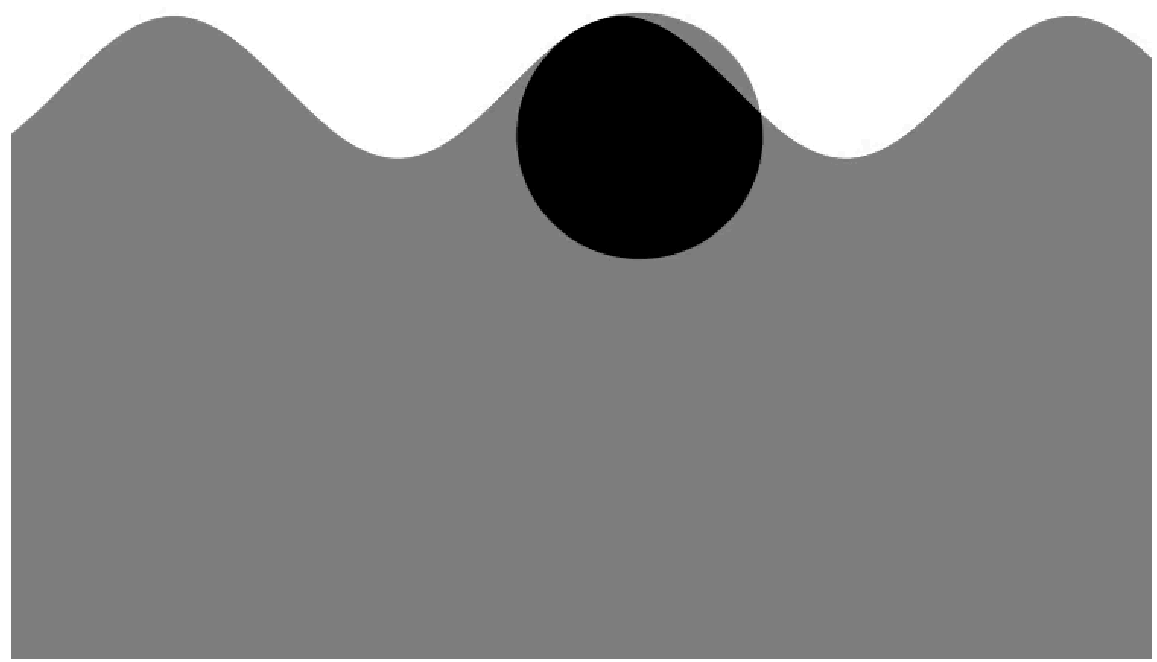


Wikipedia: Osculating Circle

Curvature


Osculating Circle
Tangent
Normal
Curve
Point

Curvature

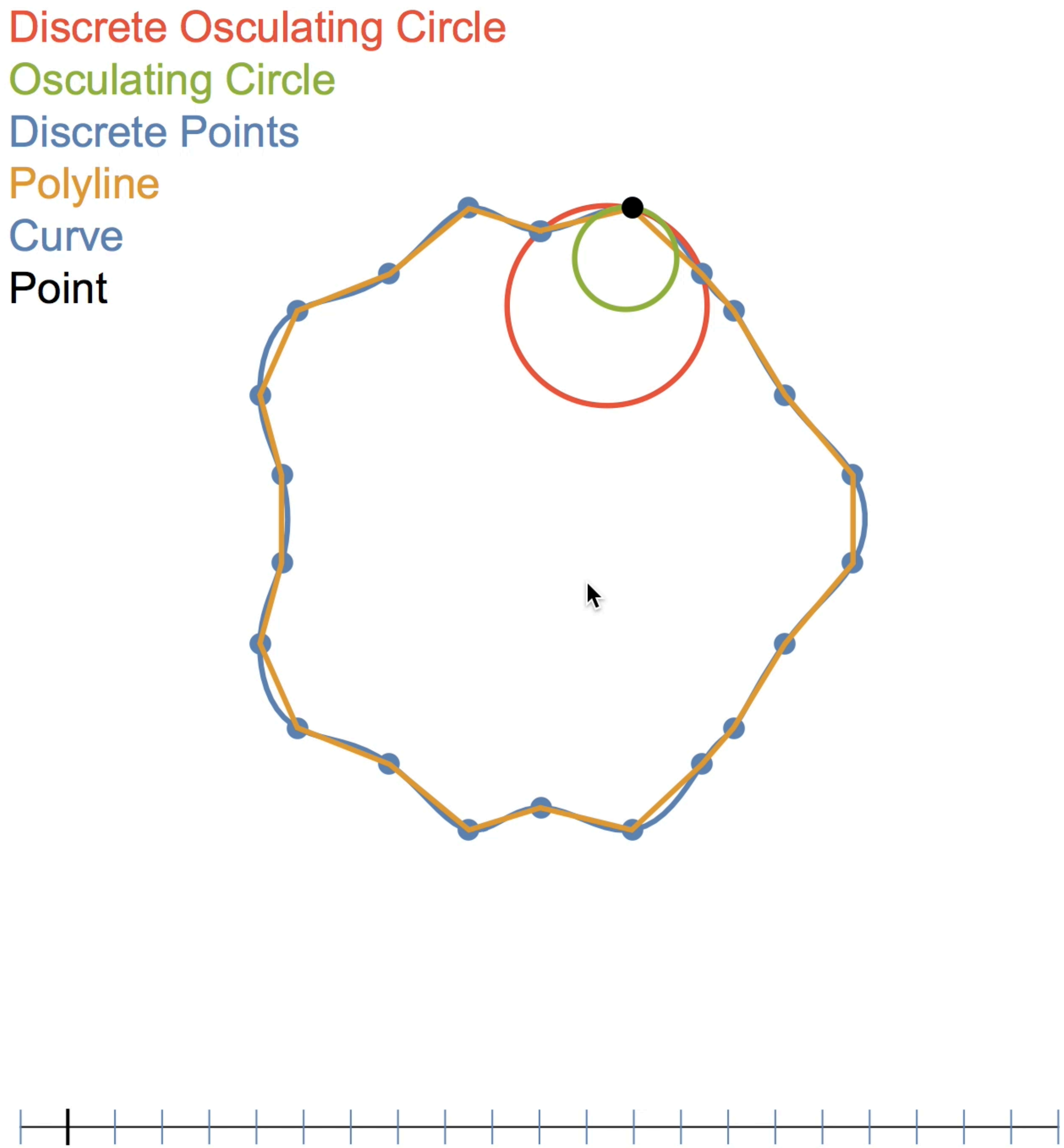


Source

Discrete Curvature

- How can we define curvature on a polyline?
- Continuous definition does not makes sense. Curvature would be zero on line segment and infinite at vertices. Polyline is not differentiable!
- Consider polyline as approximation of a smooth curve.
- Approximate osculating circle by circle passing through three adjacent points.
- Aside: Prove that any three distinct points define a unique circle!

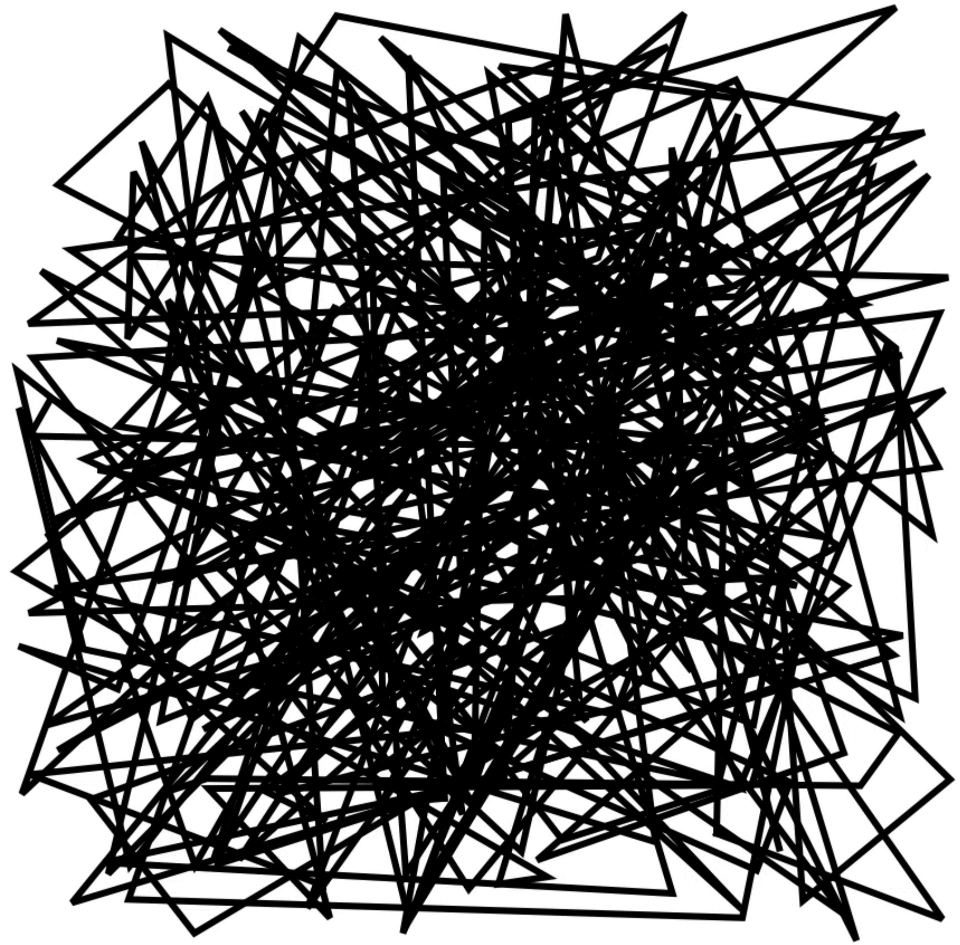
Discrete Curvature


Discrete Osculating Circle
Osculating Circle
Discrete Points
Polyline
Curve
Point

A First Application: Curve Smoothing

- Variant A: Curvature Flow
  - For each vertex, compute the discrete osculating circle
  - Move every vertex towards the center of circle
  - Iterate!
- Variant B: Laplacian Smoothing
  - For each vertex, compute the average of its two neighbors
  - Move every vertex towards the average
  - Iterate!

Discrete Laplacian Curve Smoothing



curve is rescaled to original bounding box after each iteration
number of iterations per frame increases towards end of video

CS457 Geometric Computing

10B - Differential Geometry of Surfaces

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Overview

- Differential Geometry of parametric curves
- **Differential Geometry of parametric surfaces**
    - tangent space, metric, first fundamental form
    - normal curvature
    - principal curvatures
    - mean & Gaussian curvature

Parametric Surfaces

- Continuous surface

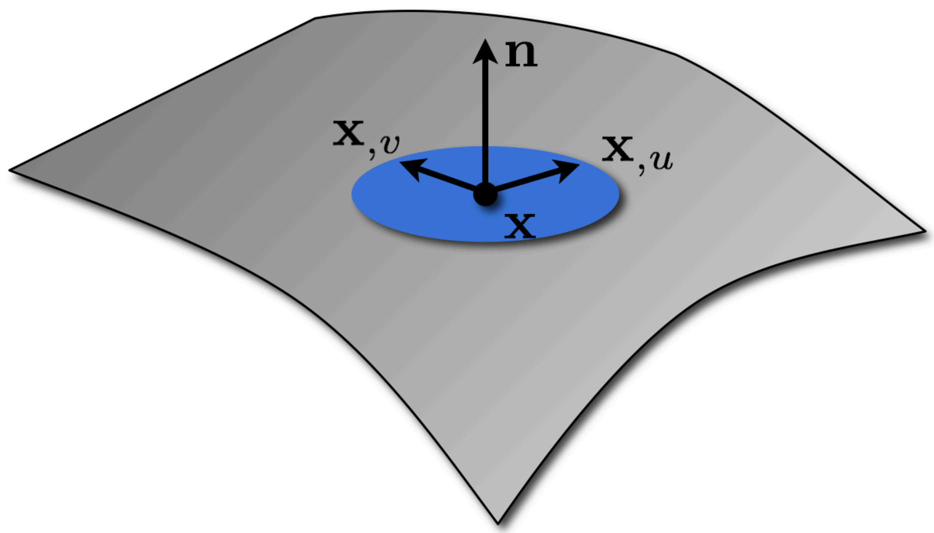
$$\mathbf{x}(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix}$$

- Normal vector

$$\mathbf{n} = \frac{\mathbf{x}_{,u} \times \mathbf{x}_{,v}}{\|\mathbf{x}_{,u} \times \mathbf{x}_{,v}\|}$$

- Assume regular parameterization

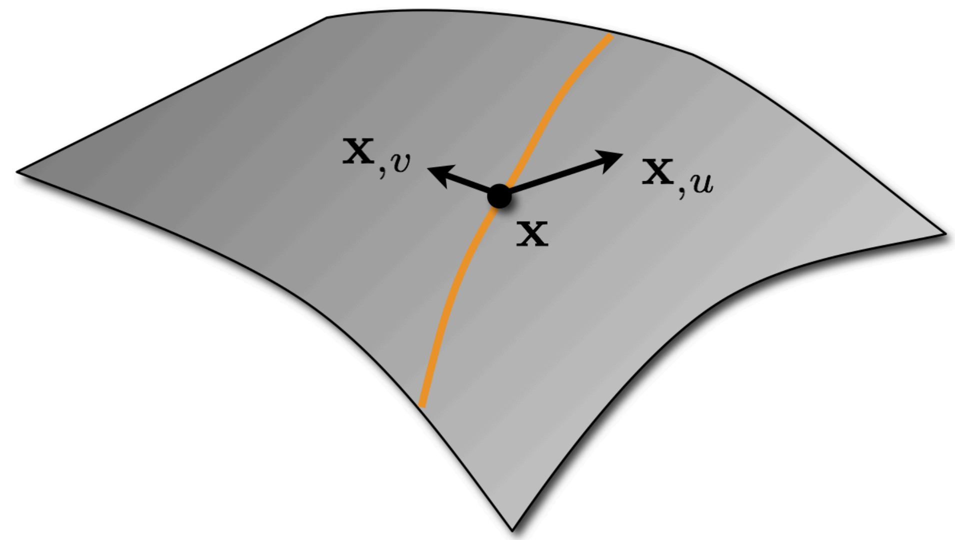
$$\mathbf{x}_{,u} \times \mathbf{x}_{,v} \neq \mathbf{0}$$



Curves on Surface

- A curve $(u(t), v(t))$ in the uv-plane defines a curve on the surface $\mathbf{x}(u, v)$:

$$\mathbf{x}(t) \ = \ \mathbf{x}(u(t), v(t))$$



Tangent Vectors of Curves on Surface

- Curve on surface $\mathbf{x}(t) = \mathbf{x}(u(t), v(t))$
- Tangent vector to curve

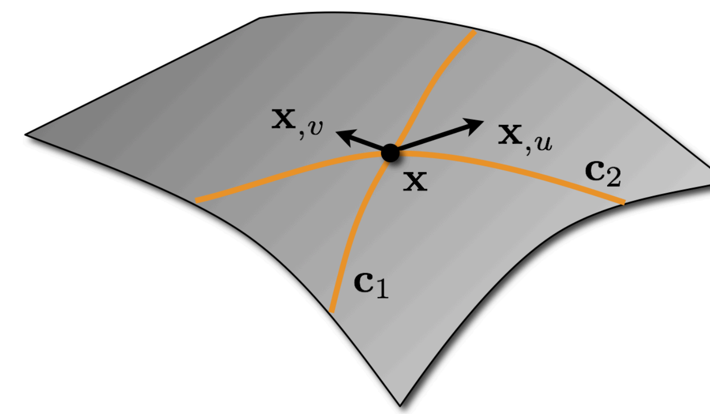
$$\begin{aligned}
\mathbf{x}(t)' = \frac{d\mathbf{x}(t)}{dt} &= \frac{d}{dt}(\mathbf{x}(u(t), v(t))) \\
&= \frac{\partial \mathbf{x}}{\partial u} \cdot \frac{du}{dt} + \frac{\partial \mathbf{x}}{\partial v} \cdot \frac{dv}{dt} \\
&= [\mathbf{x}_{,u}, \mathbf{x}_{,v}] \begin{bmatrix} u' \\ v' \end{bmatrix} \\
&= J\mathbf{u}'
\end{aligned}$$

- **Jacobian matrix** $J \in \mathbb{R}^{3\times 2}$ defines mapping of tangent vectors from the domain to the surface: $\mathbf{u}' \mapsto \mathbf{x}'$
- **Tangent plane** is formed by linear combinations of $\mathbf{x}_{,u}$ and $\mathbf{x}_{,v}$

Angles on Surface

- What is the angle of intersection of two curves $\mathbf{c}_1$ and $\mathbf{c}_2$ intersecting at $\mathbf{x}$?
- Two tangents $\mathbf{t}_1$ and $\mathbf{t}_2$

$$\mathbf{t}_i = \alpha_i \mathbf{x}_{,u} + \beta_i \mathbf{x}_{,v}$$



- Compute inner product

$$\begin{aligned}
\mathbf{t}_1{}^\top \mathbf{t}_2 \;&=\; \cos\theta \|\mathbf{t}_1\| \|\mathbf{t}_2\| \;=\; (\alpha_1 \mathbf{x}_{,u} + \beta_1 \mathbf{x}_{,v})^\top (\alpha_2 \mathbf{x}_{,u} + \beta_2 \mathbf{x}_{,v}) \\
&=\; \left([\mathbf{x}_{,u}, \mathbf{x}_{,v}] \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} \right)^\top [\mathbf{x}_{,u}, \mathbf{x}_{,v}] \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} \;=\; [\alpha_1, \beta_1] J^\top J \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} \\
&=\; (\alpha_1, \beta_1) \begin{bmatrix} \mathbf{x}_{,u}{}^\top \mathbf{x}_{,u} & \mathbf{x}_{,u}{}^\top \mathbf{x}_{,v} \\ \mathbf{x}_{,u}{}^\top \mathbf{x}_{,v} & \mathbf{x}_{,v}{}^\top \mathbf{x}_{,v} \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix}
\end{aligned}$$

First Fundamental Form

- First fundamental form

$$\mathbf{I} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} := \begin{bmatrix} \mathbf{x}_{,u}^T \mathbf{x}_{,u} & \mathbf{x}_{,u}^T \mathbf{x}_{,v} \\ \mathbf{x}_{,u}^T \mathbf{x}_{,v} & \mathbf{x}_{,v}^T \mathbf{x}_{,v} \end{bmatrix} = J^\top J$$

- Defines inner product on tangent space

$$\left\langle \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}, \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} \right\rangle := \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}^T \mathbf{I} \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix}$$

Wikipedia: First Fundamental Form

First Fundamental Form

First fundamental form allows to measure...

- Angles

$$\begin{aligned} \mathbf{t}_1^T \mathbf{t}_2 &= \langle (\alpha_1, \beta_1), (\alpha_2, \beta_2) \rangle \\ &= E\alpha_1\alpha_2 + F\left(\alpha_1\beta_2 + \alpha_2\beta_1\right) + G\beta_1\beta_2 \end{aligned}$$

- Length

$$\begin{aligned} \mathrm{d}s &= \sqrt{\langle (\mathrm{d}u, \mathrm{d}v), (\mathrm{d}u, \mathrm{d}v) \rangle} \\ &= \sqrt{E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\mathrm{d}v + G\,\mathrm{d}v^2} \end{aligned}$$

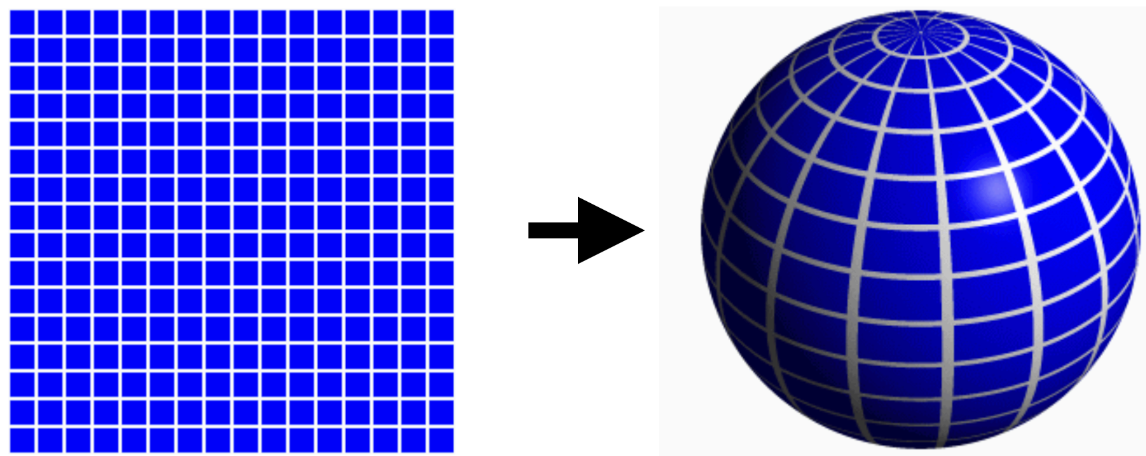
- Area

$$\begin{aligned} \mathrm{d}A &= \sqrt{\det\left(\mathbf{I}\right)}\,\mathrm{d}u\,\mathrm{d}v \\ &= \sqrt{EG - F^2}\,\mathrm{d}u\,\mathrm{d}v \end{aligned}$$

Wikipedia: First Fundamental Form

Example: Unit Sphere

- Spherical parameterization



$$\mathbf{x}(u,v) \;=\; \begin{bmatrix} \cos u \sin v \\ \sin u \sin v \\ \cos v \end{bmatrix}, \qquad (u,v) \in [0, 2\pi) \times [0, \pi)$$

Example: Unit Sphere

- Tangent vectors

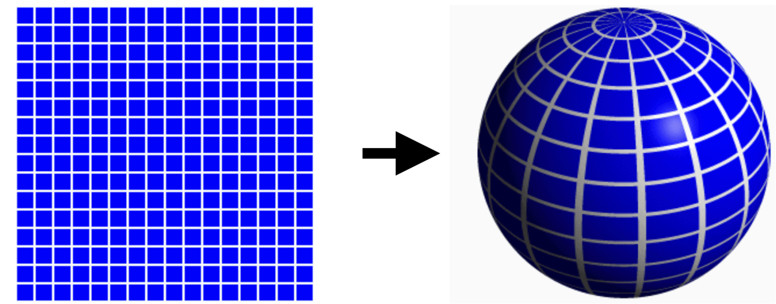
$$\mathbf{x}_{,u}(u,v) = \begin{bmatrix} -\sin u \sin v \\ \cos u \sin v \\ 0 \end{bmatrix} \quad \mathbf{x}_{,v}(u,v) = \begin{bmatrix} \cos u \cos v \\ \sin u \cos v \\ -\sin v \end{bmatrix}$$

- First fundamental form

$$\mathbf{I} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} := \begin{bmatrix} \mathbf{x}_{,u}{}^\top \mathbf{x}_{,u} & \mathbf{x}_{,u}{}^\top \mathbf{x}_{,v} \\ \mathbf{x}_{,u}{}^\top \mathbf{x}_{,v} & \mathbf{x}_{,v}{}^\top \mathbf{x}_{,v} \end{bmatrix} = \begin{bmatrix} \sin^2 v & 0 \\ 0 & 1 \end{bmatrix}$$

Example: Unit Sphere

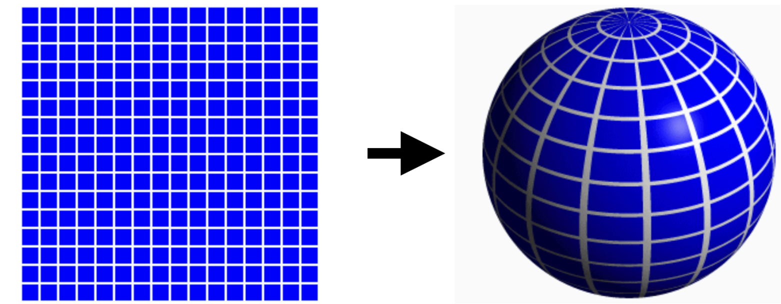
- Length of equator $\mathbf{x}(t, \pi/2)$
  - $u(t) = t$ and $u'(t) = 1$
  - $v(t) = \pi/2$ and $v'(t) = 0$



$$\begin{aligned}
\int_0^{2\pi} 1\,\mathrm{d}s \;=\; & \int_0^{2\pi} \sqrt{E\,u_{,t}^2 + 2F\,u_{,t}v_{,t} + G\,v_{,t}^2}\,\mathrm{d}t \\
=\; & \int_0^{2\pi} \sin v\,\mathrm{d}t \\
=\; & 2\pi \sin v \;=\; 2\pi
\end{aligned}$$

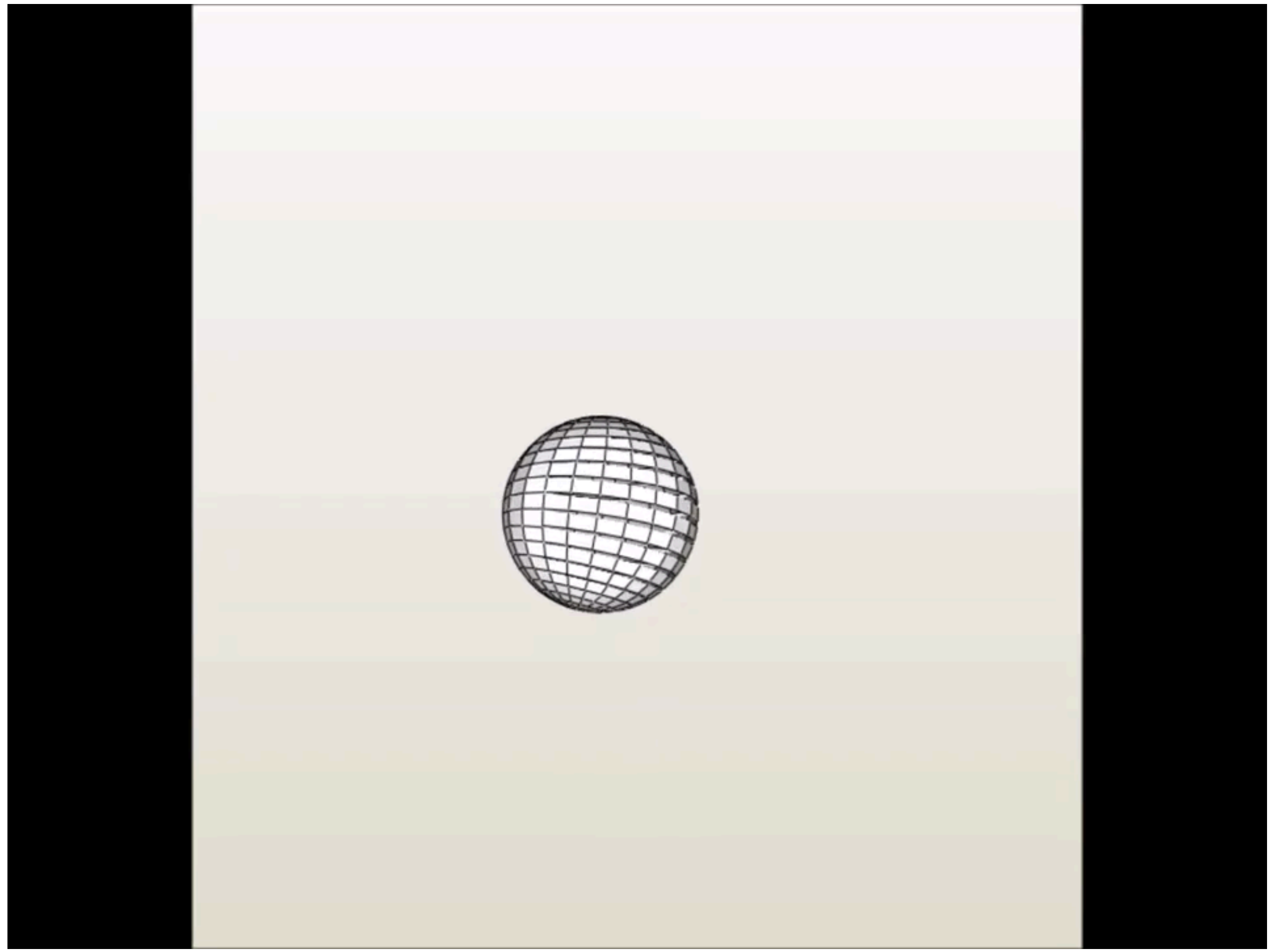
Example: Unit Sphere

- Area of sphere



$$
\begin{aligned}
\int_0^{\pi}\int_0^{2\pi} 1 \,\mathrm{d}A &= \int_0^{\pi}\int_0^{2\pi} \sqrt{EG - F^2}\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_0^{\pi}\int_0^{2\pi} \sin v \,\mathrm{d}u\,\mathrm{d}v \\
&= 4\pi
\end{aligned}
$$

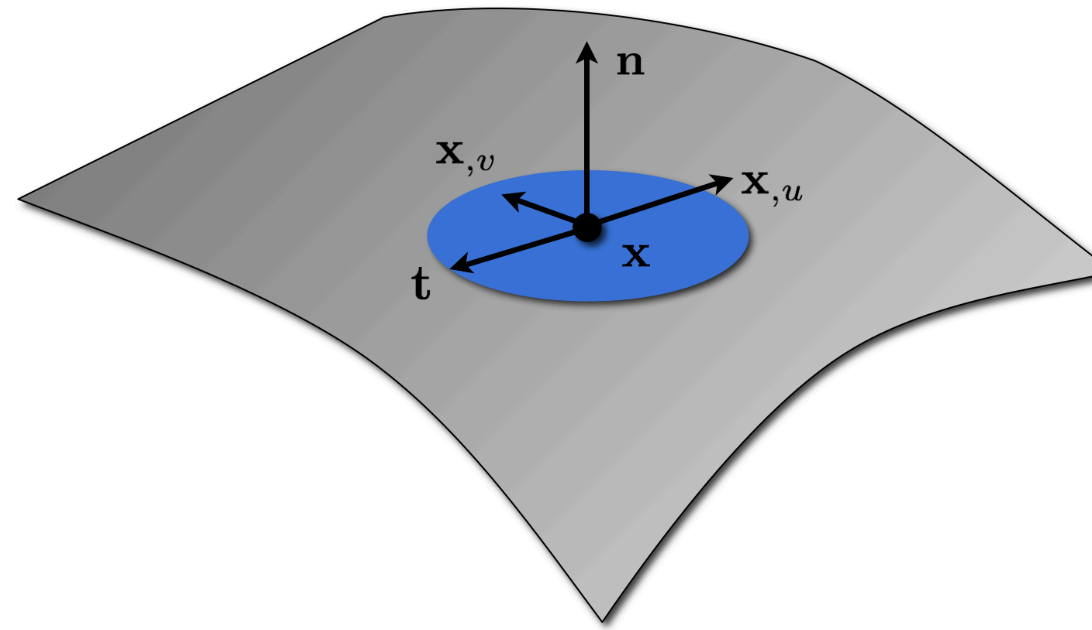
Example: Unit Sphere



source

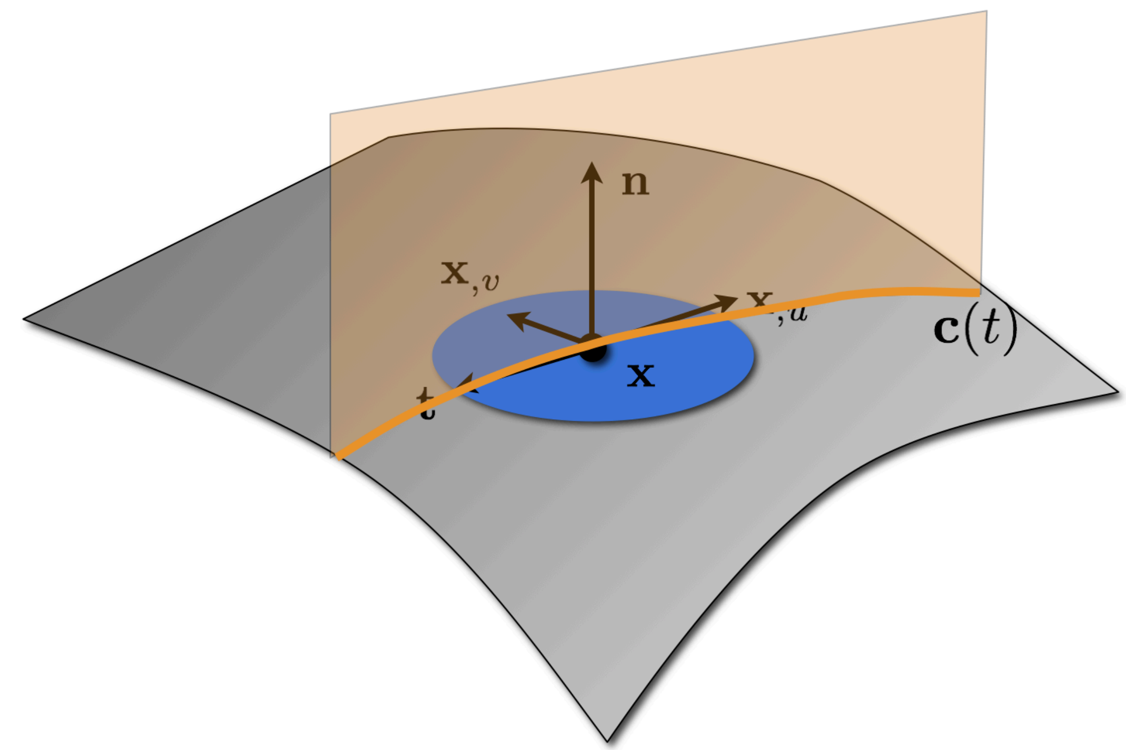
Normal Curvature

- Let $\mathbf{t}$ be a tangent vector at $\mathbf{x}$.



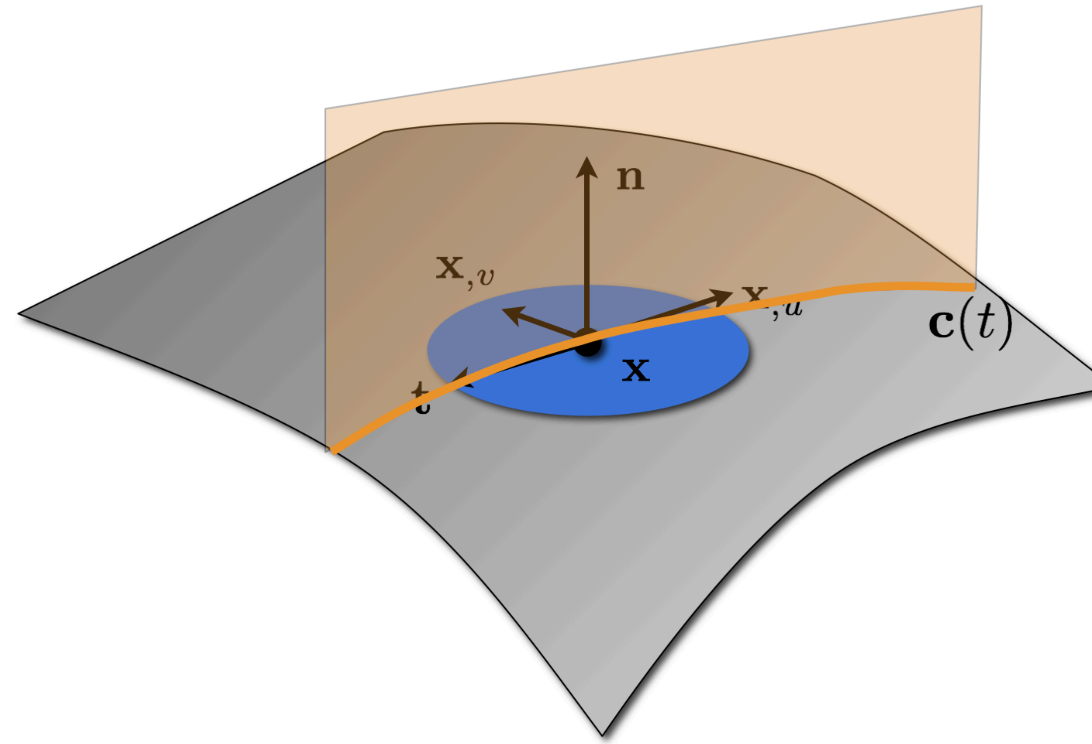
Normal Curvature

- Let $\mathbf{t}$ be a tangent vector at $\mathbf{x}$.
- $\mathbf{x}$, $\mathbf{n}$, and $\mathbf{t}$ define a *normal plane*. The intersection of this plane with the surface yields a curve $\mathbf{x}(t)$, called a *normal section*.



Normal Curvature

- Let $\mathbf{t}$ be a tangent vector at $\mathbf{x}$.
- $\mathbf{x}$, $\mathbf{n}$, and $\mathbf{t}$ define a *normal plane*. The intersection of this plane with the surface yields a curve $\mathbf{x}(t)$, called a *normal section*.
- Normal curvature $\kappa_n(\mathbf{t})$ is defined as the curvature of the normal section $\mathbf{x}(t)$ at the point $\mathbf{x}(u, v)$.



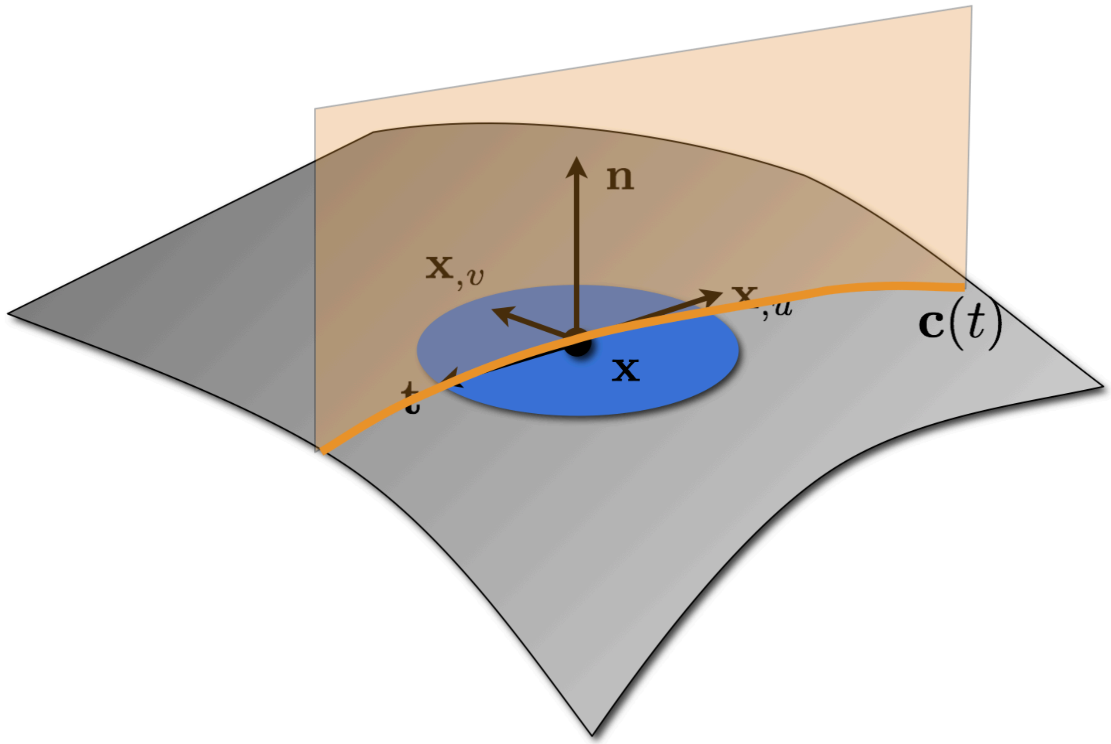
Normal Curvature

- If we write $\mathbf{t} = a\mathbf{x}_{,u} + b\mathbf{x}_{,v}$, the normal curvature can be computed as

$$\kappa_n(\mathbf{t}) = \frac{(a,b)\,\mathbf{II}\,(a,b)^\top}{(a,b)\,\mathbf{I}\,(a,b)^\top} = \frac{ea^2 + 2fab + gb^2}{Ea^2 + 2Fab + Gb^2}$$

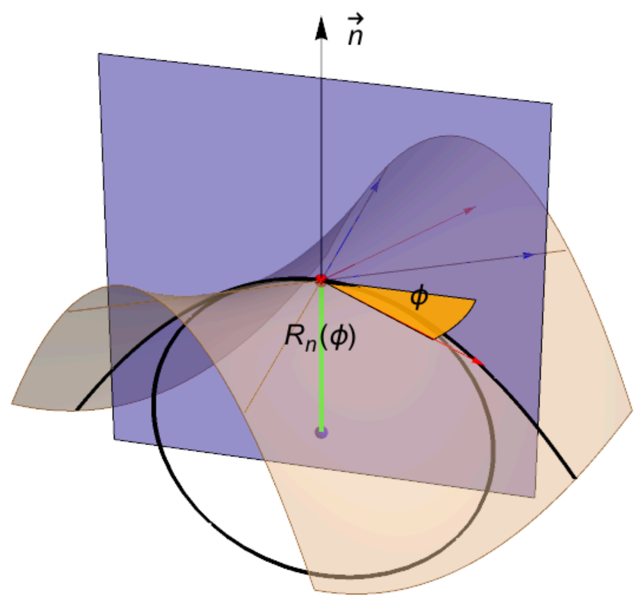
with the *second fundamental form*

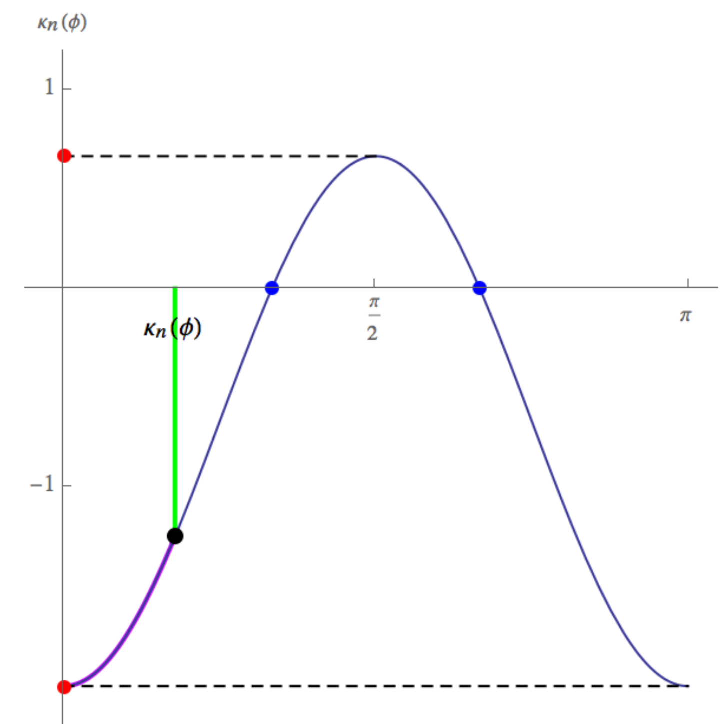
$$\mathbf{II} = \begin{bmatrix} e & f \\ f & g \end{bmatrix} := \begin{bmatrix} \mathbf{x}_{,uu}{}^\top \mathbf{n} & \mathbf{x}_{,uv}{}^\top \mathbf{n} \\ \mathbf{x}_{,uv}{}^\top \mathbf{n} & \mathbf{x}_{,vv}{}^\top \mathbf{n} \end{bmatrix}$$



Normal Curvature

- Let $\mathbf{t}(\phi) = \cos\phi\,\mathbf{x}_{,u} + \sin\phi\,\mathbf{x}_{,v}$ be a tangent vector at $\mathbf{x}$ and assume that $\mathbf{x}_{,u}$ and $\mathbf{x}_{,v}$ are orthonormal.
- We can plot $\kappa_n(\mathbf{t}(\phi))$ as a function of the tangent angle ϕ

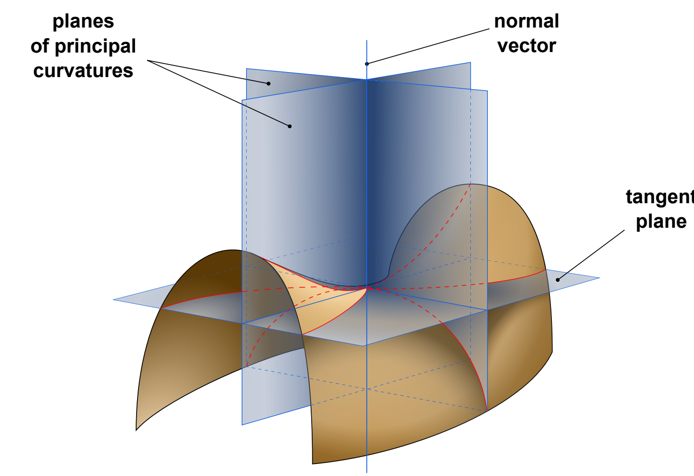




Surface Curvature(s)

Principal curvatures

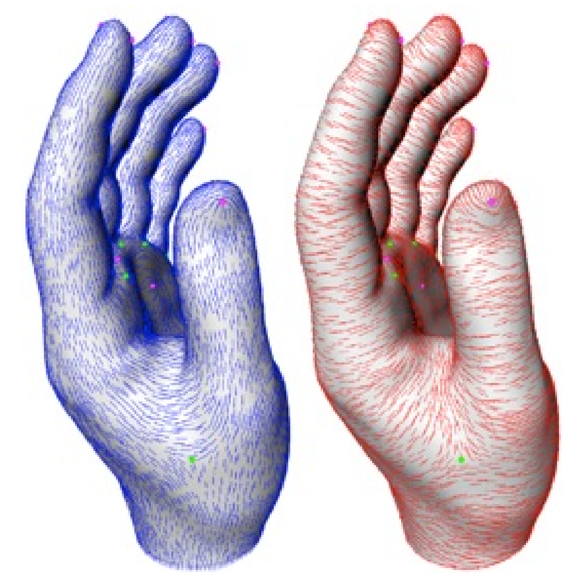
- Maximum curvature $\kappa_1 = \max_\phi \kappa_n(\phi)$
- Minimum curvature $\kappa_2 = \min_\phi \kappa_n(\phi)$



- Corresponding principal directions $\mathbf{e}_1$, $\mathbf{e}_2$ are orthogonal.
- Can be computed from eigenvectors of the *shape operator*

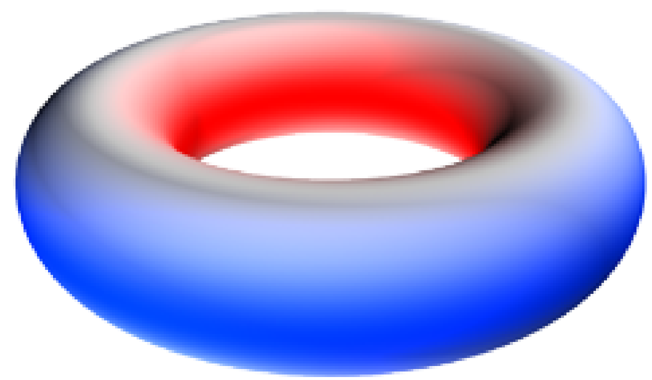
$$S = \mathbf{I}^{-1}\mathbf{II}$$

  - (see handout for details).



Surface Curvature(s)

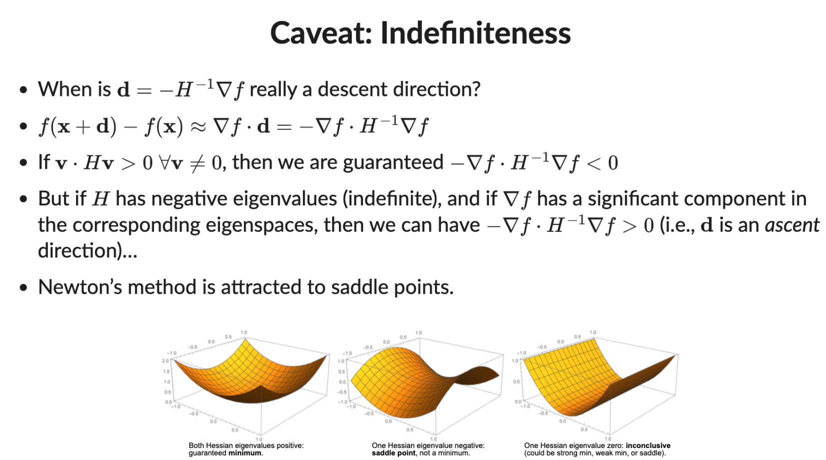
- Euler theorem: $\kappa_n(\phi) \;=\; \kappa_1 \cos^2\phi + \kappa_2 \sin^2\phi$
  - Proof: See handout
- Gaussian curvature $K = \kappa_1 \cdot \kappa_2$
- Example: Torus



Classification

A point $\mathbf{x}$ on the surface is called

- *elliptic*, if $K > 0$
- *hyperbolic*, if $K < 0$
- *parabolic*, if $K = 0$
- *umbilic*, if $\kappa_1 = \kappa_2$



Caveat: Indefiniteness

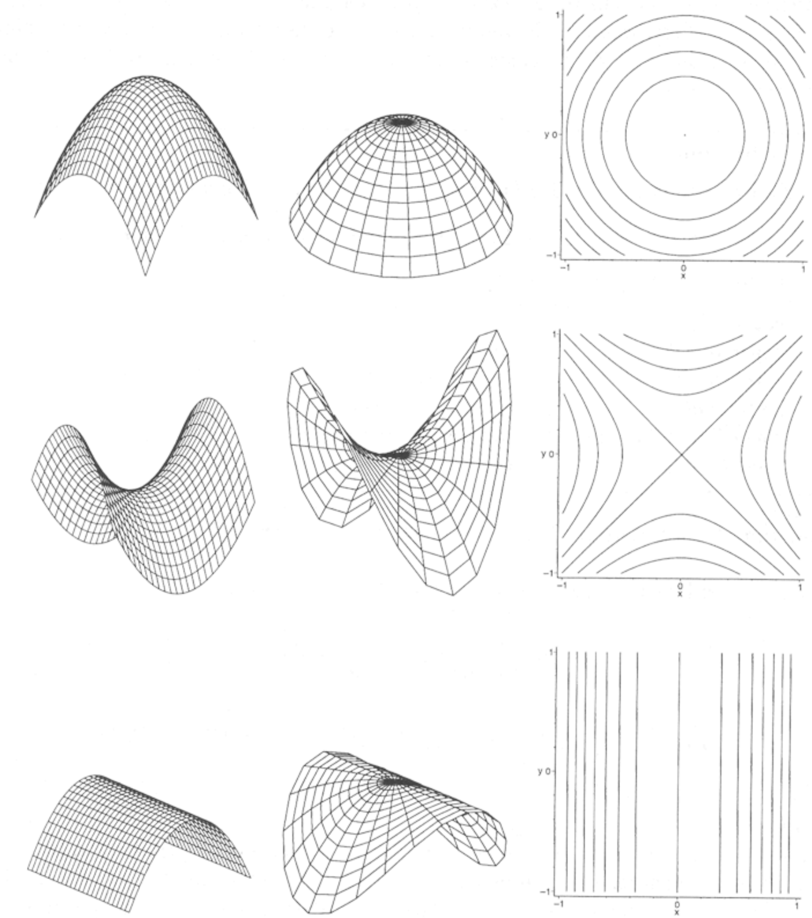
- When is $\mathbf{d} = -H^{-1}\nabla f$ really a descent direction?
- $f(\mathbf{x}+\mathbf{d}) - f(\mathbf{x}) \approx \nabla f \cdot \mathbf{d} = -\nabla f \cdot H^{-1}\nabla f$
- If $\mathbf{v} \cdot H\mathbf{v} > 0 \ \forall \mathbf{v} \neq 0$, then we are guaranteed $-\nabla f \cdot H^{-1}\nabla f < 0$
- But if H has negative eigenvalues (indefinite), and if ∇f has a significant component in the corresponding eigenspaces, then we can have $-\nabla f \cdot H^{-1}\nabla f > 0$ (i.e., $\mathbf{d}$ is an *ascent* direction)...
- Newton's method is attracted to saddle points.

Both Hessian eigenvalues positive: guaranteed **minimum**.

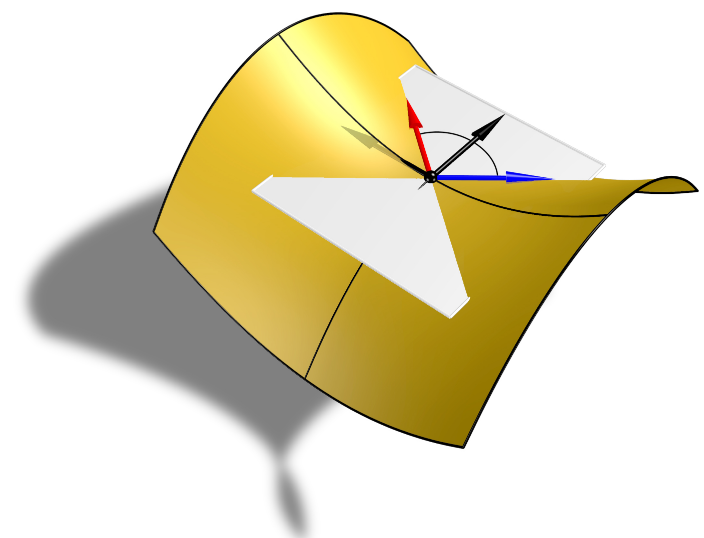
One Hessian eigenvalue negative: **saddle point**, not a minimum.

One Hessian eigenvalue zero: **inconclusive** (could be strong min, weak min, or saddle).

Slide 10 from 04B



Asymptotic Directions



- Which curves on the surface have zero normal curvature?
- Euler theorem: $\kappa_n(\phi) \;=\; \kappa_1 \cos^2\phi + \kappa_2 \sin^2\phi$
- Solutions for $\kappa_n(\phi) = 0$

$$\cos\theta = \sqrt{1 - \frac{\kappa_1}{\kappa_1 - \kappa_2}}, \quad \sin\theta = \pm\sqrt{\frac{\kappa_1}{\kappa_1 - \kappa_2}}.$$

- Observation: Only if $K \leq 0$ can we have $\kappa_n(\phi) = 0$ for some ϕ.
- For a point on the surface, a direction defined by $\mathbf{t}(\phi) = \cos\phi \mathbf{x}_{,u} + \sin\phi \mathbf{x}_{,v}$ with $\kappa_n(\phi) = 0$ is called an *asymptotic direction*.

Asymptotic Curves

- A curve on the surface that is at each point tangent to an asymptotic direction is called an *asymptotic curve*.
- Asymptotic curves only bend in the tangent plane of the surface, but not in the orthogonal direction.