Exercise Set III, Algorithms II

These exercises are for your own benefit. Feel free to collaborate and share your answers with other students. **This exercise set contains many problems.** So solve as many problems as you can and ask for help if you get stuck for too long. Problems marked * are more difficult but also more fun :).

These problems are taken from various sources at EPFL and on the Internet, too numerous to cite individually.

1 *(*)* Consider an undirected graph $G = (V, E)$ and let $s \neq t \in V$. Recall that in the min s, t-cut problem, we wish to find a set $S \subseteq V$ such that $s \in S$, $t \notin S$ and the number of edges crossing the cut is minimized. Show that the optimal value of the following linear program equals the number of edges crossed by a min s, t-cut:

$$
\begin{array}{lll}
\textbf{minimize} & \displaystyle\sum_{e \in E} y_e & \\
\textbf{subject to} & y_{\{u,v\}} \geq x_u - x_v & \text{for every } \{u,v\} \in E \\
 & y_{\{u,v\}} \geq x_v - x_u & \text{for every } \{u,v\} \in E \\
 & x_s = 0 & \\
 & x_t = 1 & \\
 & x_v \in [0,1] & \text{for every } v \in V
\end{array}
$$

The above linear program has a variable x_v for every vertex $v \in V$ and a variable y_e for every edge $e \in E$.

Hint: Show that the expected value of the following randomized rounding equals the value of the linear program. Select θ uniformly at random from $[0, 1]$ and output the cut $S = \{v \in V : x_v \leq \theta\}$.

2 *(*)* Consider the linear programming relaxation for minimum-weight vertex cover:

$$\begin{aligned} \text{Minimize} \quad & \sum_{v \in V} x_v w(v) \\ \text{Subject to} \quad & x_u + x_v \geq 1 \quad \forall \{u, v\} \in E \\ & 0 \leq x_v \leq 1 \qquad \forall v \in V \end{aligned}$$

In class, we saw that any extreme point is integral when considering bipartite graphs. For general graphs, this is not true, as can be seen by considering the graph consisting of a single triangle. However, we have the following statement for general graphs:

Any extreme point x^* satisfies $x^*_v \in \{0, \frac{1}{2}, 1\}$ for every $v \in V$.

Prove the above statement.

3 Write the dual of the following linear program:

$$\begin{aligned} \text{Maximize} \quad & 6x_1 + 14x_2 + 13x_3 \\ \text{Subject to} \quad & x_1 + 3x_2 + x_3 \leq 24 \\ & x_1 + 2x_2 + 4x_3 \leq 60 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Hint: How can you convince your friend that the above linear program has optimum value at most z?

4 Consider the min-cost perfect matching problem on a bipartite graph $G = (A \cup B, E)$ with costs $c : E \to \mathbb{R}$. Recall from the lecture that the dual linear program is

$$\begin{aligned} \text{Maximize} \quad & \sum_{a \in A} u_a + \sum_{b \in B} v_b \\ \text{Subject to} \quad & u_a + v_b \leq c(\{a, b\}) \qquad \text{for every edge } \{a, b\} \in E. \end{aligned}$$

Show that the dual linear program is unbounded if there is a set $S \subseteq A$ such that $|S| > |N(S)|$, where $N(S) = \{v \in B : \{u, v\} \in E \text{ for some } u \in S\}$ denotes the neighborhood of S. This proves (as expected) that the primal is infeasible in this case.

5 *(half a *)* Prove Hall's Theorem:

"An n-by-n bipartite graph $G = (A \cup B, E)$ has a perfect matching if and only if $|S| \leq |N(S)|$ for all $S \subseteq A$."

(Hint: use the properties of the augmenting path algorithm for the hard direction.)

6 Consider the Maximum Disjoint Paths problem: given an undirected graph $G = (V, E)$ with designated source $s \in V$ and sink $t \in V \setminus \{s\}$ vertices, find the maximum number of edge-disjoint paths from s to t. To formulate it as a linear program, we have a variable x_p for each possible path p that starts at the source s and ends at the sink t. The intuitive meaning of x_p is that it should take value 1 if the path p is used and 0 otherwise[1]. Let P be the set of all such paths from s to t. The linear programming relaxation of this problem now becomes

$$
\begin{array}{lll}
\text{Maximize} & \displaystyle\sum_{p \in P} x_p & \\
\text{subject to} & \displaystyle\sum_{p \in P : e \in p} x_p \leq 1, & \forall e \in E, \\
& x_p \geq 0, & \forall p \in P.
\end{array}
$$

What is the dual of this linear program? What famous combinatorial problem do binary solutions to the dual solve?

[1] The number of variables may be exponential, but let us not worry about that.