ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 12 — Modern Digital Communications

Midterm Solutions — November 9, 2022

Problem 1. 13 points (Paper and Pencil)

1. The signal at the output of the matched filter is

$$\begin{aligned} Y(t) &= R(t) \star h_{\mathrm{MF}}(t) \\ &= R(t) \star \psi^*(-t) \\ &= \int R(\alpha)\psi^*(\alpha - t)d\alpha \\ &= \langle R(\alpha), \psi(\alpha - t)\rangle. \end{aligned}$$

2.

$$\begin{aligned} c(\alpha) &= \int R(t)s^*(t-\alpha)dt \\ &= \langle R(t), s(t-\alpha)\rangle \\ &= \left\langle R(t), \sum_k s_k \psi(t-\alpha-kT) \right\rangle \\ &= \sum_k s_k^* \langle R(t), \psi(t-\alpha-kT)\rangle \\ &= \sum_k s_k^* Y(\alpha + kT). \end{aligned}$$

3.

$$\begin{aligned} c[m] := \sqrt{T_s}c(mT_s) &= \sum_k s_k^* \sqrt{T_s} Y(mT_s + kT) \\ &= \sum_k s_k^* Y[m + kN]. \end{aligned}$$

4.

$$\begin{aligned} c[m] &= \sum_k Y[m+kN]s_k^* \\ &= \sum_k Y[m+k]\hat{s}_k^*, \end{aligned}$$

where $\hat{s}_k$ is the upsampled sequence with upsampling factor N, i.e.,

$$\hat{s}_k := \begin{cases} s_{k/N} & \text{if } k/N \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

5. The formula above expresses $c[m]$ as being the correlation between $Y[\,]$ and $\hat{s}_k$. It can be also expressed as the convolution between $Y[\,]$ and p_k, where p_k is obtained by flipping $\hat{s}_k$ and complex conjugating it:

$$\begin{aligned} c[m] &= \sum_k Y[m+k]\hat{s}_k^* \\ &= \sum_k Y[k]\hat{s}_{k-m}^* \\ &= \sum_k Y[k]p_{m-k} \\ &= (Y \star p)[m], \end{aligned}$$

where $p_k = \hat{s}_{-k}^*$.

Problem 3. 4 points (Bonus Question)

1. Let $x(t) = caPulse(t)$ and $y(t) = caPulseDelayedWithDoppler(t)$.
   From the problem statement, we have that $y(t) = x(t-\tau)e^{j2\pi f_d t}$. Since

   $$\mathcal{F}\{x(t-\tau)\} = x_{\mathcal{F}}(f)e^{-j2\pi f\tau},$$

   we can easily relate their Fourier transforms as follows:

   $$y_{\mathcal{F}}(f) = \mathcal{F}\{x(t-\tau)e^{j2\pi f_d t}\} = x_{\mathcal{F}}(f-f_d)e^{-j2\pi(f-f_d)\tau}.$$

   Hence,

   $$|y_{\mathcal{F}}(f)| = |x_{\mathcal{F}}(f-f_d)|.$$

   The correlation between $|y_{\mathcal{F}}(f)|$ and $|x_{\mathcal{F}}(f)|$ will have its maximum at f_d.