ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 21 Modern Digital Communications
Final Exam December 22, 2021

Name:

Note:

- You have 2 hours to work on the exam.
- The exam is closed book, but you are allowed one double-sided A4 page of handwritten notes. Resources from the internet as well as code written outside this exam are not allowed (unless the code is written on the sheet of handwritten notes).
- The code will be evaluated according to the usual criteria, namely correctness, speed, form, and readability. Short comments that allow us to follow what you are doing will improve readability.
- The problems can be solved in any order.
- You will upload (to Moodle) your solution to the problems that require writing MATLAB or Python code. Do so in a single archive.

To get started with the exam, do the following:

1. Close all the windows and programs on your laptop.
2. Launch the MATLAB/Python editor and close all the tabs (previously written code).
3. From Moodle, download the file `mdc_final_2021.zip`. Unzip the file to create the directory `mdc_final_2021`. For the rest of the exam you are required to work inside that directory. The MATLAB/Python files for Problem n, are found in subfolder pn.
4. Turn your WiFi off until you are ready to upload your solutions.
5. Wait until you receive the go-ahead signal.

PROBLEM 1. 20 points (Paper and Pencil)

Consider a communication system with t transmitter and r receiver antennas, where $t, r \geq 2$ are integers. We assume that there is a single path between each transmit-receive antenna pair. The inputs and the outputs of the symbol-level channels in such a system will be related as

$$\boldsymbol{y}[n] = H\boldsymbol{x}[n] + \boldsymbol{z}[n], \text{ for } n = 1, 2, \cdots$$

where

- $\boldsymbol{x}[n] = (x_1[n], x_2[n], \cdots, x_t[n])^T$ is the vector of t transmitter antenna inputs;
- $H = \begin{bmatrix} H_{11} & H_{12} & \cdots & H_{1t} \\ H_{21} & H_{22} & \cdots & H_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ H_{r1} & H_{r2} & \cdots & H_{rt} \end{bmatrix}$ is the $r \times t$ channel matrix, which in assumed to be constant over time;
- $\boldsymbol{y}[n] = (y_1[n], y_2[n], \cdots, y_r[n])^T$ is the vector of r receiver antenna outputs; and
- $\boldsymbol{z}[n] = (z_1[n], z_2[n], \cdots, z_r[n])^T$ is the vector of r i.i.d. white Gaussian noise samples.

We would like to estimate H. To this end, on each transmitter antenna, we send training sequences of the form $\boldsymbol{x}_i = (x_i[1], x_i[2], \cdots, x_i[N])$, where $1 \leq i \leq t$ and N is the length of each training sequence. The training sequences are different for each transmitter antenna, and they are known to the receiver. We assume that $N \geq t$.

Consider the p-th receiver antenna, for some fixed p, where $1 \leq p \leq r$.

1. Let $\boldsymbol{y}_p = (y_p[1], \cdots, y_p[N])^T$ be the received sequence at this antenna as a result of the transmission of the training sequences . Let $\boldsymbol{h}_p = (H_{p1}, \cdots, H_{pt})^T$ and $\boldsymbol{z}_p = (z_p[1], \cdots, z_p[N])^T$. Specify the matrix S such that
$$\boldsymbol{y}_p = S\boldsymbol{h}_p + \boldsymbol{z}_p.$$
2. Write down the objective function which is being minimized by the least squares (LS) estimate $\hat{\boldsymbol{h}}_p$ of $\boldsymbol{h}_p$ based on $\boldsymbol{y}_p$.
3. The vector $S\hat{\boldsymbol{h}}_p$ is an element of an inner-product space $\mathcal{V}$. What is $\mathcal{V}$?
4. The projection theorem specifies the conditions that need to be satisfied by the LS estimate $\hat{\boldsymbol{h}}_p$. What are those conditions? Express them in terms of a matrix equation.
5. Find the expression for $\hat{\boldsymbol{h}}_p$ as a function of S and $\boldsymbol{y}_p$.

In the following, assume that $\boldsymbol{z}_p \sim \mathcal{N}_{\mathcal{C}}(0, \sigma^2 I_N)$, $E\left[\boldsymbol{h}_p \boldsymbol{h}_p^\dagger\right] = I_t$, and that $\boldsymbol{h}_p$ and $\boldsymbol{z}_p$ are statistically independent.

6. Write down the expression of the LMMSE estimator $\hat{\boldsymbol{h}}_{p,\texttt{LMMSE}}$ as a function of S, σ^2 and $\boldsymbol{y}_p$.
7. Assume that $N = t$. Specify the condition on σ^2 such that $\hat{\boldsymbol{h}}_{p,\texttt{LMMSE}} \approx \hat{\boldsymbol{h}}_p$.

Problem 2. 20 points (MATLAB/Python)

The file `rx_signal.mat` contains the (noisy and filtered) received samples of an OFDM signal transmitted over a discrete-time AWGN channel. However, the receiver is not yet synchronized, in the sense that the first received sample does not correspond to the start of the cyclic prefix of an OFDM block. The OFDM transmitter uses a total number of $N = 256$ carriers and a cyclic prefix of length $L = 25$. The transmitter sends as well a vector (of length N) of training symbols which are known to the receiver, used to estimate the channel coefficients (the lambdas). This vector is stored in the file `training_symbols.mat`.

Please follow the instructions in the script `p2.m[py]`.

PROBLEM 3. 20 points (MATLAB/Python)

Consider the communication system of Figure 1. You are asked to produce the samples of $R(t)$. The sequence $\{s_k\}$ consists of $N = 10^4$ BPSK symbols. The symbol period is $T = 10^{-6}$ seconds and we use SPS $= 5$ samples per symbol. The pulse shaping filter $\psi(t) = \text{sinc}(\frac{t}{T})$, and is truncated to a total length of 20 symbols (symmetric around the origin). The channel impulse response is $h(t) = \frac{1}{T}\,\text{sinc}(\frac{t}{T})$. The noise $N(t)$ is zero-mean, real-valued AWGN and its variance should be chosen such that we obtain an SNR of 20 dB.

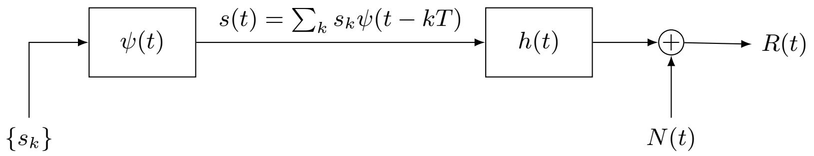

Figure 1: Communication System of Interest

Please follow the instructions in the script `p3.m[py]`.