ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 23 Modern Digital Communications
Final Solutions December 18, 2019

Problem 2.

1. $T_s = \frac{T}{N}$.
2. $h(t) = \text{sinc}\left(\frac{t}{T_s}\right)$.
3. We want to activate only one carrier, and the symbols on that carrier are $b_0 \dots b_9$. Hence $\mathbf{X}$ has 0's everywhere except for one row where it contains $b_0 \dots b_9$. Recall that the top row activates the carrier with frequency 0, the next row activates the carrier with frequency $1/T$, etc. In general, the ith row, $i = 1, \dots, N/2$ activates the carrier with frequency $(i-1)/T$. Hence the row of interest is the one with index i that satisfies $(i-1)/T = f_c$, i.e., $i = f_c T + 1$.
4. The purpose of the cyclic prefix is to make sure that each carrier acts as an eigenfunction (when we discard the channel output part associated to the cyclic prefix). We have no such requirement here. All we want is to create a BPSK signal.
5. By keeping the cyclic prefix, we extend the symbol interval from T to $T + T_L$, where T_L is the duration of the cyclic prefix. This leads to the desired result (BPSK), as long as we modify the parameters accordingly.

Problem 3.

1. We can write $\mathbf{Y} = M\mathbf{X} + \mathbf{Z}$, with
$$M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$
Then
$$B = M^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$
2. As derived in class, $\hat{\mathbf{X}}_{\text{MMSE}}(\mathbf{y}) = \mathbb{E}[\mathbf{X}|\mathbf{Y} = \mathbf{y}] = \mathbf{K_{XY}K_Y^{-1}y}$.
$K_{\mathbf{XY}} = \mathbb{E}[\mathbf{XY}^T] = \mathbb{E}[\mathbf{XX}^T]M^T = M^T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$,
$K_{\mathbf{Y}} = \mathbb{E}[\mathbf{YY}^T] = M\mathbb{E}[\mathbf{XX}^T]M^T + \mathbb{E}[\mathbf{ZZ}^T] = MM^T + I = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$.
Thus,
$$\begin{aligned} K_{\mathbf{XY}}K_{\mathbf{Y}}^{-1} &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}^{-1} \\ &= \frac{1}{5} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}. \end{aligned}$$
3. For jointly-Gaussian random vectors, the LMMSE and the MMSE estimators are the same (because the MMSE estimator is linear).
4. As derived in class,
$$\hat{\mathbf{X}}_{\text{LS}}(\mathbf{y}) = (M^T M)^{-1} M^T \mathbf{y}.$$
Since M has the inverse M^{-1}, we can further write
$$\hat{\mathbf{X}}_{\text{LS}}(\mathbf{y}) = (M^T M)^{-1} M^T \mathbf{y} = M^{-1} (M^T)^{-1} M^T \mathbf{y} = M^{-1}\mathbf{y}.$$